LEC 10 : CHANGE OF COORDS 2/14

LAST TIME

KILLING VECTORS & SYMMETRIES

(we'll need this

homework explores this a bit

HW 5 POSTED

TODAY & THU :

understanding Schw. black holes

(study

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

STUDY GEODESICS ... WHAT KIND?

- cross r_s
- null geodesic $\leftrightarrow$ light trajectories

THESE DEMARCATE THE <u>CAUSAL STRUCTURE</u> OF THE SPACETIME

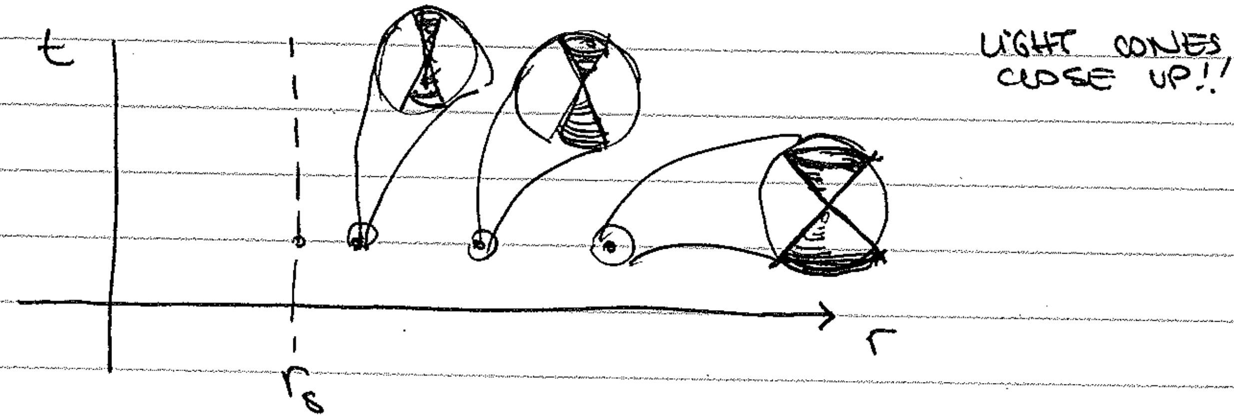
$ds^2 = 0$

- RADIAL geodesic : $d\Omega^2 = 0$
  - ↳ invoke spherical symmetry

(0) $ds^2 = 0 \Rightarrow \boxed{\frac{dt}{dr} = \pm\left(1 - \frac{r_s}{r}\right)^{-1}}$ ← not a geodesic

$$= \frac{\pm 1}{1 - r_s/r}$$

BLOWS UP AS $r \to r_s$



COORDINATE TIME

WHAT DOES $dt/d\tau$ ACTUALLY MEAN?

Definitely not the coordinates of a free-falling observer entering $r = r_s$.

TEXTBOOK: "observer far away" measures $dt/d\tau$
WHY? 'FAR AWAY' THE SPACETIME is $\approx$ FLAT, SO NO ACCELERATION.

ALTERNATIVELY, THE r, t COORDS ARE AN ACCELERATED FRAME S.t. BH STAYS @ $r = 0$.

SO THE COLLAPSE OF THE LIGHTCONE IN r, t COORDINATES SHOULDN'T BOTHER US.

ASIDE. We want to understand it.

FIRST CHECK: DUST FALLING INTO r_s FROM OUTSIDE. MASSIVE
geodesic (finally)

$$\overset{ds^2}{d\tau^2} = (1 - r_s/r)\, dt^2 - (1 - r_s/r)^{-1}\, dr^2$$

① $$\Rightarrow \boxed{1 = (1 - r_s/r)\, \dot{t}^2 - (1 - r_s/r)^{-1}\, \dot{r}^2}$$

