→ BASED ON gr-qc/0103044
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LEC 14: UNDERSTANDING EINSTEIN'S EQ. 2 MAR.

EINSTEIN EQ: $\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} = \underline{\underline{8\pi G}}\, T_{\mu\nu}$

↑ the origin of many conventions for the grav. coupling

motivated by Maxwell's eqs
more complicated ... more indices

ALSO MORE COMPLICATED – *NOT* <u>LINEAR</u>

↳ sum of solutions is <u>not</u> a solution
COMPLETELY DIFFERENT FROM NEWTON

Φ_1 Φ_2

stuff$_1$ stuff$_2$

$$\Phi_{tot} = \Phi_1 + \Phi_2$$

b/c $\vec{g}(x) = -\nabla \Phi(x)$

generally: $\Theta\, \Phi$

LINEAR OPERATOR — POTENTIAL

THIS MAKES FINDING SOLUTIONS <u>HARD</u>.

eg CANNOT SUPERIMPOSE SCHWARZSCHILD W/ "STUFF IN OTHERWISE FLAT SPACE

MORE INTUITIVELY: EARTH & SUN

separately: $E_\odot = m_\odot$ $E_\oplus = m_\oplus$

together: forms a BOUND STATE: solar system

very seperated solar sys:
$E_{ss} \approx m_\odot + m_\oplus \leftarrow$ "m_{ss}"

but bring them closer:
$E_{ss} = m_\odot + m_\oplus - \Delta E_{BINDING} \overset{?}{\leftarrow}$ "m_{ss}"

cf STRONG (OR EINSTEIN) EQUIV. PRINC. which was about free fall

Eh?! this violates [weak] Equivalence Principle

RECALL: MASS $\longrightarrow$ GRAV. CHARGE $F = m(-\nabla\Phi)$
$\searrow$ $F = \underline{m}a$

EQUIVALENCE: these are the same
(ie GRAV. ACC. IS INDEP OF TYPE OF MASS)

BUT if $\underset{\text{"CHARGE"}}{m_{grav}} = m_\odot + m_\oplus$, $\neq m_\odot + m_\oplus \underline{- \Delta E_{BIND}}$

SOLUTION: W.E.P. HOLDS, BUT ONLY BC GRAVITY COUPLES TO GRAVITATIONAL FIELD OF BOUND STATE

i.e. grav. self coupling.

contrast to electromagnetism:
sure, <u>changing</u> E field → B field
& so forth ⇒ EM waves

⌞ same thing for grav waves...

[ENERGY OF]
BUT DIFFERENT FROM "<u>GRAV FIELD ALSO SOURCES GRAVITY</u>"

So: it's complicated.

WHAT ABOUT ALL OF THOSE COMPONENTS!!?
$G_{\mu\nu} \sim T_{\mu\nu}$ ← 16 EQUATIONS.

BUT: $G_{..}$ & $T_{..}$ are <u>symmetric</u> : only <u>10 eqns</u>

<u>further</u>: they are <u>covariantly constant</u>

$$D_{\cdot} T^{\cdot\mu} = 0 \,, \quad D_{\cdot} G^{\cdot\mu} = 0$$

GIVES 4 "CONSTRAINTS"
really: redundancies — the tensors
are constructed to be cov. const,
so some of these eqs are
redundant. analogous to
symmetry — REDUNDANT

so only <u>6 indep eqs</u>

<u>intuition</u> $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ is a statement about physics — INDEPENDENT OF COORDINATES

so we have a "change of coordinates" redundancy

$$x^\mu \to y(x)^\mu \leftarrow \underline{4 \text{ EQS}}$$

WON'T CHANGE PHYSICS

... still a lot of components
+ still nonlinear —— all hard to solve
... typically

1. APPEAL TO <u>SYMMETRY</u>
2. WORK IN <u>LINEAR</u> APPROX.
3. NUMERICAL RELATIVITY

↑

eg. one of the dudes that wrote Numerical Recipes

BUT WE CAN REDUCE FURTHER IF WE ONLY ASK SPECIFIC, ILLUMINATING QUESTIONS.

RECALL OLD HW:

EQUIV PRINCIPLE: CAN SET $g_{\mu\nu} \to \eta_{\mu\nu}$ @ one point
"no gravity" @ that point

BUT: feel gravity when you look away from that one point.

