no more HW :)

LEC 15 : GRAV. WAVES — 7 MARCH

0709.4682

4 more lectures : GRAV. RADIATION , DIFF. FORMS

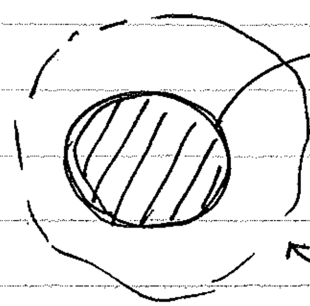
LAST TIME : A CONVENIENT EQUATION TO DIAGNOSE THE PHYSICS OF EINSTEIN EQ

$$\frac{\ddot{V}}{V} = -R_{tt} = -4\pi G\left(\rho + 3P\right)$$

from Geodesic deviation (SMALL, SPHERICAL BALL OF TEST PARTICLES INIT @ REST) → Geometry

from EINSTEIN EQ. (Physics) ; action principle

eg



PLANET, FULL OF CONST DENSITY STUFF

$T \neq 0$

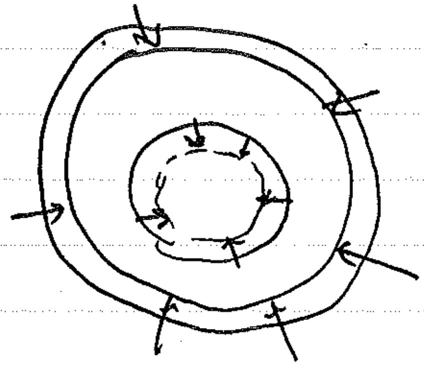
Gaussian surface

fill this w/ imaginary marbles (test balls)

inside the planet: volume shrinks

outside —"— : shape deformation

SHARON — Lifetimes & Resonances



$$\frac{\delta V}{V} = \underbrace{\frac{1}{2}\frac{\ddot{V}}{V}}_{-2\pi G\rho}(\delta t)^2$$

OUTER SHELL: $\underbrace{\delta V}_{4\pi r^2 \delta r} = -2\pi G \underbrace{\rho\,(\delta t)^2\, V}_{M}$

$$\Rightarrow \underbrace{\delta r = \frac{-2\pi G M}{4\pi r^2}\,\delta t^2 \equiv \frac{1}{2} a\, \delta t^2}$$

$$\boxed{a = -\frac{GM}{r^2}}$$

Newtonian limit

ANOTHER APPLICATION: shortcut to "baby cosmology" $\longrightarrow$ see Yonah's class for grown-up cosmology!

ANSATZ FOR SPH. SYM BUT t-DEP UNIVERSE:

$$ds^2 = dt^2 - a(t)^2 d\underline{x}^2$$

USUAL PATH TO COSMOLOGY:
PLUG IN THIS METRIC INTO EINSTEIN EQ, ~~SEE~~ WHAT HAPPENS.

$\hookrightarrow$ this case is even do-able by hand (w/ some patience) — metric only has one function... that depends on one coord.

RESULT: FRIEDMANN EQ.

we already bypassed this for cosmo. const when we explored EINST EQ. from ACTION PRINCIPLE.

NOW TRY IT FOR A UNIVERSE OF MATTER.

$$V \sim r^3$$

$$\frac{\ddot{V}}{V} = \frac{3\ddot{r}}{r} = -4\pi G \rho$$ ← ignore pressure

CAVEAT: seems like we're cheating ... in exp. universe, the initial test particles are not at rest... so our "$\nabla \cdot E = \rho$, $\nabla \cdot B = 0$" scenario seems invalid!

BUT: even though $\dot{r}(0) \neq 0$, it doesn't show up here.

CAN IMAGINE A SECOND TEST BALL W/ $R(0) = r(0)$, BUT TEST PARTICLES INIT @ REST w/rt ea other, $\dot{R}(0) = 0$. then: $\ddot{R}(0) = \ddot{r}(0)$ ... SO IT MAKES NO DIFF

CONSERVATION OF MASS: $\rho r^3 = \text{const}$.