$\dot{\circ} = d/d\tau$ CLOCK OF INFALLING DUST

WE ALSO HAVE: $g_{\mu\nu}$ is t-INDEPENDENT

so $\partial/\partial t = K_{(t)} = (1,0,0,0)$ is A KILLING VEC

and $K \cdot p$ is CONSERVED $\longrightarrow$ ENERGY

$$= g_{00}(m\dot{x})^0$$
$$= m(1 - r_s/r)\dot{t}$$

② $\Rightarrow \boxed{\mathcal{E} \equiv E/m = (1 - r_s/r)\dot{t}}$

A GOOD CHOICE: @ $r = \infty$, no KE

("drop from ∞")

then: $E = M \rightarrow \boxed{\mathcal{E} = 1}$

① + ② USE SHORTHAND: $1 - r_s/r = \heartsuit$

$$\hookrightarrow \heartsuit\dot{t} = 1$$

$$\heartsuit\dot{t}^2 - \frac{1}{\heartsuit}\dot{r}^2 = 1$$

$1/\heartsuit$

$\Rightarrow$ ~~[illegible]~~

$$\frac{1}{\heartsuit}\dot{r}^2 = \frac{1}{\heartsuit} - 1$$

$$\boxed{\dot{r}^2 = 1 - \heartsuit = r_s/r}$$

perfectly well behaved @ $r = r_s$

<u>IN FACT</u>: RECALL THE ANALOGOUS NEWTONIAN PROBLEM:

$$r\ {}^{d}\!/dt$$

$$E = 0 = \frac{1}{2} m \dot{r}^2 - \frac{GMm}{r}$$

$$\Downarrow$$

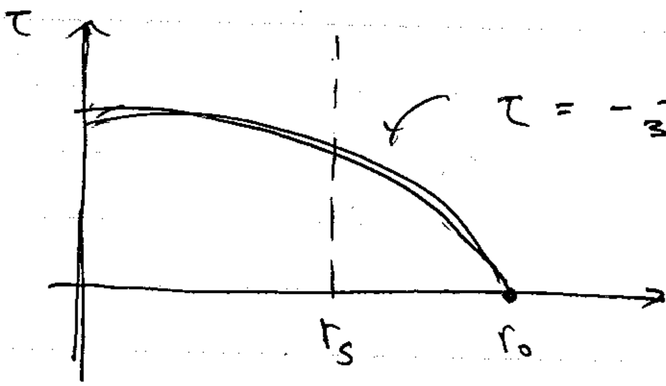
$$\left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r} \leftarrow r_s$$

EXACTLY THE SAME! nothing fishy!

<u>Solution</u>: $\dot{r} = \sqrt{r_s}/\sqrt{r}$

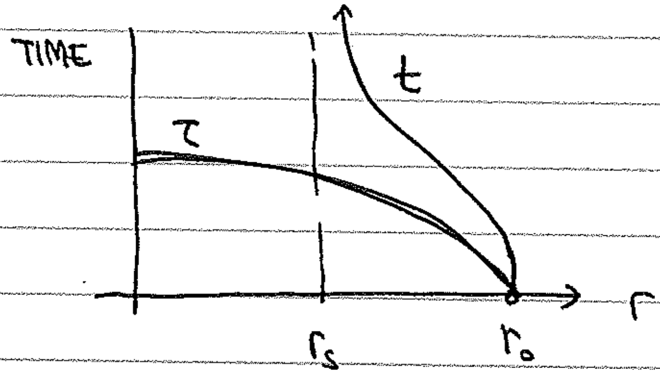
$$\sqrt{r}\, dr = \overset{\text{infalling}}{-}\sqrt{r_s}\, d\tau$$

$$\boxed{\frac{2}{3}\left(r^{3/2} - r_0^{3/2}\right) = -\sqrt{r_s}\,(\tau - \underset{=0}{\tau_0})}$$



$$\tau = -\frac{2}{3\sqrt{r_s}}\left(r^{3/2} - r_0^{3/2}\right)$$

IN YOUR HW:



$$\boxed{\frac{dt}{dr} = \frac{-\sqrt{r/r_s}}{1 - r_s/r}}$$

for massive test particle

↪ compare to massless:

BACK TO NULL GEODESIC:

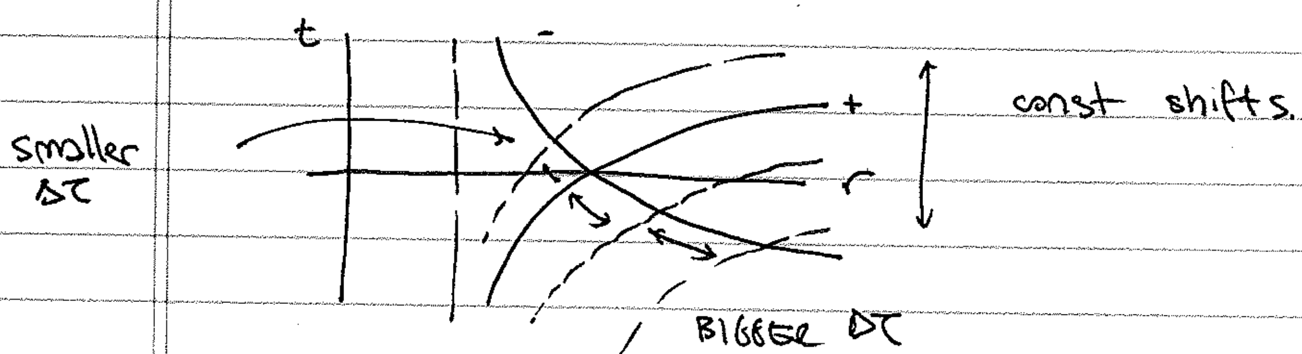
② → $\dfrac{dt}{dr} = \pm \dfrac{1}{1 - r_s/r}$

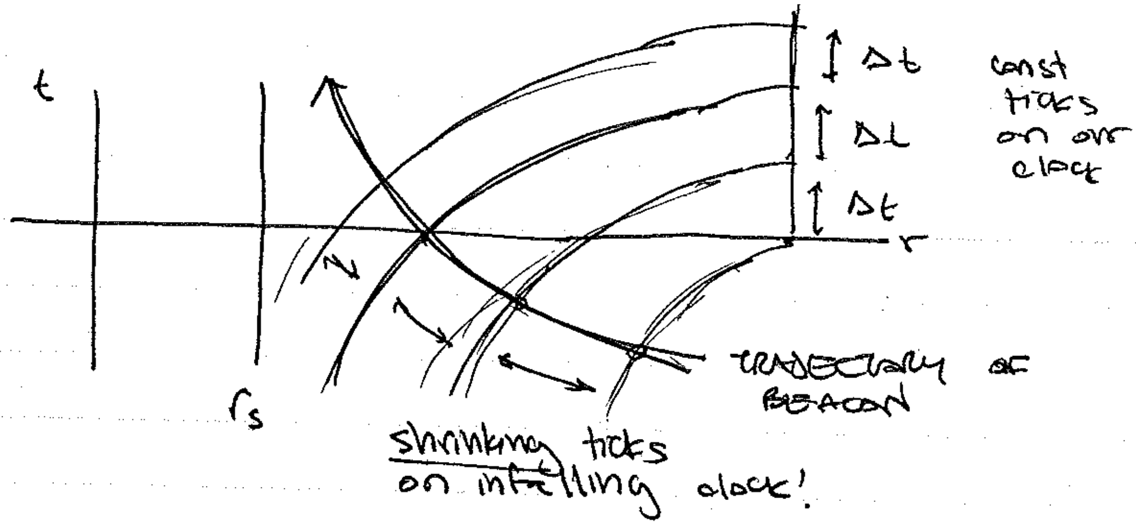
③ $$\boxed{t - t_0 = \pm \left(r + r_s \log\left(\frac{r}{r_s} - 1\right) \right) + \text{const.}}$$

− sign
AS $t\uparrow$, $r\downarrow$

DOMINATES FOR $r \gtrsim r_s$

DOM. for $r \gg r_s$





SO AS TEST PARTICLE APPROACHES SINGULARITY, IT LOOKS TO US THAT IT SLOWS DOWN!

of course, this is not at all what the particle sees.

alternatively: in above diagram: if the beacon sent a blip every $\Delta\tau$, the Δt would increase w/ each blip.

SO OUR COORDINATES FAIL US AS TEST PARTICLE FALLS IN. GOAL: BETTER COORDS.