GEODESIC DEVIATION: free fall for nearby particles

$$\underbrace{\frac{D^2}{d\tau^2}}_{\text{covariant deriv. along } x(\tau)} \underbrace{\delta x^\lambda}_{\text{rel. displacement of two test particles}} = \underbrace{R^\lambda{}_{\nu\mu\rho}}_{\text{Riemann tensor}} \delta x^\mu \dot{x}^\nu \dot{x}^\rho$$

like a relative acceleration

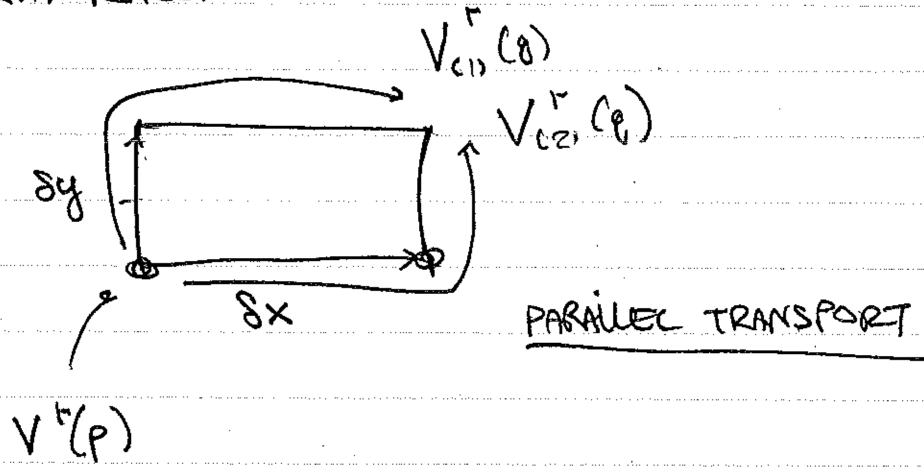
this is the 'more useful concept'

TIDAL FORCES THAT YOU FEEL AS YOU FREE FALL.

AKA: WHY EQUIV. PRINCIPLE WON'T SAVE YOU AS YOU FALL INTO BLACK HOLE

RECALL HOW THIS WORKED:

Riemann Tensor:



HOW ARE $V_{(1)}$ & $V_{(2)}$ RELATED?

if $\delta x = \epsilon W$

$\delta y = \epsilon u$

ORIGINAL VEC.

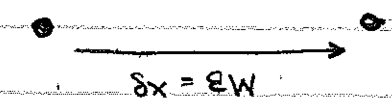
then: AS $\varepsilon \to 0$, $V^{\mu}_{(2)} - V^{\mu}_{(1)} = \varepsilon^2 \underbrace{R^{\mu}{}_{\alpha\beta\gamma} W^{\alpha} u^{\beta}}_{R(W,u)^{\mu}{}_{\gamma}} V^{\gamma}$

$R(W,u)^{\mu}{}_{\gamma} V^{\gamma}$

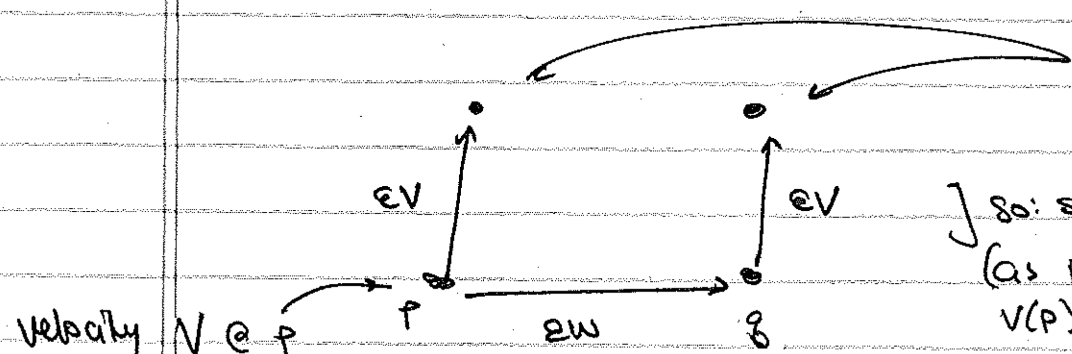
Use this machinery for relative acceleration

