LEC 9 SYMMETRIES, formally — 9 FEB

- HW 4 — in progress still ~
- CORRECTIONS: Jon asked: effect of DM on PERIHELION OF ☿?
  → CONTRIBUTES TO 1/r POTENTIAL BUT not const (GAUSS' LAW)

REMINDER: ALL MY SIGNS ARE SUBJECT TO ERROR!

$$p^2 = E^2 - \vec{p}^2 = m^2 \quad \text{vs.} \quad -E^2 + \vec{p}^2 = -m^2$$

LAST TIME: circuitous discussion of different topics

SCHW. METRIC: $(1 - r_s/r)\,dt^2 - (1 - r_s/r)^{-1} dr^2 \ldots$

↳ think about black holes ...
r_s significance?

HOW TO STUDY: geodesic into black hole!
... BUT WE WANTED MORE MACHINERY:
CONS. QUANTITIES ⟷ SYM.

application: PERIHELION OF ☿

TODAY: DIG IN & DEVELOP TOOLS TO GET BACK TO BH BUSINESS (& other GR)

SYMMETRY — why?

1. allows us to carry tools to solve ordinary mechanics problems to relativistic mechanics
   ↳ eg. PERHELION OF MERCURY

2. SYMMETRIES ARE POWERFUL
   ↳ best eg: AdS / CFT correspondence

isometries of this metric ...

spacetime sym. of this thy

LAST TIME: hint of Noether's thm in curved space

$$L(x,\dot{x}) = \underbrace{(ds/d\tau)^2}_{=1} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

$\dot{\ } = d/d\tau$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}$$

SO IF METRIC IS INDEP OF A COORDINATE, THEN $\partial L/\partial \dot{x}^\mu$ IS τ-INDEP.

BUT: $\frac{\partial L}{\partial \dot{x}^{\bar{F}}} = 2 g_{\bar{F}\nu} \dot{x}^{\nu}$ is not <u>invariant</u>

$\bar{F}$ means NOT a tensor index, rather: SOME FIXED VALUE, eg $\bar{F} = 0$ for ENERGY CONS.

CONSERVED & INVARIANT ARE DIFFERENT!

SO WE DEFINED A "HELPER VECTOR" TO LET US PICK OUT A DIRECTION:

$$K_{(F)}{}^{\nu} = \delta_{\bar{F}}{}^{\nu} \quad \text{eg } (1,0,0,0) \text{ for } K_{(t)}{}^{\nu}$$

then $\partial L / \partial \dot{x}^{\bar{F}} = \partial L / \partial \dot{x}^{\nu} \, K_{(F)}{}^{\nu} \sim \dot{x} \cdot K_{(F)}$ is <u>conserved</u> & <u>invariant</u>.

<u>KILLING VECTOR</u>.

WE INTRODUCED KILLING VECTOR AS A HACK, BUT THEY'RE TELLING US SOMETHING:

$$\partial L / \partial x^{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x^{F}} g_{..} = 0$$

"the metric is constant along this direction"

<u>iso</u> <u>metry</u>

CONSERVED QUANTITIES:

for massive particles: $p = \dot{x} m$

for massless particles: CAN CHOOSE AFFINE PARAMETER s.t. $p = \dot{x}$

geodesic eq is 2^{ND} O
so inv't. if $\tau \to a\tau + b$

SO A NICE PHYSICAL QUANTITY THAT IS CONSERVED is $\boxed{p \cdot K_{(F)}}$

carroll ch. 3

"along geodesic"

up to m

$$0 = \frac{d}{d\tau}(p\cdot K) \propto p\cdot D(K\cdot p)$$

$\uparrow$ $\dot{x}\cdot D$

$$= p^\mu (D_\mu K_\nu) p^\nu + \underbrace{p^\mu K_\nu (D_\mu p^\nu)}$$

$(p\cdot D)p^\nu = 0$ by geodesic eq (geodesic: we parallel transport $\dot{x} \propto p$)

$$= \underbrace{p^\mu p^\nu} D_\mu K_\nu$$

BUT THIS IS TOTALLY SYMMETRIC $\longrightarrow$ PROJECTS OUT ANTISYMMETRIC PART OF $D_\mu K_\nu$

$$= \frac{1}{2} p^\mu p^\nu \underbrace{D_\mu K_\nu + D_\nu K_\mu}$$

$$\equiv D_{(\mu} K_{\nu)}$$

momentum cons in k dir $\rightarrow$ no grav. force.