1. UNDERSTAND WHAT HAPPENS PAST r_s
2. EXTEND COORDS

We proceed in a series of steps.

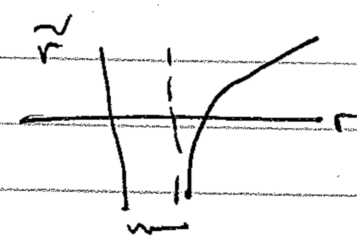
"Achilles & tortoise"

1. **TORTOISE COORDINATES**

one interp.

$\frac{dt}{dr} \to \infty$ b/c "dr shrinks as $r \to r_s$ & dr does not"

so choose new coordinate



no $\tilde r$ descr. $r < r_s$.

$$\star \quad \tilde{r} = \underbrace{r + r_s \ln\left(\tfrac{r}{r_s} - 1\right)}$$

in ③ $\longrightarrow$ WE SAW EARLIER THAT Null geodesics satisfy

$$\boxed{t = \pm \tilde{r} + \text{const.}}$$

④ $$d\tilde{r} = dr + \frac{1}{r/r_s - 1}\, dr = \frac{1}{\heartsuit}\, dr$$

$$\star \quad \boxed{d\tilde{r} = \frac{1}{1 - r_s/r}\, dr}$$

where $r = r(\tilde{r})$

$$ds^2 = \underbrace{\left(1 - \frac{r_s}{r}\right)}_{\heartsuit}\left(dt^2 - d\tilde{r}^2\right) - r^2 d\Omega$$

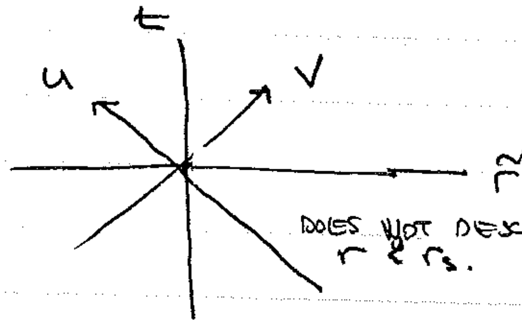
- now light cones don't squish @ r_s
- no divergent metric coords (inverse though...)
- $r = r_s \longleftrightarrow \tilde{r} = \infty$

so we're hiding r_s ...

2. $\boxed{\text{EDDINGTON - FINKELSTEIN}}$ coords.

↳ "lightcone coords" wrt tortoise

★ $$V = t + \tilde{r}$$
$$u = t - \tilde{r}$$



DOES NOT DESCRIBE $r \leq r_s$.

$$\boxed{\begin{array}{l}\text{infalling light: } v = \text{const} \\ \text{outgoing light: } u = \text{const}\end{array}}$$

BUT NOW USE HYBRID COORDS: (EF)

USE ORIGINAL r COORD $\longrightarrow$ $r = r_s$ is finite

USE V for "time" coord.

$$dV = dt + d\tilde{r}$$
$$dt = dV - \frac{1}{\heartsuit} dr \qquad \leftarrow \text{from } ④$$

$$ds^2 = \heartsuit\, dt^2 - \frac{1}{\heartsuit} dr^2 - r^2 d\Omega^2$$

(EF) $$\boxed{ds^2 = \heartsuit\, dv^2 - 2\, dv\, dr - r^2 d\Omega^2}$$

↳ $\heartsuit = 0$ @ $r = 0$

BUT METRIC IS STILL INVERTIBLE

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{in } v\text{-}r \text{ block}$$

$$g = \det g_{\mu\nu} = \overset{\text{check}}{-r^4 \sin^2\theta}, \quad \text{regular @ } r_s$$

