LEC 12: EINSTEIN'S EQ. 23 FEB 17

TODAY: ENERGY-MOMENTUM TENSOR
GUESSING A FIELD EQUATION
SKETCHING AN "HONEST" DERIVATION

btw: "conformal" $\longrightarrow$ $\cos\theta = \frac{a \cdot b}{\sqrt{a^2 b^2}}$
& ask Kmill: mag monopoles in BEC

CHENG 286

REVIEW: (CLASSICAL) EM IN COVARIANT FORMULATION

everything tensorial
w/rt Minkowski space
s.t. Lorentz transformation
properties are manifest

$\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$

$dF = 0 \quad \longleftrightarrow \quad \partial_{[\mu} F_{\nu\lambda]} = 0$ BIANCHI

$*d*F = j \quad \longleftrightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu$ from ACTION

F is the tensor
that 'contains' the
EM field

CURRENT 4-VECTOR

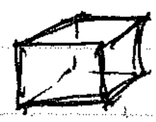
CURRENT: $j^\mu = (\rho, \underline{j})$

ρ ↑ CHARGE DENSITY

$\underline{j}$ ↑ CURRENT

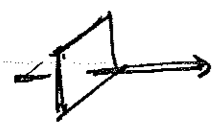
CHARGE PER VOL

$$\rho = \frac{dq}{dx\,dy\,dz}$$



CHARGE ~~FLOW~~ PER TIME, PER CROSS SEC. AREA

$$j^x = \frac{dq}{dt\,dy\,dz}$$



OR: $$j^\mu = \frac{dq}{dS^\mu}$$

dS^μ: infinit 3-volume perpendicular to $\hat{e}_{(\mu)}$

CONSERVATION OF CHARGE: $\partial_\mu j^\mu = 0$

$$\boxed{\dot{\rho} + \underline{\nabla} \cdot \underline{j} = 0}$$

SIMILARLY: MASS IS CONSERVED, COULD WRITE

$j_{MASS} = (\rho_{MASS}, \underline{j}_{MASS})$

4 MOMENTUM CONSERVATION $\longrightarrow$ ENERGY
$p^\mu = (E, \vec{p})$ $\qquad$ P_x, P_y, P_z

$\underbrace{\hphantom{P_x, P_y, P_z}}$ FOUR QUANTITIES

not a current, each of these is a charge

THE CURRENT is: $\boxed{T^{\mu\nu} = dp^\mu / dS^\nu}$

forgive my placement of indices.

BIG PIC: WILL WANT A FIELD EQ. OF THE FORM

$$\underset{\Gamma \sim \partial\Gamma}{(\text{curvature})} = (\text{energy})$$

ANALOGOUS TO:

$$\underset{\text{DYNAMICS}}{\partial F} = \underset{\text{SOURCE}}{j}$$

INTUITION: $T^{00} = \dfrac{dE}{dx\,dy\,dz}$ energy density

$T^{01} = \dfrac{dE}{dt\,dy\,dz}$ energy flux

ENERGY PER TIME CROSSING $dy\,dz$

$T^{10} = \dfrac{dP_x}{dx\,dy\,dz} =$ x-momentum density

T^{i0} T^{0i}

OBSERVE: momentum density = E flux

Why: **Velocity**: $p^i/E = v^i$

$$\frac{dp^i}{dE} = \frac{dx^i}{dt}$$ } think of these as infinitesimals, not diff. ops

$$\Rightarrow \boxed{\frac{dp^i}{dx^i} = \frac{dE}{dt}}$$

$$T^{01} = \frac{dE}{dt\,dy\,dz} = \frac{dP_x}{dx\,dy\,dz} = T^{10}$$ ✓

SIMILARLY: $\dfrac{dP_x}{dy} = \dfrac{dP_y}{dx}$

from, say: $p^i = E v^i$

then $p^i/p^j = v^i/v^j$

s.t. $T^{ij} = T^{ji}$

$$\Rightarrow \boxed{T^{\mu\nu} = T^{\nu\mu}}$$ stress-energy tensor is symmetric

Meaning of T^{ij}

$$\begin{matrix}\text{momentum} \\ \text{flux}\end{matrix} = \frac{\text{momentum}}{\underset{\substack{\uparrow \\ \text{perp.}}}{\text{area}} \cdot \text{time}} = \frac{\text{Normal force}}{\text{area}} = \text{PRESSURE}$$

