· FROM LAST TIME: why $\sqrt{g}$

LEC 13: EINSTEIN EQ. / ADVANCED TENSORS

21 FEB

LAST TIME: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \; T_{\mu\nu}$

how? CURVATURE $\sim$ ENERGY-MOMENTUM

RIEMANN TENSOR "G" $T_{\mu\nu}$

SYM 2-TENSOR

$\sim R_{\mu\nu} + \alpha R g_{\mu\nu}$

COVARIANTLY CONSTANT: $D_\mu T^{\mu\nu} = 0$

$\sim R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

$8\pi G$ from matching Newtonian limit

SO WE HACKED TOGETHER A PLAUSIBLE ~~STUFF~~ EQUATION OF MOTION FOR SPACETIME & ~~MATTER~~ stuff.

TODAY: SYSTEMATIC DERIVATION from AN ACTION PRINCIPLE

$$S = S_{EH} + S_{STUFF} \qquad g = |\det g_{..}|$$

want to study

ALL OUR USUAL ACTIONS, BUT:
(1) $d^4x \to d^4x\sqrt{g}$
(2) $\partial_\mu \to D_\mu$

COUPLES STUFF TO GRAVITY.

$$S_{EH} = \int d^4x \sqrt{g}\, ?$$

(EINSTEIN-HILBERT)

SINGLET, not tensor

THERE'S ONE OBVIOUS CHOICE:

$$? = \beta R$$

RICCI SCALAR

SOME CONSTANT WHICH CONTROLS S_{EH} SIZE RELATIVE TO S_{STUFF} (related to coupling)

In fact: ? $\propto R$ is <u>unique</u> term that includes up to 2 derivatives of $g_{\mu\nu}$.

$$R \sim R^{\alpha\beta}{}_{\alpha\beta} \sim \partial\Gamma,\ \Gamma^2 \sim \partial^2 g,\ (\partial g)^2$$

DIMENSIONAL ANALYSIS

$$[S] = 0 \quad \text{(DIM-LESS)}$$

$$S = \int \underbrace{d^4x}_{L^4}\ \underbrace{\sqrt{g}}_{\text{DIM LESS}}\ \beta\ \underbrace{R}_{L^{-2}}$$

L^{-2} from two $\partial/\partial x$'s.

$$\Rightarrow \beta \sim L^{-2}$$

WE HAVE ONE DIMENSIONFUL PARAMETER: G

$$G\frac{Mm}{r} \sim \underbrace{V}_{\sim E \sim M}$$

$$\Rightarrow G \sim L/\underbrace{M}_{M_{Pl}} \sim \underbrace{\hbar/M_{Pl}^2}_{L^2} \qquad \text{USING } \hbar \sim E\,t$$

so: $\beta \sim G^{-2}$

nb: dividing by a ~~tiny~~ number!
NOT A BIG DEAL. JUST SHOWS THAT THERE IS A LARGE HIERARCHY BETWEEN S_{EH} & S_{stuff} PREFACTORS

<u>OBSERVE</u>: WE COULD HAVE ESCHEWED OCCAM'S RAZOR & PROPOSED, eg

$$S' = \int d^4x \sqrt{g}\, \beta\, a\, R^2$$

$\sim G^{-2}$; to compare size to S_{EH}.

$[R^2] = L^{-4}$
similarly for $\partial^2 R, \ldots$

$$\Rightarrow [\beta a] = 0 \Rightarrow [a] = L^2$$
$$\Rightarrow \boxed{a \sim G}$$

so: next simplest term is suppressed by G!

SO EINSTEIN'S GRAVITY, WHERE WE USE THE SIMPLEST S_{EH}, IS ROOTED IN THE NOTION OF <u>EFFECTIVE THEORY</u>.

<u>NATURE</u> may have a more complicated gravitational action:

$$\underbrace{S_{EH} + S' + S'' + \cdots}_{} + S_{stuff}$$

BUT ALL THIS ~~STUFF~~ is SUPPRESSED RELATIVE TO S_{EH} BY POWERS OF $G \sim \hbar/M_{Pl}^2$.

they MODIFY THE EOM from the Einstein eq., <u>BUT</u> only with a tiny coefficient.

[ASSUMPTION: DIMENSIONLESS NUMBERS ARE $\mathcal{O}(1)$ $\leftrightarrow$ naturalness]

cf Nathaniel Craig's colloqurum two weeks ago.

$$\delta S_{EH} = \int d^4x \Big[\sqrt{g}\, g^{\mu\nu} (\delta R_{\mu\nu})^{\text{(I)}} + \sqrt{g}\, (\delta g^{\mu\nu}) R_{\mu\nu}^{\text{(II)}} + (\delta\sqrt{g}) R^{\text{(III)}} \Big]$$

two pieces in variation of R... metric matters.

