LEC 11 : continued exploration of Schw. 2/16

LAST TIME :

@ $r = r_s$, lightcones "tighten up" in Schw. coordinates

... BUT INFALLING STUFF DOESN'T KNOW, JUST FALLS IN

then: sequence of coordinate changes to investigate $r < r_s$

① TORTOISE COORDS

$$\tilde{r} = r + r_s \log\left(\frac{r}{r_s} - 1\right)$$

$$d\tilde{r} = \frac{1}{\Psi} dr \qquad \Psi = \left(1 - \frac{r_s}{r}\right)$$

$$\text{then } ds^2 = \Psi\,(dt^2 - d\tilde{r}^2) - r^2 d\Omega^2$$

LIGHTCONES (null geodesics) are 45°

fixed: lightcones don't squish

PROBLEMS: $g^{\mu\nu} \to \infty$ @ $r = r_s$

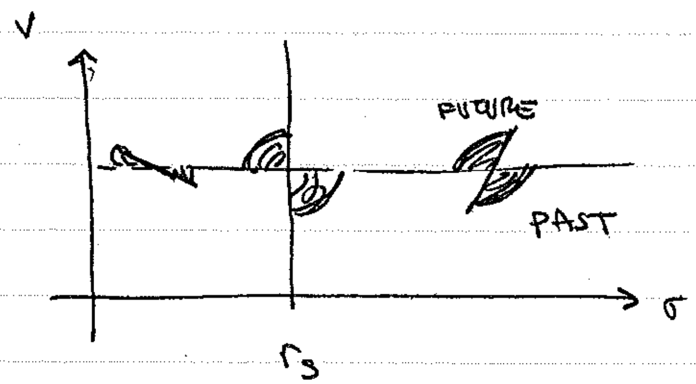
... but really: $r = r_s$ is located @ $\tilde{r} = \infty$

② EDDINGTON - FINKELSTEIN : v & r

$$v = t + \tilde{r}$$

$$ds^2 = \not{V}\, dv^2 - \underbrace{2\,dv\,dr}_{\text{OFF DIAGONAL}} - r^2 d\Omega^2$$

→ metric is everywhere invertible, even when $\not{V} = 0$

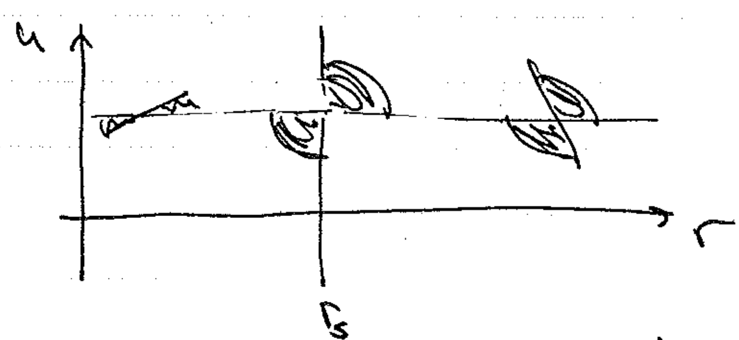


LIGHT CONES

⇒ EVENT HORIZON separates regions of spacetime according to causality

we noticed: ORTHOG TO v IS $u = t - \tilde{r}$

COULD HAVE USED (r, u) COORDS.



"WHITE HOLE"
OR:
stuff that came from inside r_s

not schwarzschild

$*: ds^2 = \heartsuit(dt^2 - d\tilde{r}^2) - r^2 d\Omega^2$

this gave us a sense of the <u>Maximal Extension</u> of schw. SPACETIME : (u, v) COORDS.

$ds^2 = \heartsuit\, du\, dv - r^2 d\Omega^2 \quad *$

FIRST, A PROBLEM: $r = r_s \Rightarrow \tilde{r} \to \infty$

THEN $\begin{cases} u \to \infty \\ v \to -\infty \end{cases}$

we've pushed r_s out of our space. so (u, v) COORDS ARE NO GOOD.