<u>So</u>: T^{xx} is a <u>pressure</u> in x-dir
T^{xy} is a <u>shear force</u>

<u>EXAMPLES</u>

uniform density

<u>DUST</u>: non interacting particle cloud
modeled as a continuous medium

<u>COMOVING COORDINATES</u>: rest frame of the cloud

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

not tensorial

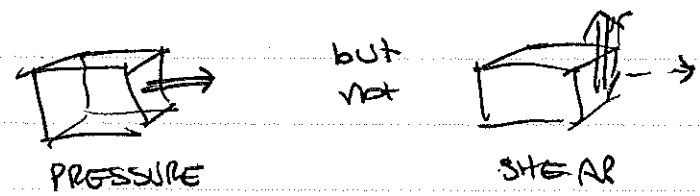
$$= \rho U^{\mu} U^{\nu}$$

$U = (1, \vec{0})$
4-velocity of cloud

tensor eq.

IDEAL FLUID : "dust" w/ only perpendicular interactions

→ no shear



$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix} \qquad \text{not covariant}$$

PRESSURE

$$= \rho U^\mu U^\nu - P\eta^{\mu\nu} + P U^\mu U^\nu$$

$$- \begin{pmatrix} P & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$

$$= (\rho + P) U^\mu U^\nu$$

maybe do field thy?

CONSERVATION OF $p^\mu \rightarrow \boxed{\partial_\mu T^{\mu\nu} = 0}$

eg. rederiving fluid mechanics
from nonrelativistic limit of ideal fluid

$$\begin{aligned}\partial_\mu T^{\mu\nu} = &\ \partial_\mu (\rho + P) U^\mu U^\nu \\ &+ (\rho + P)(\partial_\mu U^\mu) U^\nu \\ &+ (\text{---}\,''\,\text{---})\, U^\mu \partial_\mu U^\nu \\ &+ \\ &- \partial^\nu P\end{aligned}$$

Strategy: many components. let's project
onto 2 subspaces:

$\parallel$ to U^μ (1D)

$\perp$ to U^μ (3D)

① U is already a normalized projector
ie $U \cdot V$ gives projection of V onto U

$$\begin{aligned}U_\nu (\partial_\mu T^{\mu\nu}) = &\ \partial_\mu (\rho + P) U^\mu \\ &+ (\rho + P) U_\nu [U^\nu \partial_\mu U^\mu \\ &\qquad + U^\mu \partial_\mu U^\nu] \\ &- U_\nu \partial^\nu P\end{aligned}$$

USE: $U^2 = 1$

$$\partial_\nu (U^\mu U_\mu) = 0 = 2U^\mu \partial_\nu U_\mu$$

$$\begin{aligned} U_\nu(\partial_\mu T^{\mu\nu}) = \; & U\cdot\partial(\rho + P) \\ & + (\rho + P)\, U^2\, \partial\cdot U \\ & + (\rho+P)\, U^\mu \underline{U_\nu \partial_\mu U^\nu} \quad = 0 \\ & - \underline{(U\cdot\partial)P} \end{aligned}$$

$$= \underbrace{U\cdot\partial\rho + \rho(\partial\cdot U)} + P(\partial\cdot U)$$

$$\partial_\mu(\rho U^\mu) - P(\partial\cdot U) \qquad = 0 \quad \partial T = 0$$

NR LIMIT: $U = (1, \underline{v})$ $\qquad |\underline{v}| \ll 1$

$U^2 = 1 + \mathcal{O}(\underline{v}^2)$ $\qquad P \ll \rho$ (low press.)

$$\boxed{\partial_t \rho + \underline{\nabla}\cdot(\rho \underline{v}) = 0}$$

CONTINUITY EQ FOR ENERGY DENSITY

(2) ALSO WANT TO PROJECT $\partial_\mu T^{\mu\nu}$ onto 3D SUBSPACE $\perp$ to U.