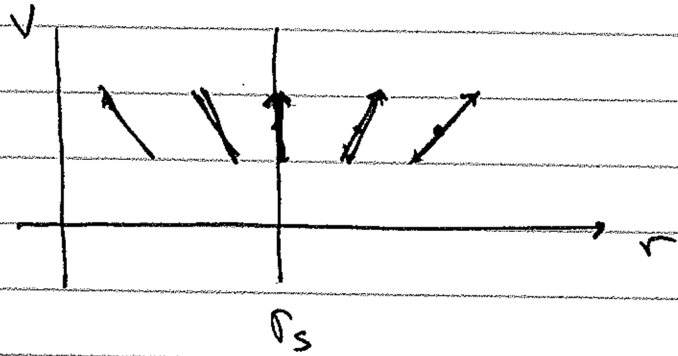
NULL GEODESICS IN EF: $ds^2 = 0$

$$\frac{dv}{dr} = \frac{2}{\heartsuit} \leftarrow \heartsuit dv^2 - 2\,dv\,dr = 0$$

$$= \boxed{\frac{2}{1 - r_s/r}}$$

that's <u>one</u> slope

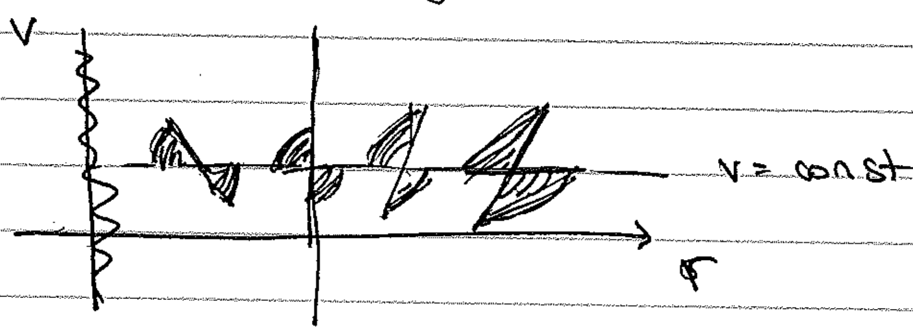
corresponds to ingoing or outgoing?



$\frac{dv}{dr} = \frac{2}{\heartsuit}$

outgoing light

$dv = 0 \leftrightarrow$ infalling light,



LIGHT CONES TILT OVER: DOOMED TO HIT SINGULARITY @ $r = 0$

$\hookrightarrow$ r_s IS AN <u>EVENT HORIZON</u>, no coming back nb : HORIZON SEPARATES REGIONS OF SPACETIME ACCORDING TO CAUSAL STRUCTURE.

nb : Newtonian "black hole"

↳ $V_{esc} = \sqrt{\frac{r_s}{r}} = 1$ when $r = r_s = 2GM$

DOES NEWTONIAN MECH HAVE AN EVENT HORIZON?

No! you can accelerate out of it!

cf SCHW. BH: inside horizon, your light cone has tumbled over ∴ you accel. w/in your light cone. YOU'RE STUCK & DOOMED.

↑ tidal forces kill you.

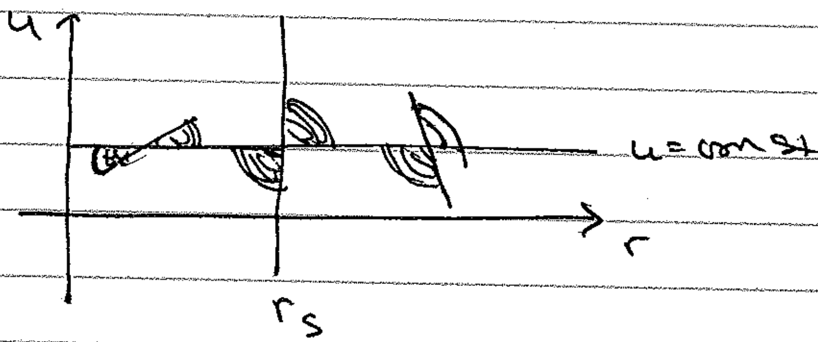
next : EXTEND SPACETIME

SO FAR: $0 < r \leq r_s$ & $r_s \leq r < \infty$ included

EF : USED v & r ← const on infalling null geodesics

another coord sys : EF′ : u & r
← const on outgoing null geodesics

$$ds^2 = V\,du^2 + 2\,du\,dr - r^2 d\Omega$$



NOW THINGS FALL OUT OF THE SINGULARITY,
BUT NOT INTO IT! → white hole

OR "FALLING" FROM $r > r_s$:
CAN ONLY CROSS r_s ON PAST-DIRECTED CURVES

spacetime now extended in two directions → future (EF)
→ past (EF′)

ANALYTIC CONTINUATION – like TAYLOR EXP in $\mathbb{C}$

TRY TO MAX EXTEND (all geodesics end on real sing.)