BUT: $v = t + r + r_s \log\left(\frac{r}{r_s} - 1\right)$

$u = t - r - r_s \log\left(\frac{r}{r_s} - 1\right)$

so the ∞ is a logarithmic ∞

USE COORDS s.t. we pull this $\log \infty$ to finite value

$\tilde{v} \equiv e^{v/2r_s}$ ← why $\underline{2}\, r_s$? this is $\sqrt{e^{v/r_s}}$

$\tilde{u} \equiv -e^{-u/2r_s}$

NATURAL THING TO DO: exp takes dimless arg, so norm. by r_s. WHY $\sqrt{\ }$? CONVENIENCE... METRIC WILL HAVE A SQUARE.

eg $\tilde{v} = e^{(t+r)/2r_s} \underbrace{\sqrt{\frac{r}{r_s} - 1}}_{\substack{\text{finite } (=0) \\ @\, r = r_s}}$

$\tilde{u} = -e^{-(t-r)/2r_s} \sqrt{\frac{r}{r_s} - 1}$

$$d\tilde{v} = \frac{1}{2r_s}\tilde{v}\,dv \longrightarrow dv = \frac{2r_s}{\tilde{v}}\,d\tilde{v}$$

$$d\tilde{u} = \frac{-1}{2r_s}\tilde{u}\,du \longrightarrow du = \frac{-2r_s}{\tilde{u}}\,d\tilde{u}$$

then: $ds^2 = \heartsuit\, du\,dv - r^2 d\Omega^2$

$$= \heartsuit\, \frac{-4r_s^2}{\tilde{v}\tilde{u}}\, d\tilde{v}\,d\tilde{u} - r^2 d\Omega^2$$

$$\boxed{\tilde{v}\tilde{u} = -e^{r/r_s}\left(\frac{r}{r_s} - 1\right)}$$

def $r(\tilde{v},\tilde{u})$ implicitly

$$ds^2 = \heartsuit\, \frac{4r_s^2}{\left(\frac{r}{r_s} - 1\right)}\, e^{-r/r_s}\, d\tilde{v}\,d\tilde{u} - r^2 d\Omega^2$$

$$= \cancel{\heartsuit}\, \frac{4r_s^3/r}{\cancel{(1 - r_s/r)}}\, e^{-r/r_s}\, d\tilde{v}\,d\tilde{u} - \ldots$$

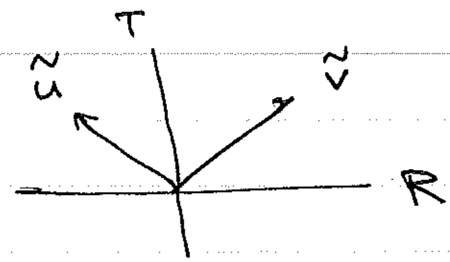
$$= \boxed{\frac{4r_s^3}{r}\, e^{-r/r_s}\, d\tilde{v}\,d\tilde{u}} \ldots$$

$r = r(\tilde{u},\tilde{v})$

well behaved @ r_s

invertible, everything

LAST STEP: RATHER THAN LIGHTCONE COORDS, GO TO TIME & SPACE



Kruskal coordinates:

$$T = \tfrac{1}{2}(\tilde{v} + \tilde{u})$$
$$R = \tfrac{1}{2}(\tilde{v} - \tilde{u})$$

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \left(dT^2 - dR^2\right) - r^2 d\Omega^2$$

again: LIGHTCONES ARE 45°
CAUSAL STRUCTURE EASY TO READ OFF SPACETIME DIAGRAM.