WHAT ARE WE VARYING WITH RESPECT TO? $g_{\mu\nu}(x)$

note: $\delta g_{\mu\nu}$ & $\delta g^{\mu\nu}$ are different :

in (II)

POSTPONE TO P. 4

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu}$$ CONSTANT

$$\Rightarrow \delta g^{\mu\lambda} g_{\lambda\nu} + g^{\mu\lambda} \delta g_{\lambda\nu} = 0$$

mult by: $g^{\nu\rho}$

$$\Rightarrow \delta g^{\mu\lambda} \delta^{\rho}_{\lambda} + g^{\mu\lambda} (\delta g_{\lambda\nu}) g^{\nu\rho} = 0$$

$$\Rightarrow \delta g^{\mu\rho} = - g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\nu}$$

mult by: $g_{\rho\mu}$

$$\Rightarrow g_{\rho\mu} (\delta g)^{\mu\lambda} g_{\lambda\nu} + \delta^{\lambda}_{\rho} \delta g_{\lambda\nu} = 0$$

$$\Rightarrow \boxed{\delta g_{\rho\nu} = - g_{\rho\mu} g_{\lambda\nu} (\delta g^{\mu\lambda})}$$

LET US WRITE OUR VARIATIONS W/RT THIS

good exercise in index manipulation

NOW LET'S GET TO WORK

nb $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$

(I): $\delta R_{\mu\nu} = \delta R^{\rho}{}_{\mu\lambda\nu} \quad \delta^{\lambda}{}_{\rho}$

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu)$$

then the variation of $R^{\cdot}{}_{\cdots}$ is expressed as a variation of $\Gamma^{\cdot}_{\cdot\cdot}$:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \delta\Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - (\lambda \longleftrightarrow \nu) \qquad (*)$$

HW

2180: see cosmo

CLAIM: $= D_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - D_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \qquad (**)$

BUT WAIT! $\Gamma^{\cdot}_{\cdot\cdot}$ is not a tensor... $D_{\cdot}$ is a deriv. that preserves covariance ... does it make sense?

YES, because $\delta\Gamma^{\cdot}_{\cdot\cdot}$ is a tensor.

IT'S THE DIFFERENCE OF 2 CONNECTIONS

(3.10) recall: $$(\Gamma')^{\nu}_{\mu\lambda} = \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\lambda}\frac{\partial x'^\nu}{\partial x^\beta}}_{\text{tensorial trans}}\Gamma^{\beta}_{\alpha\delta} - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\lambda}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta}$$

non tensorial part INDEP OF CONNECTION

By the rules of covariant differentiation $D_\lambda(\delta\Gamma^\rho_{\nu\mu})$ has 4 terms: "$\partial + \delta\Gamma$"

$$= \partial_\lambda \underline{\delta\Gamma^\rho_{\nu\mu}}$$

$$+ \Gamma^\rho_{\lambda\sigma} \underline{\delta\Gamma^\sigma_{\nu\mu}}$$

$$\left.\begin{aligned} &- \Gamma^\sigma_{\lambda\underline{\nu}} \underline{\delta\Gamma^\rho_{\sigma\mu}} \\ &- \Gamma^\sigma_{\lambda\underline{\mu}} \underline{\Gamma^\rho_{\nu\sigma}} \end{aligned}\right\}$$ minus sign: lower index

then you can check that indeed ⊛ = (⊛⊛)

D'Inverno p-145

<u>THERE IS A SLICKER DERIVATION</u>:

GO TO FREE FALLING FRAME : $\Gamma = 0$, but derivatives $\neq 0$

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} - (\lambda \leftrightarrow \nu)$$

THINK: $\delta\Gamma^{\cdot}_{\cdot\cdot}$ INDUCES $\delta R^{\cdot}_{\cdots}$

$$\delta R^{\cdot}{}_{\cdots} = \partial_\lambda (\delta\Gamma^\rho_{\nu\mu}) - (\lambda \leftrightarrow \nu)$$
$$= D_\lambda(\text{—''—}) - (\text{—''—})$$

↑ PROMOTE TO COVARIANT DERIV.