KILLING'S EQ: $D_{(\mu}K_{\nu)} = 0 \Rightarrow (K\cdot p)$ conserved

actually a Killing field (@ ea point)

$$\text{HW:} \quad \underbrace{D_\mu D_\sigma K^\rho}_{\text{2nd DERIV of } K} = R^\rho{}_{\sigma\mu\nu} \underbrace{K^\nu}_{\text{0th DERIV.}}$$

WEINB 6.5, 11.1

THIS RESULT IS OF MATHEMATICAL INTEREST:

GIVEN K & DK, WE CAN NOW BOOTSTRAP D^2K & ALL HIGHER DERIVATIVES

SO THE DATA IN A KILLING FIELD IS GIVEN BY

$$\underset{@x}{K^\mu} = \underbrace{A(x)^\mu{}_\nu\, \underline{\underline{K(0)^\nu}} + B(x)^{\mu\nu}{}_\rho\, \underline{\underline{D_\nu K(0)^\rho}}}_{\text{FUNCTIONS OF POS.}} + \underbrace{\cdots}_{\text{DETERM. FROM } A \,\&\, B}$$

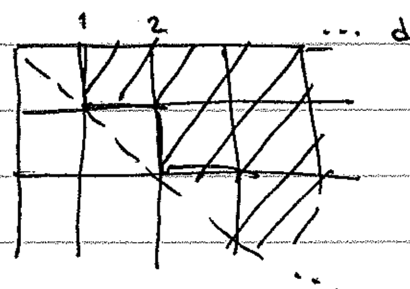
$K(0)^\nu$ HAS d DOF IN d DIMENSIONS

$D_\nu K(0)^\rho$?

two indices $\longrightarrow$ d^2 dof

BUT $D_{(\nu} K_{\rho)} = 0$, s.t. symmetric part vanishes

only antisymmetric piece left



$\frac{1}{2} d^2 - \frac{1}{2} d = \boxed{\frac{1}{2}(d-1)d}$

so: total DOF: $d + \frac{1}{2}d(d-1) = \boxed{\frac{1}{2}d(d+1)}$

A SPACE w/ A FULL SET OF KILLING VECTORS (fields) is <u>MAXIMALLY SYMMETRIC</u> (implications below).

CHECK: $d = 4$, max # KILLING VEC = 10 ... ?!

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

then said: look for <u>ISOMETRIES</u>

$$\frac{\partial g_{..}}{\partial x^\mu} = 0$$

only <u>four</u> of these!

so: what are the isometries of Minkowski?

4 translations $(\partial/\partial x^\mu)$ HOMOGENEITY

6 rotations $(x^\nu \partial/\partial x^\mu - x^\mu \partial/\partial x^\nu)$ ISOTROPY

not linearly independent??
BUT CLEARLY A SYMMETRY

@ any point K's may be lin. <u>dependent</u>

& a lin comb $a K_{(1)} + b K_{(2)}$ is also
a Killing vector... lin. dep.

BUT WE'RE TALKING ABOUT VECTOR <u>FIELDS</u>

$$\underline{\underline{a(x)}}\, K_{(1)}(x) + \underline{\underline{b(x)}}\, K_{(2)}(x)$$

this resulting ~~KILLING~~ VECTOR <u>FIELD</u>
is <u>not</u> nec. dependent...
... <u>nor</u> is it nec. a Killing field !

IN FACT, FINDING KILLING VECTORS FROM
A METRIC is TRICKY ... not clear
when to stop looking

part of why MAXIMALLY SYMMETRIC
SPACES ARE NICE.

is SCHWARZSCHILD MAX SYM? <u>NO</u>.

BUT COUSINS OF MINKOWSKI ARE:
DE SITTER & ANTI-de SITTER

PUNCHLINE: if space is <u>maximally symmetric</u> then geometry looks the same everywhere.

→ CURVATURE is the <u>same everywhere</u>
calculate $R = g^{\mu\nu} R_{\mu\nu} = R^{\lambda\mu}{}_{\lambda\mu}$ @ one place & you're done.

R encodes everything there is to know about the geometry (LOCAL) of the space!
(GIVEN dimensionality, # time dir, ...)

(eg not topo.)