$$T = \frac{1}{2}\sqrt{\frac{r}{r_s}-1}\left(e^{\frac{t+r}{2r_s}} - e^{-\frac{t-r}{2r_s}}\right)$$

$$= \sqrt{\frac{r}{r_s}-1}\, e^{r/2r_s} \sinh\left(\frac{t}{2r_s}\right)$$

$$R = \sqrt{\frac{r}{r_s}-1}\, e^{r/2r_s} \cosh\left(\frac{t}{2r_s}\right)$$

keep this on board

Use $\cosh^2 x - \sinh^2 x = 1$

$$\boxed{R^2 - T^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s}}$$

gives implicit definition $r(R,T)$

event horizon: $\boxed{R^2 - T^2 = 0}$

constant $r \rightarrow \boxed{R^2 - T^2 = \text{const}}$

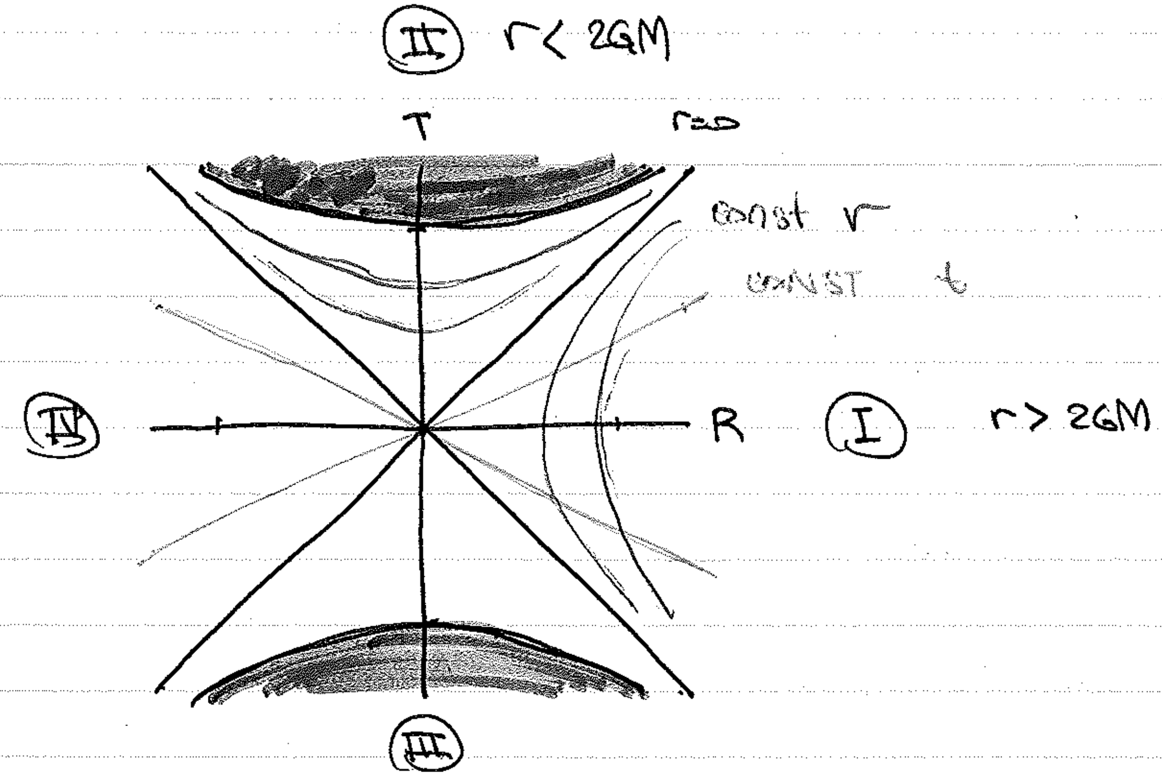
SINGULARITY: $r = 0 \rightarrow \boxed{T^2 = R^2 + 1}$

$T^2 \leq R^2 + 1$

similarly:

$$\frac{T}{R} = \tanh \frac{t}{2r_s}$$

$\rightarrow$ constant $t \rightarrow \boxed{T/R = \text{const}}$



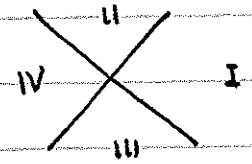
I + II $\longleftrightarrow$ SCHWARZSCHILD