PROJECTION TENSOR

$$(\text{Proj})^\alpha{}_\nu = \delta^\alpha{}_\nu - U^\alpha U_\nu$$

eg. CHECK IN

$(\mathrm{Proj})^{\alpha}{}_{\nu}\, \partial_\mu T^{\mu\nu} =$

$$\cancel{\partial_\mu(\rho+P)U^\mu U^\alpha} - \cancel{\partial_\mu(\rho+P)U^\mu U^\alpha}$$
$$+ \cancel{(\rho+P)(\partial_\mu U^\mu)U^\alpha} - \cancel{(\rho+P)(\partial_\mu U^\mu)U^\alpha}$$
$$+ (\;-\!\!\shortparallel\!\!-\;)U^\mu \partial_\mu U^\alpha - (\;-\!\!\shortparallel\!\!-\;)U^\mu \; U^\alpha \underbrace{\cancel{(U^\nu \partial_\mu U_\nu)}}_{0}$$
$$- \partial^\alpha P \qquad + U^\alpha U_\nu \partial^\nu P \qquad = 0$$

<u>NR limit</u>

$$\underbrace{\rho (U\cdot\partial)U^\alpha}_{} - \partial^\alpha P + U^\alpha (U\cdot\partial) P = 0$$

$\alpha = i$ $$\rho\left(\partial_t \underline{v} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) - \underline{\nabla} P + \underbrace{\underline{v}\,(\partial_t + \underline{v}\cdot\underline{\nabla})P}_{} = 0$$

higer O in v & P

$$\boxed{\rho\,(\underbrace{\partial_t \underline{v} + (\underline{v}\cdot\underline{\nabla})\underline{v}}_{}) = \underline{\nabla} P}$$

EULER EQ.

CONVECTIVE DERIV.

ANALOG OF $\underline{F} = m\underline{a}$ in fluids

$\underline{a} = \dot{\underline{v}}$

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underbrace{\frac{dx^i}{dt}\frac{\partial \underline{v}}{\partial x^i}}_{(\underline{v}\cdot\underline{\nabla})\underline{v}}$$

THIS COMES UP IN GALAXY EVOLUTION

REM: to go to curved space,
good rule of thumb: $\partial \to D$

eg: MAXWELL IN CURVED SPACE

$$\frac{dU^\mu}{d\tau} = qF^{\mu\nu}U_\nu \qquad \text{FORCE LAW}$$

$$\left(\frac{DU^\mu}{d\tau} \right.$$

analog in GRAV is Bianchi, too

BIANCHI: $\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \to D_\mu \cdots = 0$

EULER-LAG: $\partial_\mu F^{\mu\nu} = j^\nu \to D_\mu \cdots = j^\nu$

eg $\partial_\mu + \Gamma^\mu_{\mu\lambda} F^{\lambda\nu} + \Gamma^\nu_{\mu\nu} F^{\mu\lambda} = j^\nu$

So: NOW WE HAVE THE SOURCE OF GRAVITY, $T^{\mu\nu}$. WANT TO WRITE LHS

something like $\partial\Gamma$ ($\leftarrow$ analog of $F \sim \partial A$)

this is $\sim R$!

$f(R\ldots) = \kappa\, T_{\mu\nu}$

RANK - 2, SYMMETRIC
CONSERVED ($D_\mu T^{\mu\nu} = 0$)

4 indices!
HAVE TO REDUCE TO 2

WHAT WE HAVE:

RICCI TENSOR: $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$

HW?

recall: by the symmetry properties of $R\ldots$, this is the "unique" contraction of $R\ldots$ into a 2 tensor.

(ALL OTHER NONZERO CONTRACTIONS CONTAIN THE SAME DATA)

fact: $R_{\mu\nu} = R_{\nu\mu}$ ← good!

but: not, in general, conserved

~~further: want something that is zero in the absence of sources.~~

there's another ~~way to make a~~ symmetric tensor floating around: $g_{\mu\nu}$.

BUT $g_{\mu\nu}$ ISN'T A CURVATURE...
that's fine. JUST SLAP A SCALAR CURVATURE MEASURE NEXT TO IT.

$\hookrightarrow R g_{\mu\nu}$

$R = R_{\mu\nu} g^{\mu\nu}$, RICCI SCALAR

this is sym, related to curvature...
also not obviously covariantly constant
~~but is it zero? no~~

CONSIDER, THEN,

$$\boxed{A R_{\mu\nu} + B R g_{\mu\nu}}$$

in fact: sufficient to consider

$$\boxed{R_{\mu\nu} + D R g_{\mu\nu}}$$

WANT D s.t. $D_\mu (R_{\mu\nu} + D R g_{\mu\nu}) = 0$

WHAT you'll SHOW ON HW: $\boxed{D = -\frac{1}{2}}$

SO WE DEFINE THE <u>EINSTEIN TENSOR</u>

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}}$$

symmetric, conserved / cov. const.
goes like $\partial\Gamma$

SO WE CAN GUESS:

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$$

curvature — energy (stuff)

coupling $\sim G$

<u>OBSERVE</u>: $T_{\mu\nu} = 0 \longrightarrow G_{\mu\nu} = 0$

this means: $\boxed{R_{\mu\nu} = 0}$

vacuum einstein eq.

eg. $R_{\mu\nu} = 0$ for SCHWARZSCHILD
(no sources!)