BUT NOW THIS IS A TENSORIAL EQ., VALID IN <u>ANY</u> FRAME. □

continuing:

$$\delta R_{\mu\nu} = \delta^{\lambda}{}_{\rho}\, \delta R^{\rho}{}_{\mu\lambda\nu}$$
$$= D_{\lambda}\, \delta\Gamma^{\lambda}_{\nu\mu} - D_{\nu}\delta\Gamma^{\lambda}_{\lambda\mu}$$

$$\delta S_1 = \int d^4x \sqrt{g}\, g^{\mu\nu}\, \delta R_{\mu\nu}$$

$$= \text{---''---}\; g^{\mu\nu}\left(D_{\lambda}\delta\Gamma^{\lambda}_{\nu\mu} - D_{\nu}\delta\Gamma^{\lambda}_{\lambda\mu}\right)$$

USING METRIC COMPATIBILITY

$$= \int \text{---''---}\; D_{\sigma}\left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\lambda}_{\lambda\mu}\right)$$

but this is just the divergence of some vector field.

USE: stokes' theorem

DIVERG $\rightarrow$ SURFACE TERM

... but no surface!

$\longrightarrow$ vanishes.

$$\boxed{\delta S_1 = 0}$$ a lot of work for zero.

II $\delta S_{II} = \int d^4x \sqrt{g}\,(\delta g^{\mu\nu})\, R_{\mu\nu}$

↑ already in the form we want.

→ look: we have the first term in the Einstein tensor.

III for δS_{III}: $\delta \sqrt{|\det g_{..}|}$

A USEFUL MATRIX IDENTITY:

$$\ln \det M = \operatorname{tr} \ln M$$

sketch pf. for sym. M (what we have),

$M = R^{-1} \hat{M} R$; $\det M = \cancel{\det R^{-1}} \det \hat{M}\, \cancel{\det R}$

↑ DIAGONAL

then: write $\hat{M} = (e^{m_1}, e^{m_2}, \ldots)$

$\det \hat{M} = \exp(\Sigma m_i)$

RHS: DIAGONALIZE: $\ln M = S^{-1} (\widehat{\ln M}) S$

cyclicity

$\operatorname{tr} \ln M = \operatorname{tr}[S S^{-1} (\widehat{\ln M})]$

$= \Sigma (\widehat{\ln M})_i$

then: values of $(\widehat{\ln M})$ are precisely m_i

$$\underbrace{\delta \ln \det M} = \delta \operatorname{Tr} \ln M$$

$$\frac{1}{\det M}\,\delta \det M = \operatorname{Tr}\left(M^{-1}\delta M\right)$$

$$\delta \operatorname{Tr} A = \delta A_{11} + \delta A_{22} = \operatorname{Tr} \delta A$$

THERE'S AN AMBIGUITY:

$$\delta \ln M \overset{?}{=} \begin{cases} (M^{-1})_{ab}\,\delta M_{bc} \\ \delta M_{ab}\,(M^{-1})_{bc} \end{cases}$$

BUT TRACE MAKES THIS IRREL.

$$\rightarrow \delta \det M = \det M \,\operatorname{Tr}(M^{-1}\delta M)$$

$$\Rightarrow \quad \delta \det g_{..} = (\det g)\,\underbrace{g^{\mu\nu}}_{(g_{..})^{-1}}\,\delta g_{\nu\mu} = -\overset{\det g}{\downarrow}\, g_{\mu\nu}\,\delta g^{\mu\nu}$$

$\boxed{\text{from P.6}}$

then:
$$\delta\sqrt{g} = \delta\sqrt{-\det g}$$

I SHOULD FESS UP. PREVIOUSLY WROTE $\sqrt{|\det g|}$ ... but now it's important that $\det g$ is neg.

$$= -\frac{1}{2}\frac{1}{\sqrt{g}}\,\delta g$$

$$= +\frac{1}{2}\frac{g}{\sqrt{g}}\,g_{..}\,\delta g^{..}$$

really: $\dfrac{\det g}{\sqrt{-\det g}}$

$$= -\frac{1}{2}\sqrt{-\det g}\; g_{..}\,\delta g^{..} = \boxed{-\frac{1}{2}\sqrt{g}\; g_{..}\,\delta g^{..}}$$

$$\delta S_{III} = \int d^4x \, (\delta\sqrt{g}) R$$

$$= \int d^4x \sqrt{g} \; (-\tfrac{1}{2}) R \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

PLUGGING IN:

$$\delta S_{EH} = \cancel{\delta S_I}^{\,0} + \delta S_{II} + \delta S_{III}$$

$$= \int d^4x \sqrt{g} \left[R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \right] \delta g^{\mu\nu}$$

LHS of EINSTEIN'S EQ!