$r = r(t)$ ... so r gets bigger → ρ dilutes

("REDSHIFT")

$$\frac{3\ddot{r}}{r} = -4\pi G \frac{\text{CONST}}{r^3}$$

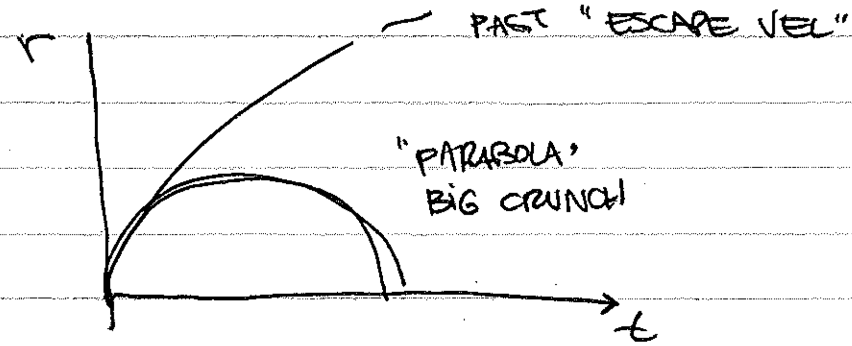
$$\Rightarrow \boxed{\ddot{r} = \frac{-4\pi G}{3} \frac{\text{const}}{r^2}}$$

as if matter felt Newtonian pot!

(BUT THIS IS AN ODE FOR LENGTH SCALES THEMSELVES)

We understand Newtonian trajectories

↳ either escapes or crashes back



cf: ADD DARK ENERGY:

battle btwn crunch & accel. expansion

$$\ddot{r} = \frac{-4\pi G}{3}\left(\frac{\text{const}}{r^2} + \Lambda r\right)$$

expon. expansion

LINEARIZED GRAVITY & GRAV. WAVES

$G_{\mu\nu} \sim T_{\mu\nu}$: NONLINEAR ⇒ ☹

1. HIGHLY SYMMETRIC SYSTEMS (eg FRIEDMANN EQ, SCHWARZSCHILD, ...)
2. NUMERICAL SOLUTIONS
3. <u>LINEAR LIMIT</u>

weak field st nonlinear terms assumed small

FAMILIAR FROM NEWTONIAN LIMIT

$$\boxed{g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)}$$

$$|h_{\mu\nu}(x)| \ll 1$$

nb this is: ASSUME $\exists$ COORDS s.t. $g_{\mu\nu} \approx \eta_{\mu\nu}$ (eg SPHR. COORDS CHANGES THIS!) SHOULDN'T WE BE COORD-INDEP? yes... but let's work in a very convenient choice!

LINEAR THEORY: EVERYTHING TO $\mathcal{O}(h_{\mu\nu})$, NO HIGHER

so, eg. $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$\uparrow$$

$$\text{\& } h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$$

UNDER A LORENTZ TRANSFORMATION, $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

$$g_{\cdot\cdot} = \underbrace{\Lambda^{\cdot}{}_{\cdot}\Lambda^{\cdot}{}_{\cdot}\eta_{\cdot\cdot}}_{=\eta} + \underbrace{\Lambda^{\cdot}{}_{\cdot}\Lambda^{\cdot}{}_{\cdot}h_{\cdot\cdot}}_{\text{transforms like a "2 lower index tensor" on flat space}}$$

ie a SPECIAL RELATIVITY TRANSFORM

But: wasn't the whole point that GR is invariant under general coordinate transforms?

Yes — BUT RESTRICTING TO "APPROXIMATELY SR" IS A USEFUL LIMIT.

SO WEAK FIELD GRAV IN $\approx$ MINKOWSKI COORDS
IS $\underline{\text{FLAT SPACE}}$ W/ SYM. $\underline{\text{TENSOR}}$ $\underline{h_{\mu\nu}(x)}$

$\uparrow$
transf. properties

CARROLL 7.1

THE CURVATURE PIPELINE:

$$\Gamma \sim \partial g \longrightarrow \partial h$$

$$\hookrightarrow = \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right)$$

$$R_{\dots} \sim \partial\Gamma + \Gamma\Gamma - (\cdot \leftrightarrow \cdot)$$

$\uparrow$
higher $\mathcal{O}$ in h $\rightarrow$ ignore

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho\Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^\lambda_{\nu\rho}$$

$\eta^{\mu\rho}$

$$= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - (\rho \leftrightarrow \sigma)\right)$$

$\eta^{\nu\sigma}$

$$R_{\nu\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\sigma + \partial_\sigma\partial_\mu h^\mu{}_\nu - \partial^2 h_{\nu\sigma} - \partial_\sigma\partial_\nu h\right)$$

$$\boxed{R = \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h}$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R$$

$$= \tfrac{1}{2}\big(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu$$
$$- \partial_\mu\partial_\nu h - \partial^2 h_{\mu\nu}$$
$$- \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\partial^2 h\big)$$

<u>OR</u>: CONVENIENT TO DEFINE
TRACE REVERSE TENSOR:

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h}$$ (similarly w/ $\bar{h} \leftrightarrow h$)

$$\text{s.t. } \bar{h} = \bar{h}^\mu{}_\mu = -h^\mu{}_\mu$$

then Einstein tensor simplifies a bit

$$G_{\mu\nu} = -\tfrac{1}{2}\big(\partial^2\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}$$
$$- \partial^\alpha\partial_\mu\bar{h}_{\alpha\nu}$$
$$- \partial^\alpha\partial_\nu h_{\alpha\mu}$$
$$+ \mathcal{O}(h^2)\big)$$

$\partial^2 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$

.. still kind of a mess ...

so: that's what we get turning the crank.
set this to $8\pi G\, T_{\mu\nu}$ & that's
einstein's eq.
{ GRAND REVIEW OF TECHNICAL MACHINERY.

something slightly different:
GAUGE TRANSFORMATIONS

WE ARE RESTRICTING TO A CLASS OF COORDINATES

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{|h_{\mu\nu}| \ll 1} \qquad (\star)$$

We gave up on completely general coord. invariance, these coords made a ~~Taylor~~ physically meaningful Taylor expansion.

BUT THIS IS NOT A UNIQUE DEF OF COORDS! ∃ COORD TRANSFORMS THAT PRESERVE $(\star)$

$$x'^{\mu} = x^{\mu} + \xi(x)^{\mu} \qquad (9)$$

$$\left(\frac{\partial x'}{\partial x}\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\xi^{\mu} + \mathcal{O}(\xi^2)$$ UPPER INDEX TRANSF

$$\left(\frac{\partial x}{\partial x'}\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \partial_{\nu}\xi^{\mu} + \mathcal{O}(\xi^2)$$ LOWER INDEX TRANSFORM

then: $g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \mathcal{O}(h^2, \xi^2)$

$$(\delta^\mu{}_\alpha - \partial_\alpha \xi^\mu)(\delta^\nu{}_\beta - \partial_\beta \xi^\nu)\, g_{\alpha\beta}$$

So for $|\partial\xi| \ll 1$, (a) PRESERVES OUR CONVENIENT COORDINATES

call this a Gauge transform.

why "GAUGE"?

↳ analogous to GAUGE transf. in EM:

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$F_{\mu\nu} \to F_{\mu\nu}$ ("COHOMOLOGY")

doing this transf. does not change the physics derived from (*)

↳ STRONGER STATEMENT:

IN EM: $\partial\alpha$ IS A REDUNDANCY IN MATH. DESCR OF PHYS

IN GR: $y(x)$ PLAYS ANALOGOUS ROLE

(a) IS A RESTR. SUBSET FOR THE WEAK FIELD LIMIT

see, eg, intro of 1702.00319

BY THE WAY: this $\partial_{(\mu}\xi_{\nu)}$ structure may look familiar from when we discussed KILLING VECTORS

$$\boxed{\mathcal{L}_\xi \eta_{\mu\nu}}$$

$g_{\mu\nu}$ @ LO

transformation of $g_{\mu\nu}$ as we flow along ξ

COMPARATOR OF $g_{\mu\nu}$ @ 2 diff points $\leftrightarrow$ 2 diff coords.

SCHUTZ 8.4

NOW RETURN TO $G_{\mu\nu}$ in (☺) (p.9)

MANY TERMS DISAPPEAR IF ONLY $\boxed{\partial_\alpha \bar{h}^{\mu\alpha} = 0}$ (☺)

↳ CAN WE MAKE THIS TRUE w/ OUR GAUGE FREEDOM?

(☺) is 4 CONDITIONS
(☺) IS 4 FREEDOMS $\Big\}$ LOOKS GOOD!

(☺) IS CALLED LORENTZ GAUGE, ANALOGOUS TO EM ∂A^{α}

SUPPOSE $h_{\mu\nu}$ s.t. $\partial_\alpha h^{\mu\alpha} \neq 0$

RECALL $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

$$\downarrow$$

$$\bar{h}'_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\xi_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\partial_\beta \xi^\beta\right)$$

$$= \bar{h}_{\mu\nu} - \partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_\beta \xi^\beta$$

$$\partial_\alpha \bar{h}'^{\mu\alpha} = \underbrace{\partial_\alpha \bar{h}^{\mu\alpha}}_{\text{some non-zero thing}} \underbrace{- \partial_\alpha \partial^{(\mu}\xi^{\alpha)} + \partial_\alpha \eta^{\mu\alpha}(\partial\cdot\xi)}_{-\partial^2\xi^\mu - \cancel{\partial^\mu(\partial\cdot\xi)} + \cancel{\partial^\mu(\partial\cdot\xi)}}$$

so: $\boxed{\partial^2 \xi^\mu = \partial_\alpha \bar{h}^{\mu\alpha}}$

this is the 3D wave eq.