CAN FIND MORE DIRECTIONS. (spacelike)

$$2t = v + u \qquad v = t + r + r_s \log\left(\frac{r}{r_s} - 1\right)$$
$$2\tilde{r} = v - u \qquad u = t - r - r_s \log\left(\frac{r}{r_s} - 1\right)$$

$$\heartsuit\, dt^2 = \heartsuit\, \tfrac{1}{4}(dv^2 + du^2)$$
$$\tfrac{1}{\heartsuit} dr^2 = d\tilde{r}^2$$
$$= \heartsuit\, \tfrac{1}{4}(dv^2 - du^2)$$

$$ds^2 = \heartsuit\, dv\, du - r^2 d\Omega$$
$$\uparrow\ r = r(u,v)$$

BUT: $r = r_s$ is PUSHED to $u, v \to \pm\infty$

SO PULL THAT TO A FINITE VALUE.

TRICK: log "divergence" $\longrightarrow$ exponentiate

$$\tilde{v} = e^{v/2r_s} = \sqrt{r/r_s - 1}\; e^{\frac{r+t}{2r_s}} \qquad \tilde{u} = -e^{-u/2r_s} = -\sqrt{r/r_s - 1}\; e^{r - t/2r_s}$$

$$d\tilde{v} = \frac{1}{2r_s} e^{v/2r_s} dv \qquad d\tilde{u} = \frac{1}{2r_s} e^{-u/2r_s} du$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$dv = \frac{2r_s}{\tilde{v}} d\tilde{v} \qquad du = -\frac{2r_s}{\tilde{u}} d\tilde{u}$$

<u>then</u>

$$ds^2 = \frac{-4r_s}{\tilde{v}\,\tilde{u}}\, d\tilde{v}\, d\tilde{u} - r^2 d\Omega^2$$

$$= \underbrace{\frac{1 - r_s/r}{r/r_s - 1}}_{\frac{\cancel{\left(\frac{r}{r_s} - 1\right)}}{\frac{r}{r_s}\cancel{\left(\frac{r}{r_s} - 1\right)}}} \cdot (+4r_s^2)\, e^{-r/r_s}\, d\tilde{v}\, d\tilde{u} - r^2 d\Omega^2$$

$$\boxed{ds^2 = \frac{4r_s^3}{r} e^{-r/r_s}\, d\tilde{v}\, d\tilde{u} - r^2 d\Omega^2}$$

well behaved @ $r = r_s$.

<u>LAST STEP</u>: go from lightcone coords to time & space coords. u & v have null partials.

$$\left.\begin{aligned} T &= \tfrac{1}{2}(\tilde{v} + \tilde{u}) \\ R &= \tfrac{1}{2}(\tilde{v} - \tilde{u}) \end{aligned}\right\} \text{ \underline{KRUSKAL} COORDS.}$$

$$\boxed{ds^2 = \frac{4r_s^3}{r} e^{-r/r_s}\left(dT^2 - dR^2 - r^2 d\Omega^2\right)}$$

<u>almost looks "FLAT"</u>

HW: $T(t,r)$, $R(t,r)$ $\Big\{$ ~~$t\sqrt{\frac{r}{r_s} - 1}\, e^{r/2r_s} \sinh \frac{t}{2r_s}$~~