SO WE CAN SEE WHAT WE NEED:

$$\delta S_{stuff} \propto \int d^4x \sqrt{g} \, T_{\mu\nu} \, \delta g^{\mu\nu}$$

stress energy tensor.

we haven't pinned down relative constants

SO: $\dfrac{\delta S_{stuff}}{\delta g^{\mu\nu}} \dfrac{1}{\sqrt{g}} \sim T_{\mu\nu}$.

in FLAT SPACE

<u>LAST TIME</u>: WE DERIVED for ensemble of particles

$$\boxed{T^{\mu\nu}(x) = \sum_a \int d\tau_a \; m_a \dot{q}_a^{\mu} \dot{q}_a^{\nu} \; \delta^{(4)}(x - q_a(\tau_a))}$$

@ POINT x: <u>if</u> ∃ particle @ x

then the 4-momentum current is: $\dot{q}_a^{\mu} (\underbrace{m_a \dot{q}_a^{\nu}}_{p_a^{\nu}})$

4 velocity along geodesic

zee p.226 this came from # density current

$$n^{\mu}(x) = \sum_a \int d\tau_a \; \dot{q}_a^{\mu} \; \delta^{(4)}(x - q_a(\tau))$$

eg if all particles @ rest, just gives

$n^{\mu} = (\text{\# particles @ } x, \underline{0})$

BUT for such an ensemble, we also know the action: $S = \sum_a S_a$

$$S_{\text{stuff}} = +\sum_a m_a \underbrace{\int d\tau_a \sqrt{g_{\mu\nu}(x_a)\, \dot{x}_a^\mu \dot{x}_a^\nu}}$$

for mostly minus metric

dimensionless action, $\int m \sqrt{ds^2}$

$$\frac{\delta S_{\text{stuff}}}{\delta g_{\mu\nu}} = +\sum_a m_a \int d\tau_a \frac{1/2}{\sqrt{g_{..}\dot{x}^{.}\dot{x}^{.}}} \dot{x}_a^\mu \dot{x}_a^\nu \delta^{(4)}(x - x_a(\tau))$$

SPACETIME POINT

SPACETIME POSITION OF a^{th} PARTICLE

where'd this come from?

$$\frac{\delta S_{\text{stuff}}}{\delta g_{\mu\nu}(x)}$$

@ some point $\underline{\underline{x}}$

action only has support where there are particles

$\sqrt{g_{..}\dot{x}^{.}\dot{x}^{.}} = 1$ for proper time

$$= \frac{1}{2} \sum_a m_a \int d\tau\, \dot{x}_a^\mu \dot{x}_a^\nu \delta^{(4)}(x - X^a(\tau)) \checkmark$$

I MAY BE CHEATING W/ UPPER v. LOWER INDICES

SEE ZEE P. 380

WE MATCH in flat space case!

WHAT WE'VE DEMONSTRATED:

$$2 \frac{\delta S_{stuff}}{\delta g_{\mu\nu}} \frac{1}{\sqrt{g}} = T^{\mu\nu}.$$

b/c $\delta\sqrt{\ }$ gave us an unwanted 1/2

to compensate the $\sqrt{g}$ in curved space

<u>REMARK</u> WE VARIED $\delta/\delta g_{\mu\nu}$ to get $T^{\mu\nu}$

recall $\delta g_{\rho\nu} = - g_{\rho\mu} g_{\lambda\nu} \delta g^{\mu\lambda}$

so

$$T_{\mu\nu} = -2 \frac{\delta S_{stuff}}{\delta g^{\mu\nu}} \frac{1}{\sqrt{g}}$$

see Zee p. 386

or: $\delta S_{stuff} = -\int d^4x \sqrt{g}\, \frac{1}{2} T_{\mu\nu}\, \delta g^{\mu\nu}(x)$

WHERE WE ARE

$$\underset{\uparrow}{A}\,\underbrace{\delta S_{EH}} + \delta S_{stuff} = 0$$

some rel. prefactor

$$\int d^4x \sqrt{g}\,[\underbrace{R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu}}_{G_{\mu\nu}}]\,\delta g^{\mu\nu}$$

$$\cancel{\frac{\delta S_{stuff}}{\delta g^{\mu\nu}}}\quad \int d^4x \sqrt{g}\,\left(\tfrac{-1}{2}\right) T_{\mu\nu}\,\delta g^{\mu\nu}$$

EOM :

$$A\, G_{\mu\nu} = \tfrac{1}{2} T_{\mu\nu}$$

if $A = \frac{1}{16\pi G}$, then we get

EINSTEIN'S EQ : $\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$