(WE'RE BASICALLY THERE)

a very similar picture



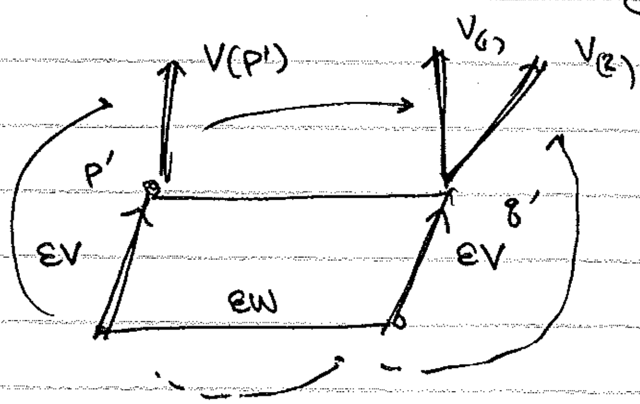
two test masses, m/t @ rest relative to each other



flows to these points in $\delta t = \varepsilon$

so: SAME VELOCITY (as if we transported $V(p) \rightarrow V(q)$)

WHAT IS THE RELATIVE VELOCITY AFTER $\delta t = \varepsilon$?



then: $$\frac{V_{(2)} - V_{(1)}}{\varepsilon^2} = R(W,V)V$$

$$\frac{\Delta V / \Delta t}{\varepsilon} = \boxed{\frac{a}{\varepsilon}}$$

REL. ACCELERATION

SLIGHTLY MORE CONVENIENT TO WRITE

$R(W,U)V = -R(U,W)V$

SO: $$\boxed{\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} a^{\mu} = -R^{\mu}{}_{\alpha\beta\gamma} V^{\alpha} W^{\beta} V^{\gamma}}$$

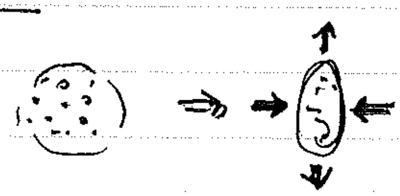
~ displacement in space

WE CAN USE THIS TO ASK A QUESTION THAT DISTILLS THE KEY INFO OF EINSTEIN'S EQ.

CONSIDER A ~~VOLUME~~ BALL OF TEST PARTICLES W/ VOLUME $V(t)$.

↳ start @ relative rest

RECALL: TIDAL FORCE:
(@ 2ND @ in time)



align axes along ellipsoid axes.

$r^j(t)$

no rel. vel. @ $t=0$

$$r^i(t) = r^i_{(0)} + \frac{1}{2} a^i t^2 + O(t^3)$$

$r^i_{(0)} = \varepsilon$, INIT RADIUS

$$\Rightarrow \lim_{t\to 0} \frac{\ddot{r}^j}{r^j} = \lim_{t \to 0} \frac{a^j}{\varepsilon} = -R^j{}_{\alpha\beta\gamma} V^\alpha W^\beta V^\gamma$$

DISPLACEMENT IN j DIR

$$= -R^j{}_{\alpha j \gamma} V^\alpha V^\gamma = \boxed{-R^j{}_{tjt}}$$

RELATIVELY @ REST,
so 4-VELOCITY is $V^\alpha = \delta^\alpha_t$
(UNIT 4-VEL IN TIME DIRECTION)

VOLUME OF ELLIPSOID $\propto r^1 r^2 r^3$

$$\frac{\ddot{V}}{V} = \sum_i \frac{\ddot{r}_i}{r_i} + \underbrace{2\sum_{ij} \frac{\dot{r}_i \dot{r}_j}{r_i r_j}}_{=0 \text{ bc } \dot{V}=0 \text{ initially}}$$

@ t=0

<u>then</u>: $\frac{\ddot{V}}{V} = \sum_i \frac{\ddot{r}^j}{r} \xrightarrow[\text{aka } r\to 0]{\varepsilon \to 0} -R^j{}_{tjt}$

$= -R^\alpha{}_{t\alpha t}$

because $R^t{}_{ttt} = 0$
by symmetry properties

this is simply $-R_{tt}$ ← ah, now we're getting close to Einstein eq!

RECALL ALTERNATE FORM OF ~~[illegible]~~ EINSTEIN EQ:

$$\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \right) g^{\mu\nu}$$

$$R - 2R = 8\pi G T^\mu{}_\mu$$ $\equiv T$, trace of T..

$$R = -8\pi G T$$

$$\boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right)}$$

then we can pare down the PHYSICS of Einstein's eq to:

WHAT HAPPENS TO A small BALL OF TEST PARTICLES IN THE PRESENCE OF gravity + stuff?

$$\lim_{V \to 0} \frac{\ddot{V}}{V}\Big|_{t=0} = -R_{tt} = 8\pi G\left(T_{tt} - \frac{1}{2} g_{tt} T\right)$$

EQUIVALENCE PRINCIPLE

$g_{\mu\nu} = \eta_{\mu\nu}$ @ A POINT

st $g_{tt} = 1$

$$R_{tt} = -8\pi G\left(T_{tt} - \frac{1}{2}(T_{tt} - T_{xx} - T_{yy} - T_{zz})\right)$$

$$= -4\pi G\,(\underbrace{T_{tt} + T_{xx} + T_{yy} + T_{zz}}_{\text{"}\rho + P_x + P_y + P_z\text{"}})$$

FANCY pf: Raychaudhuri eq. (APPENDIX OF CARROLL)

<u>one</u> equation ?!
doesn't even use off-diagonal
parts of $T_{\mu\nu}$?!