ARGUMENT: LOC. INT. FRAME: $g_{\mu\nu} = \eta_{\mu\nu}$ @ a point

no preferred direction

{ unchanged by Lorentz transforms @ that point

so want $R_{\rho\sigma\mu\nu}$ to <u>also</u> be unchanged by Lorentz transf in this frame.

→ $R_{\rho\sigma\mu\nu}$ must be constructed from the tensors that are Lorentz invariant

↳ $g_{\mu\nu}$, $\delta^{\mu}{}_{\nu}$, $\varepsilon_{\mu\nu\rho\sigma}$

So @ this point & in these coords.
$R_{\dots}$ must be made out of $g_{\cdot\cdot}$, $\delta^{\cdot}_{\cdot}$, $\varepsilon_{\dots}$

BUT (HW) R HAS VERY SPECIFIC SYMMETRIES
w/rt its indices!

$$R_{\circ\circ\circ\circ}$$

- last pair: ANTISYM w/rt interchange
- first pair: SEPARATELY ANTISYM w/rt interchange

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

UNIQUE SOLUTION:

$$R_{\rho\sigma\mu\nu} \propto g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}$$

BUT THIS IS A TENSORIAL EQ (even though
written in specific coords)

↳ both sides transform well (& consistently)
w/rt change of coords.

PROPORTIONALITY CONST:

$$R_{\rho\sigma\mu\nu} = A\,(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$$

$g^{\rho\mu} g^{\sigma\nu}$

LHS: R RHS: $(d^2 - d)$

$$\Rightarrow \boxed{A = \frac{R}{d(d-1)}}$$

so: in a maximally symmetric space $\Leftrightarrow$

$$R_{\rho\sigma\mu\nu} = \frac{R}{d(d-1)} (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$$

w/ R some <u>constant</u> over spacetime

<u>LOCALLY</u>: WHAT MATTERS IN CLASSIFICATION IS

$$R \longrightarrow \pm, 0\ ?$$

Lie Derivative

NOTIONS OF DERIVATIVE:

∂_μ PARTIAL $\longrightarrow$ not covariant

D_μ COVARIANT $\longrightarrow$ introduces connection, Γ, to "fix" non-covariance of ∂_μ

$X \cdot D$

COULD ~~USE THIS~~ DEFINE Γ, BUT ON A METRIC SPACE (Riemannian Manifold) THERE IS A NATURAL CHOICE.

important for GEODESICS

$\Uparrow$

~~Both~~ D_μ takes $T \longrightarrow \underline{DT}$

higher rank Tensor.

important for INTEGRAL CURVES
SHOWS UP IN GEOM MECHANICS

LIE DERIVATIVE: $\mathcal{L}_X$ takes tensor to same rank tensor

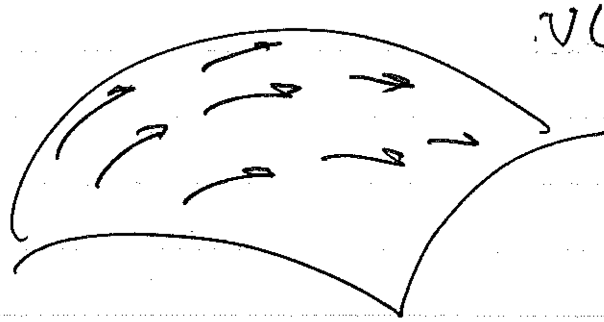
When we touched on this, we noted that the Lie derivative of a vector (field) is

$$\mathcal{L}_X Y = [X, Y] = (X \cdot \partial)Y - (Y \cdot \partial)X$$

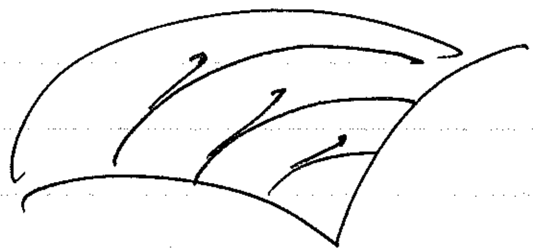
D'INVERNO
p.70

MORE INTUITIVE PICTURE : COMPARING ACTIVE VS PASSIVE

$V(x)$ vector field



DEFINES TRAJECTORIES
$$\frac{dx^\mu}{d\tau} = V^\mu(x(\tau))$$



(Reaffirms our identification of <u>vectors</u> w/ partial derivatives)

WHAT IS THE DERIVATIVE OF A TENSOR T @ x IN THE DIRECTION OF $V^\mu(x)$?