III "white hole"
REGION DESCRIBED BY (u, r)

this is the region where we continue past geodesics from I

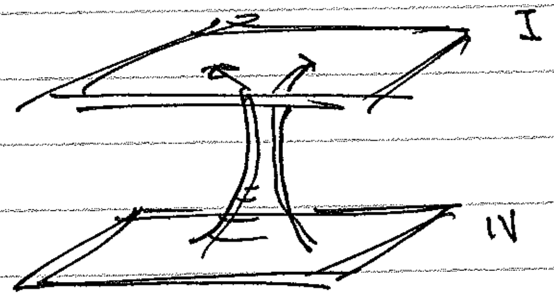
↳ Maximal extension

eg. $A_\mu = \int j^\nu(x') G_{\nu\mu}(x', x) \, d^4x' \Big|_{\text{PAST LIGHT CONE}}$



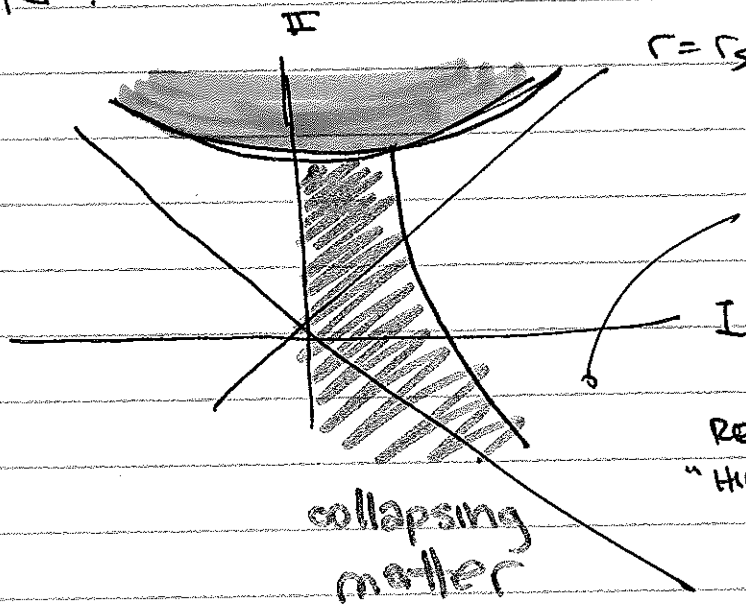
← EINSTEIN - ROSEN BRIDGE BETWEEN TWO ASYMP. FLAT, IDENTICAL SPACETIMES

Why we build this?
GIVES A "STORY" OF WHERE BH CAME FROM



START w/ I & IV (asymp flat)
& glue them together w/ white hole singularity

no reason to believe this, but it's a story that doesn't require matter.



metric is SCHW. even though no bh formed yet

REGIONS III & IV "HIDDEN" → unphysical

one more step: **Penrose Diagram** (CONFORMAL DIAG)

- time & space coords
- null / lightcone geodesics are 45° lines
- all of spacetime mapped onto a <u>finite region</u> including "∞"

$-\frac{\pi}{2} \le \tilde{T}, \tilde{R} \le \frac{\pi}{2}$

$$T + R = \tilde{v} \equiv \tan(V) \equiv \tan(\tilde{T} + \tilde{R})$$
$$T - R = \tilde{u} \equiv \tan(U) \equiv \tan(\tilde{T} - \tilde{R})$$

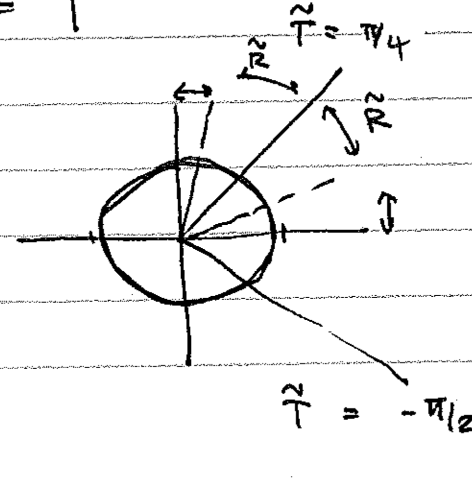
$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \underline{(dT^2 - dR^2)} - r^2 d\Omega^2$$