$\partial^2 = \partial_t^2 - \underline{\nabla}^2$

SO CAN REFER TO EXISTENCE PROOFS OF SOLUTIONS.

☺ UNIQUENESS: CAN ADD ANY HOMOGENEOUS SOLUTION TO ξ^μ.

LEFTOVER FREEDOM

RESULT: WAVE EQ. FOR EINSTEIN EQ.

$$G_{\mu\nu} \rightarrow -\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\boxed{\partial^2 \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}}$$

LINEARIZED EINSTEIN EQ.

GRAVITATIONAL WAVES: $T_{\mu\nu} = 0$

(analogous to EM waves → the waves source their own propagation)

$$(\partial_t^2 - \nabla^2)\bar{h}_{\mu\nu} = 0$$

POLARIZATION (16 elem) → 10 by sym.

WE KNOW THE ANSWER: $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik\cdot x}$

$$e^{ik^0 t - i\underline{k}\cdot\underline{x}}$$

$$(ik^0)^2 - (i\underline{k})^2 = -k^2 = 0$$

FREQ/ENERGY

WAVE VECTOR

C LIGHTLIKE MOMENTA
travels @ speed of light

IMPLICIT: LORENTZ GAUGE CONDITION

$$\partial_\alpha \bar{h}^{\mu\alpha} = 0$$

$$\underbrace{(iK_\alpha)\, A^{\mu\alpha}}\, e^{ik\cdot x} = 0$$

$$\boxed{K_\alpha A^{\mu\alpha} = 0}$$

→ transverse polarization

$10 \to 6$ DEG OF FREEDOM

PUSH FORWARD: speaking of gauge choices, we still have leftover gauge freedom: $\tilde{\xi}$

$$\cancel{\xi \to \xi + \cdots}$$

st $\partial^2 \tilde{\xi} = 0$

4 more DOF to SIMPLIFY THINGS → $\tilde{\xi}^\mu = B^\mu e^{ik\cdot x}$

different from the one that took us to LORENTZ GAUGE. $\tilde{\xi}$ LEAVES US IN LORENTZ GAUGE.

<u>then</u>: $\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{(\mu}\tilde{\xi}_{\nu)} + \eta_{\mu\nu}\,\partial\tilde{\xi}$

$\bar{h}_{\mu\nu} \leftarrow A_{\mu\nu}\, e^{ikx}$

$$A'_{\mu\nu} = A_{\mu\nu} - iB_{(\mu}K_{\nu)} + i\eta_{\mu\nu} B\cdot K$$

SCHUTZ 0709.4682

<u>CAN CHOOSE</u>

(I) B^0 s.t. $\boxed{\bar{h}^\alpha{}_\alpha = 0}$ ($A'^\alpha{}_\alpha$) traceless $\underline{(\bar{h} = h)}$

(II) B^i s.t. $\boxed{h^{i0} = 0}$ $\leftrightarrow A'^{i0} = 0$

BUT $K_\alpha A'^{\mu\alpha} = 0$ BY LORENTZ

$\mu = 0$

$K_0 A'^{00} = \boxed{0 \sim \partial_0 \bar{h}^{00}}$

can set $\boxed{\bar{h}^{00} = 0}$ for all time

BTW: MORE INVT. WAY OF SAYING THIS IS
$A'_{\mu\nu} U^\nu = 0$
$\forall$ OBS 4-VEL. U^ν

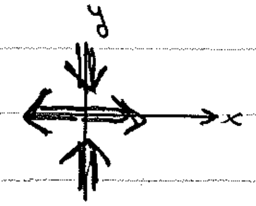
so: $\left.\begin{aligned} h^{00} &= 0 \\ h^{0i} &= 0 \\ \partial_i h^{ij} &= 0 \\ h^{ii} &= 0 \end{aligned}\right\}$ <u>transverse traceless gauge</u> (TT)

$$h^{TT}_{\mu\nu} \doteq \begin{pmatrix} 0 & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & 0 \end{pmatrix} e^{ik\cdot x}$$

ASSUMED $\underline{k}$ in $\hat{z}$ DIR.

CAN SEE WHAT THESE DO

$$\begin{pmatrix} h_+ & \\ & -h_+ \end{pmatrix} e^{ik\cdot x}$$



$$\begin{pmatrix} & h_\times \\ h_\times & \end{pmatrix} e^{ik\cdot x}$$

eigenvecs : $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$