~~FIND~~ COMPARE 2 VERSIONS OF $T(x)$:

① PASSIVE TRANSFORM OF COORDINATES
T stays "the same", it's just coord. system that's changing

eg $T^{\mu\nu}(x) \longrightarrow T'^{\mu\nu}(x')$

(same point, in different coordinates

$$x' = x + \delta\tau\, V(x) \quad + \cdots$$

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu{}_\alpha + \delta\tau \frac{\partial}{\partial x^\alpha} V^\mu(x) \quad + \cdots$$

$$T'^{\mu\nu}(x') = \left(\frac{\partial x'}{\partial x}\right)^\mu{}_\alpha \left(\frac{\partial x'}{\partial x}\right)^\nu{}_\beta\, T^{\alpha\beta}(x)$$

$$= T^{\mu\nu}(x) + \delta\tau\,(\partial_\alpha V^\mu)\, T^{\alpha\nu}(x)$$
$$+ \delta\tau\,(\partial_\beta V^\nu)\, T^{\mu\beta}(x)$$
$$+ \cdots$$

(2) ACTIVE TRANSFORMATION (maybe I got these labels mixed up...)

EVALUATE $T^{\mu\nu}(x)$ @ A DIFFERENT POINT ON THE MANIFOLD, $x' = x + \delta\tau\, V(x)$

$$T^{\mu\nu}(x') = T^{\mu\nu}(x) + \delta\tau\, V^\gamma \frac{\partial}{\partial x^\gamma} T^{\mu\nu}(x) \quad + \cdots$$

nearby point — same point, diff coord

$$\mathcal{L}_V T = \lim_{\delta\tau \to 0} \frac{T^{\mu\nu}(x') - T'^{\mu\nu}(x')}{\delta\tau}$$

$$= V^\gamma \partial_\gamma T^{\mu\nu} - T^{\mu\beta} \partial_\beta V^\nu - T^{\alpha\nu} \partial_\alpha V^\mu$$

OBS: GIVES PREV. RESULT WHEN T = vector ; $\mathcal{L}_V W = [V, W]$

now easy to generalize

* BTW: CAN REPLACE $\partial \to D$ since the connection pieces vanish (by antisym)

↳ lower index : $\partial x / \partial x'$, so FLIP SIGN

$$\mathcal{L}_V Y_\mu = V^\gamma \partial_\gamma Y_\mu + Y_\beta \partial_\mu V^\beta$$

from $Y'_\mu(x') = \left(\frac{\partial x}{\partial x'}\right)^\beta{}_\mu Y_\beta(x)$

$$= (\delta^\beta_\mu - \delta\tau\, \partial_\mu V^\beta) Y_\beta$$

PROPERTIES : $\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S$

LINEAR

$$\mathcal{L}_V(TS) = T\mathcal{L}_V S + (\mathcal{L}_V T) S$$

LIEBNIZ

~~really~~ $\mathcal{L}_V(\phi) = V^\gamma \partial_\gamma \phi$

↳ really a DERIVATIVE .

BACK TO ISOMETRIES:

ANOTHER WAY OF IDENTIFYING ISOMETRY: DERIVATIVE VANISHES

↳ why didn't we just say $D_\mu g_{\nu\rho} = 0$?

THIS IS ALWAYS TRUE FOR OUR COVARIANT DERIVATIVE!
(metric compatibility)

LIE DERIVATIVE GIVES ALTERNATIVE:

$$\mathcal{L}_V g_{\mu\nu} = V^\gamma \partial_\gamma g_{\mu\nu} + g_{\mu\beta}\partial_\nu V^\beta + g_{\alpha\nu}\partial_\mu V^\alpha$$

CAN PROMOTE $\partial \to D$
(connection terms cancel)

metric commutes w/ cov. der.

$$\mathcal{L}_V g_{\mu\nu} = \underbrace{V \cdot D g_{\mu\nu}}_{=0 \text{ BY COMPATIBILITY}} + D_{(\mu} V_{\nu)}$$

$$= D_{(\mu} V_{\nu)} = 0 \quad \text{for isom.}$$

(KILLING EQUATION)