A USEFUL REFORMULATION
for matching to Newtonian limit

$$g^{\mu\nu}\left(R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu}\right) = g^{\mu\nu} \kappa T_{\mu\nu}$$

$$R - 2R = \kappa T \leftarrow T = T^{\mu}{}_{\mu}$$

$$\rightarrow \boxed{R = -\kappa T}$$

then:

$$\boxed{R_{\mu\nu} = \kappa\left(T_{\mu\nu} + \tfrac{1}{2} T g_{\mu\nu}\right)}$$

then: (HW)

$$R_{00} = +\tfrac{1}{2}\nabla^2 \underbrace{g_{00}}_{(1+2\Phi)} \quad \text{in NEWTONIAN LIMIT}$$

sign?

RHS: $\kappa\left(T_{00} - \tfrac{1}{2} T g_{00}\right)$

$$T = g^{00} T_{00} + \text{small}$$ ~ PRESSURE (LOW VEL.)

s.t. $\tfrac{1}{2} T g_{00} = \tfrac{1}{2} \underbrace{g^{00} g_{00}}_{=1} T_{00}$

SO: $R_{00} = +\frac{1}{2}\nabla^2 g_{00} = \frac{1}{2}K T_{00} = \frac{1}{2}K\rho$

$(1+2\phi)$

$$\nabla^2\phi = \boxed{\frac{1}{2}K}\rho \equiv 4\pi G\rho$$

$$\Rightarrow \boxed{K = 8\pi G}$$

SO: EINSTEIN EQ:

CURVATURE COUPLING SOURCE

$$\boxed{\underbrace{G_{\mu\nu}}_{R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}} = 8\pi G\, T_{\mu\nu}}$$

Zee IV.1

Skip?

Another Way

FREE PARTICLE ACTION $\boxed{S = -m \int d\tau}$

$$d\tau = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}$$

elegant, covariant, ... what else could it be?

$$S \approx \int dt \Big(\underbrace{m}_{\text{overall const.}} + \underbrace{\tfrac{1}{2} m \left(\tfrac{d\vec{x}}{dt}\right)^2}_{\text{KIN E.}} + \cdots \Big)$$

IN NR LIMIT, WE GET ORDINARY MECHANICS

$$S_{NR} = \int dt \quad \tfrac{1}{2} m \left(\tfrac{d\vec{x}}{dt}\right)^2 \quad \underbrace{- V(x)}_{\text{POTENTIAL}}$$

external forces that we stick on in "ordinary" mechanics

HOW TO INCORPORATE $V(x)$ IN REL way?

inside (I) $S = -\int m\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu} + V(x)\,dt$

outside (II) $S = -m\int\sqrt{\left(1+\frac{2V}{m}\right)dt^2 - dx^2}$

$$-m\int\sqrt{1+\frac{2V}{M}}\,dt - \frac{1}{2}\frac{1}{\sqrt{1+\frac{2V}{m}}}\frac{d\vec{x}^2}{dt}$$

$$\approx -m\int\left(1+\frac{V}{M}\right)dt - \frac{1}{2}\frac{d\vec{x}^2}{dt}$$

$$= \int dt\ \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V - \underset{?}{\underset{\uparrow}{m}}$$

but: neither I nor II is really LORENTZ invt!
dt is SPECIAL vs dx.

(I)′ $S = \int -m\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu} + \underline{A_\mu(x)dx^\mu}$

aha!
from ELECTR to
ELECTROMAG.

(II') need dt & dx on equal footing:

$$S = -m \int \sqrt{g_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

inside $\sqrt{}$

this is ~~indeed~~ the action for a particle in a grav field — as we have assumed!

what about the dynamics of GRAVITY itself?