I thought EINSTEIN EQ has
6 eqns inside ?!

OUR INIT ASSUMPTION: BALL OF TEST PARTICLES
<u>AT REST</u>.

all "6 eqns" come from considering
all possible initial velocities

$\sim$ # generators of Lorentz transf

<u>ANALOG</u>: $\nabla \cdot E \sim \rho$ & $\nabla \cdot B = 0$
ELECTRO/MAGNETO STATICS

if you boost in diff dir, these
"transform into" full Maxwell's eq.

yield same info.

SOME OBSERVATIONS

SOURCES OF GRAV. COLLAPSE

assume isotropic

$$\frac{\ddot{V}}{V} = -4\pi G\left(\rho + 3P\right)$$

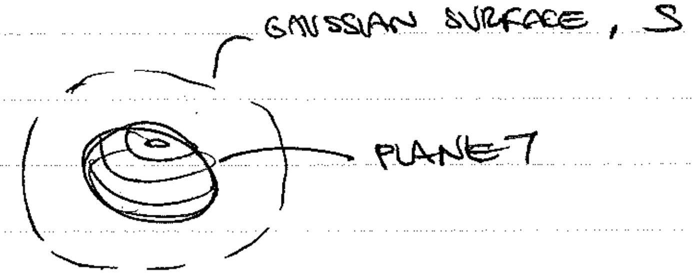
wait: isn't pressure supposed to push OUTWARD?

yes. & it does. eg in a neutron star, degeneracy pressure resists grav. collapse to a black hole.

BUT THE ENERGY OF THE PRESSURE ALSO CONTRIBUTES TO THE GRAVITY.

in a neutron star, $\rho \sim P$, MASS & PRESSURE CONTRIBUTE ON THE SAME O MAGNITUDE!

NEWTONIAN LIMIT (weak field, low pressure)

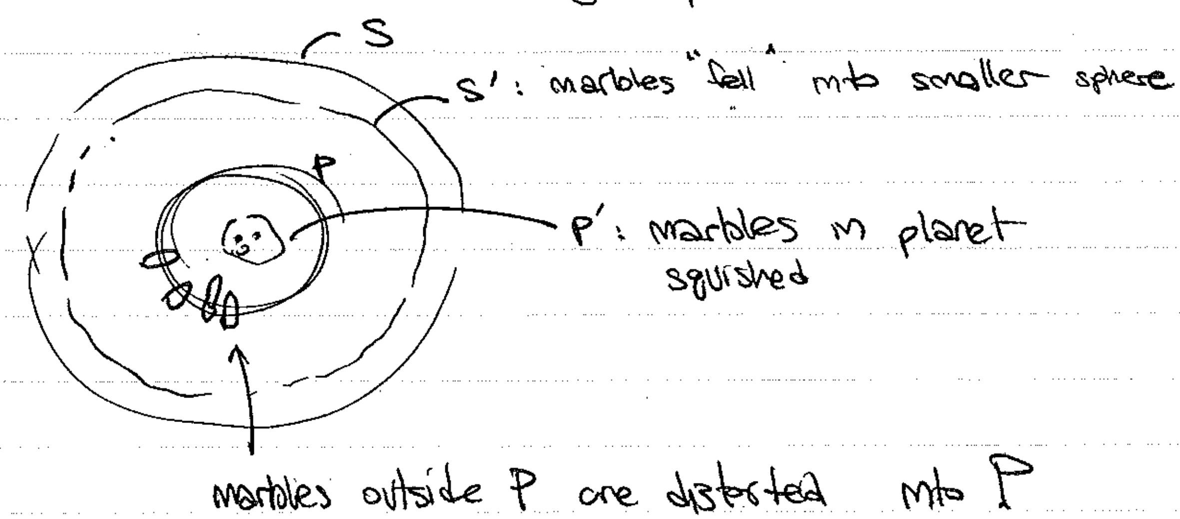


fill the gaussian surface w/ imaginary marbles (TEST VOLUMES)