NULL CURVES: $T = \pm R + \text{const.}$

$$T = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s} \sinh(t/2r_s)$$

$$R = \text{———}''\text{———} \cosh(t/2r_s)$$

$\Rightarrow r$ def. by: $\boxed{T^2 - R^2 = (1 - r/r_s)\, e^{r/r_s}}$

- then: EVENT HORIZON is $\boxed{T = \pm R}$
- constant t surface

$$\frac{T}{R} = \tanh \frac{t}{2r_s}$$

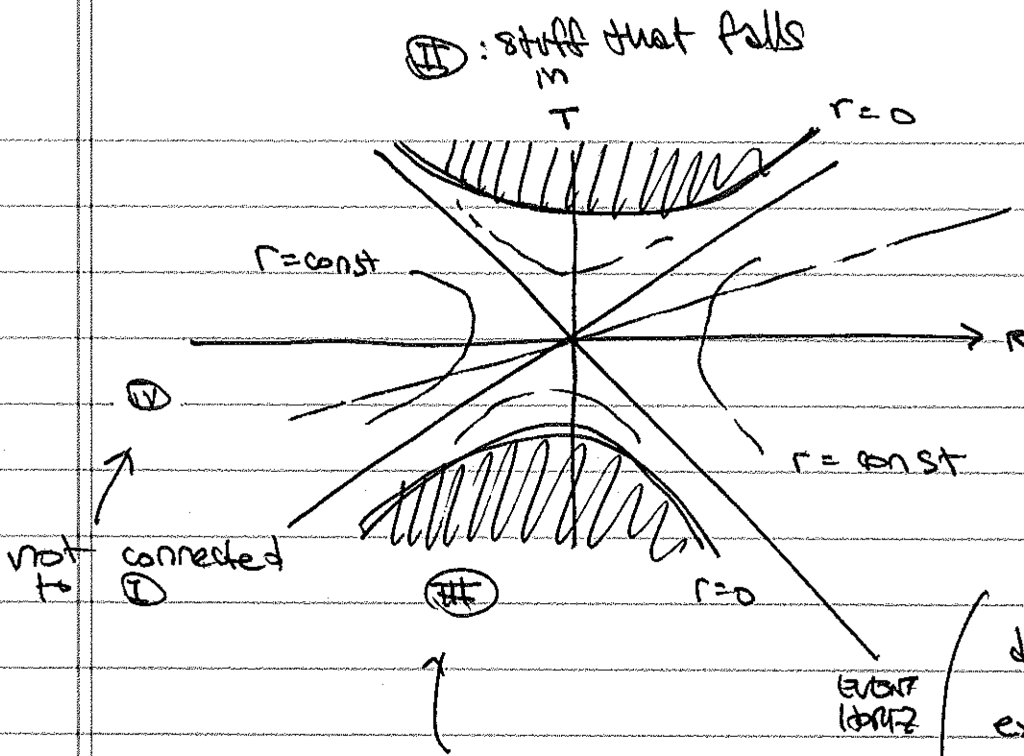
- constant r surface

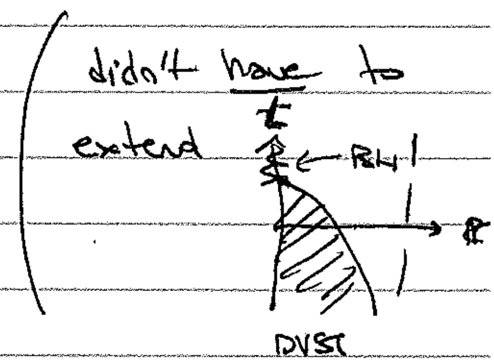
$$T^2 - R^2 = \text{const.} \quad (\text{from above})$$

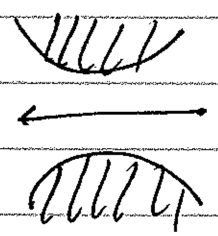
- AVOIDING SINGULARITY @ $r=0$

$$T^2 < R^2 + 1$$

LIGHT CONES:







"EINSTEIN-ROSEN BRIDGE"

LOOKS LIKE A SPACELIKE PATH BETWEEN I & IV

BUT NO TIMELIKE PATH.

Meaning of III?

eg. WHERE CHARGE OF A BH COMES FROM.

$$A_\mu = \int j^\nu(x')\, \underbrace{G_{\mu\nu}(x',x)}_{\text{RETARDED GF.}}\, d^4x'$$

or eg. GRAV FIELD even!

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stuff that eventually goes into bh.