SPACETIME

$$S \sim \int d^4x \sqrt{|g|}\, \underbrace{R(x)}$$

RICCI SCALAR
ONLY CURVATURE

UP TO
CONSTANT

Why: "UNIQUENESS" OF EINSTEIN EQ. IS MANIFEST FROM DIMENSIONAL ANALYSIS

$$R \sim \partial^2 \rightarrow \text{CONST} \sim 1/\text{LEN}^2 \sim \frac{1}{G}$$

$$\sim \text{MASS}^2$$

other terms that are covariant $\longrightarrow$ HIGHER O IN G

HACKING THE SOURCE-~~FREE~~ EINSTEIN EQN

$\delta S \sim \delta \int d^4x \sqrt{g}\, R$

VARY w/rt WHAT? $g_{\mu\nu}$

$\equiv \int d^4x \sqrt{g}\, \boxed{K^{\mu\nu}}\, \delta g_{\mu\nu}$

DEFINE

then $\delta S = 0 \Rightarrow K^{\mu\nu} = 0$

$R \sim \partial\Gamma \sim \partial^2 g$ $\longrightarrow$ so $K^{\mu\nu}$ must have two derivatives

also $\Gamma^2 \sim (\partial g)^2$

so: 2-tensor that is $\mathcal{O}(\partial^2)$

SAME AS BEFORE: $K^{\mu\nu} \sim R^{\mu\nu} + D g^{\mu\nu} R$

then $K^{\mu\nu} = 0 \Rightarrow \underbrace{g_{\mu\nu} K^{\mu\nu}}_{} = 0$

$R + 4DR = 0$

either $D = -1/4$, $\boxed{R = 0}$

$\downarrow$

$R_{\mu\nu} = 0$ ✓

matter

ADDING IN STUFF

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g}\, R$$

$$S_{stuff} = \ldots$$

$$\delta S_{stuff} = \frac{1}{2} \int d^4x \sqrt{g} \boxed{T^{\mu\nu}}\, \delta g_{\mu\nu}$$

$$= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

PROPERTIES: SYM. 2-INDEX TENSOR FIELD

P. 880

P. 226 ANOTHER PICTURE:

\# DENSITY CURRENT

q_a^{μ} is position of a^{th} PARTICLE

$$n^{\mu}(x) = \sum_a \int d\tau_a \frac{dq_a^{\mu}}{\partial \tau_a} \delta^{(4)}(x - q_a(\tau))$$

sum over dust particles

flow of particle in spacetime

worldline of stationary particle

$$\text{why: } \delta^{(3)}(\underline{x}) = \int d\tau\, \delta(x^0 - q^0(\tau))\, \delta^{(3)}(\underline{x})$$

$$= \int d\tau \frac{dq^0}{d\tau} \delta(x^0 - q^0)\, \delta^{(3)}(x - q) = n^0$$

$$\tau_1 \qquad \tau_0$$

then electric current

$$j^\mu = \sum_a e_a \int d\tau_a \frac{dq_a^\mu}{d\tau_a} \delta^{(4)}(x - q_a(\tau_a))$$

e_a charge of ath particle

then the 4-momentum current:

$$T^{\mu\nu} = \sum_a \int d\tau_a \frac{dq_a^\mu}{d\tau_a} (p_a^\nu) \delta^{(4)}(x - q_a(\tau_a))$$

p_a^ν plays role of e_a

$$= \sum_a \int d\tau_a (m_a \dot{q}_a^\mu \dot{q}_a^\nu) \delta^{(4)}(x - q_a(\tau_a))$$

$\dot{} \Rightarrow d/d\tau_a$

Then: $S_{stuff} = -\sum_a m_a \int d\tau \sqrt{g_{\mu\nu}(x_a) \frac{dx_a^\mu}{d\tau} \frac{dx_a^\nu}{d\tau}}$

$$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} S_{stuff}$$

$$= \frac{1}{\sqrt{g}} \sum_a m_a \int d\tau_a \underbrace{\frac{1}{\sqrt{g_{..} \dot{x}_a \dot{x}_a}}}_{=1 \text{ for } \tau = \text{PROP. TIME}} \dot{x}_a^\mu \dot{x}_a^\nu \delta^{(4)}(x - x_a(\tau))$$

$$= \frac{1}{\sqrt{g}} \sum_a m_a \int d\tau \, \dot{x}_a^\mu \dot{x}_a^\nu \delta^{(4)}(x - x_a(\tau))$$

PRECISELY WHAT WE FOUND BEFORE $\equiv T^{\mu\nu}$

So: $\delta S = 0 \rightarrow \delta S_{EH} + \underbrace{\delta S_{stuff}}_{T^{\mu\nu}} = 0$