INSIDE PLANET: $\ddot{V}/V = -\frac{1}{2}\rho$ ← volume of marbles shrink

OUTSIDE " : $\ddot{V}/V = 0$ ← no vol. change (BUT TIDAL DISTORTION OF SHAPE)

in time δt, the imaginary marbles reconfigure



$$\frac{\delta V}{V} = \frac{1}{2}\frac{\ddot{V}}{V}(\delta t)^2 = -2\pi G\,\rho\,(\delta t)^2$$

VOLUME "LOST", $\delta V_s = \delta V_p$

$$\delta V_s = -2\pi G\,\rho\left(\frac{\delta V}{V}\right)V$$

$$= -2\pi G\,\underbrace{\rho\,(\delta t)^2\,V}_{M} = -2\pi G\,M\,\delta t^2$$

$$= 4\pi r^2\,\delta r$$

$$\delta r = -\frac{G}{2}M\frac{1}{r^2}\delta t^2$$

$$\parallel$$

$$\frac{1}{2}a\,\delta t^2$$

$$\Rightarrow \boxed{a = -GM/r^2}$$

EARLY UNIVERSE MODELS

expanding universe: $ds^2 = dt^2 - a(t)^2 d\underline{x}^2$

BALL B

seems easy to test: BALL OF TEST PARTICLES...
JUST SEE WHAT HAPPENS — direct probe of "expansion of space"

<u>BUT</u>: we always had "ball of test particles @ rest" @ $t=0$

this is <u>not</u> the case for the expanding universe

BALL B'

NEED <u>SECOND</u> BALL OF TEST PARTICLES THAT ARE AT REST w/rt ea other.

B B'

$$R(0) = r(0)$$

$\dot{R}(0) \neq 0$ but $\dot{r}(0) = 0$

$$\ddot{R}(0) = \ddot{r}(0)$$

r of B' $V \sim r^3$

$$\frac{\ddot{V}}{V} = \frac{3\ddot{r}}{r} = -4\pi G(\rho + 3P)$$

$\frac{3\ddot{R}}{R}$ indep of 1st deriv.
(so it didn't end up mattering!)

turns out: true $\forall$ size BALL
(ASSUMING HOMOG. EXP. UNIV.)
& nothing special about $t=0$; true $\forall t$.

IN CASE OF PRESSURELESS MATTER (DM) (or GAL.

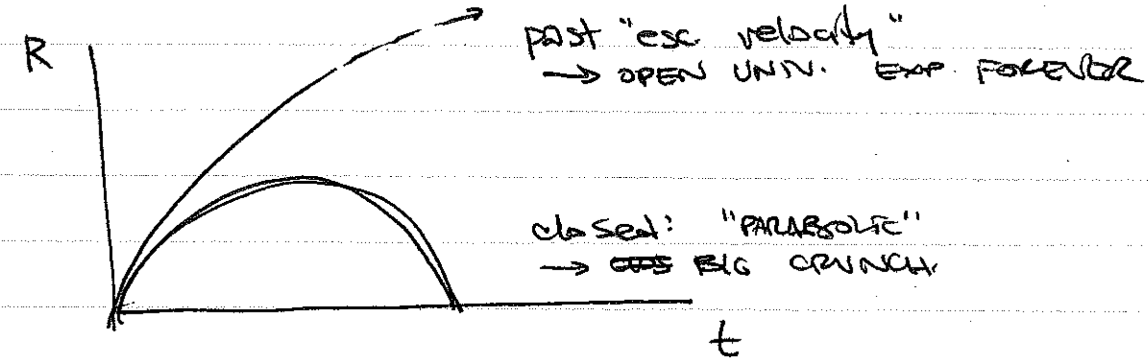
conservation of GAL/DM : $\rho R^3 = \text{CONST}$

$$\frac{3\ddot{R}}{R} = -4\pi G\left(\frac{\text{const}}{R^3}\right)$$

$$\boxed{\ddot{R} = -\frac{4\pi G}{3}\frac{\text{const}}{R^2}}$$

as if we are in
some Newtonian pot!!

So dynamics of an exp. univ
w/ only "DM":



Stuff + DE: $T_{DE} = \Lambda g_{\mu\nu} = \begin{pmatrix} \Lambda & & & \\ & -\Lambda & & \\ & & -\Lambda & \\ & & & -\Lambda \end{pmatrix}$

in inertial frame

$$\Rightarrow \ddot{R} = -\frac{4\pi}{3} G \left(\frac{\text{const}}{R^2} + \Lambda R \right)$$

If this gets smaller than $\sim \Lambda R$, then this takes over → exponential expansion (ASSUMING $\Lambda > 0$)