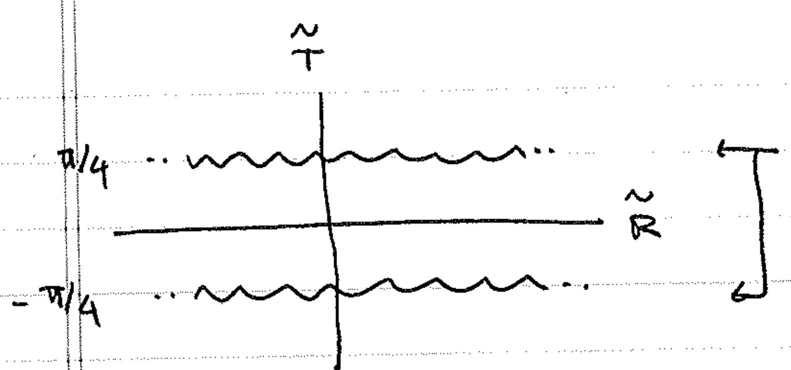
$$d\tan x = \frac{dx}{\cos^2 x} \longrightarrow \quad d\tilde{v}\, d\tilde{u} = \frac{dV\, dU}{\cos^2 V \cos^2 U}$$

$$= \frac{d\tilde{T}^2 - d\tilde{R}^2}{\cos^2(\tilde{T}+\tilde{R}) \cos^2(\tilde{T}-\tilde{R})}$$

- WHERE is $r=0$ SINGULARITY? $T^2 - R^2 = 1$

$$\Rightarrow \tan(\tilde{T}+\tilde{R}) \tan(\tilde{T}-\tilde{R}) = 1$$
$$\Rightarrow \sin(\tilde{T}+\tilde{R}) = \cos(\tilde{T}-\tilde{R})$$
$$\Rightarrow \boxed{\tilde{T} = \pm \pi/4} \leftrightarrow r = 0$$



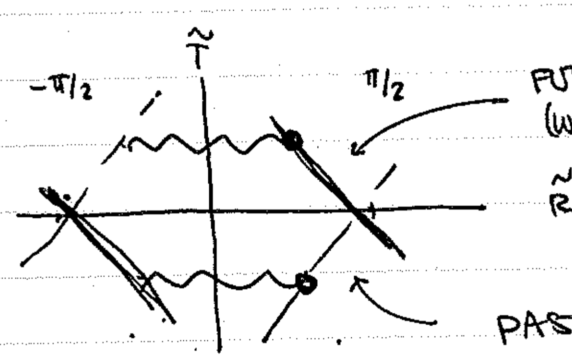


$r = 0$ SINGULARITY

- const $r \to \pm\infty \iff T^2 - R^2 = \mp\infty$

$$\tan(\tilde{T}+\tilde{R})\tan(\tilde{T}-\tilde{R}) = \mp\infty$$

$$\Rightarrow \boxed{\tilde{T} \pm \tilde{R} = \pm\pi/2} \quad \cos\theta = 0$$



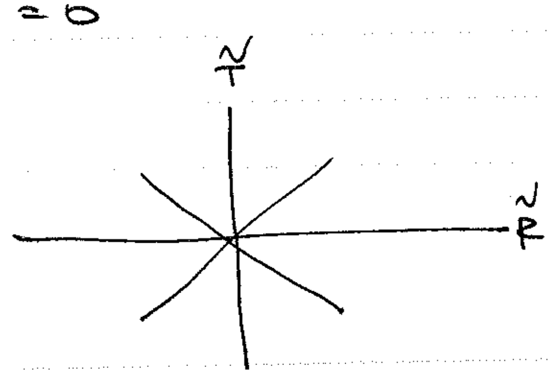
FUTURE NULL ∞
(where null geodesics go)

PAST NULL ∞

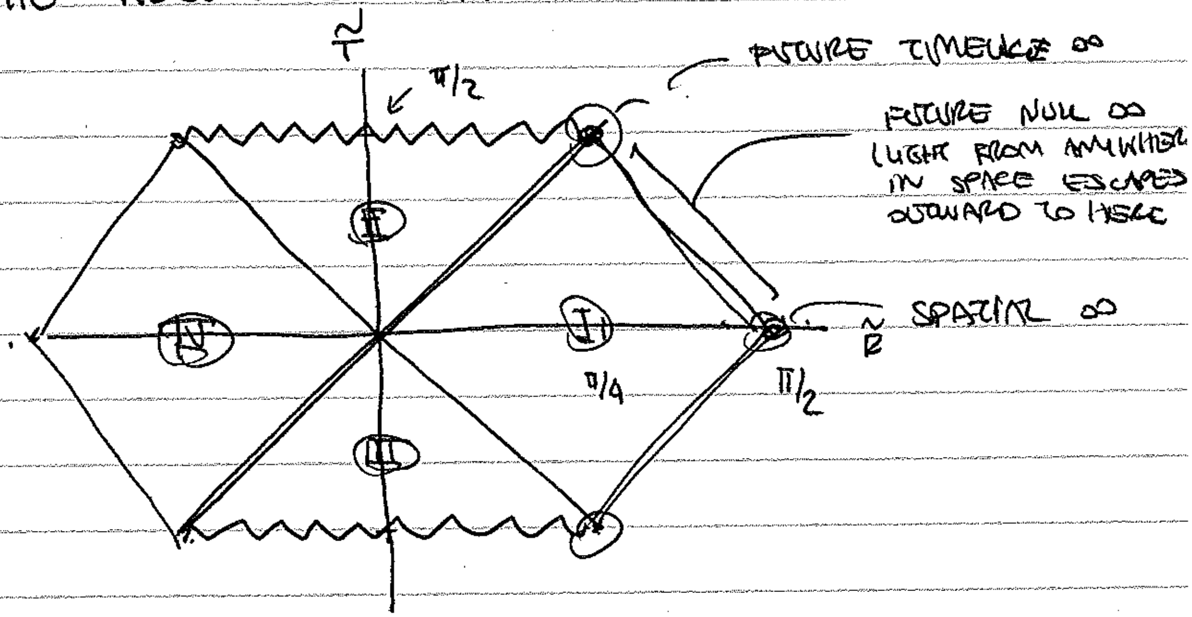
- EVENT HORIZON $R^2 - T^2 = 0$

$$\tan(\tilde{T}+\tilde{R})\tan(\tilde{T}-\tilde{R}) = 0$$

$$\Rightarrow \boxed{\tilde{T} \pm R = 0,}$$



SO THE RESULTING DIAGRAM IS



- for const r: $\tilde{v} = e^{t} C$

  $\tilde{u} = -e^{-t} C$

$$\tilde{T} + \tilde{R} = \tan^{-1}(e^{t} C) \xrightarrow{t\to\infty} \pi/2$$
$$\tilde{T} - \tilde{R} = \tan^{-1}(-e^{-t} C) \xrightarrow{t\to\infty} 0$$

$$\Rightarrow \left.\begin{array}{l} \tilde{T} = \pi/4 \\ \tilde{R} = \pi/4 \end{array}\right\} \text{top right corner}$$
FUTURE TIMELIKE ∞

similarly $\longrightarrow$ bot right : PAST TIMELIKE ∞

- $\underline{t=0 \quad r\to\infty}$ (SPATIAL ∞)

$$\tilde{T} + \tilde{R} = \tan^{-1}\left(e^{r/2r_s}\sqrt{\frac{r}{r_s}-1}\right) \xrightarrow{r\to\infty} \pi/2$$
$$\tilde{T} - \tilde{R} = \tan^{-1}\left(- \text{——} \text{''} \text{——}\right) \xrightarrow{r\to\infty} -\pi/2$$