LEC 8 : SCHWARZSCHILD METRIC 7 FEB 2017

TODAY: HW3 HIGHLIGHTS
SCHWARZSCHILD METRIC - singularities?
CONSERVED QUANTITIES
PERIHELION OF MERCURY

REVIEW: HW #3 HIGHLIGHTS

1. LIMITS ON EQUIVALENCE

$$g_{\mu\nu}(x) = g(0)_{\dots} + A_{\dots}x^{i} + B_{\dots}x^{i}x^{j}$$

$$x = K y^{i} + L_{\dots} y^{i} y^{j} + M y y y$$

CAN WE CHOOSE THIS TO MAKE $g_{\mu\nu} = \eta_{\mu\nu}$?

@ any point, yes. K even includes LORENTZ REDUNDANCY

THEN GO TO SMALL DISPLACEMENTS & ARGUE O by O. L & A HAVE SAME # DOF BUT B HAS MORE FREEDOM THAN M.

↳ SO DEVIATIONS FROM FLAT ARE $O(x^2)$

VOLUME ELEMENTS

$$d^4x\,(\cdots) \longrightarrow \sqrt{g}\, d^4x\,(\cdots)$$

ONLY IN CARTESIAN, EUCLIDEAN SP.

INVARIANT VOLUME ELEMENT IN ANY COORD SYS, SPACETIME

eg gives "shortcut" for covariant deriv. by converting invariant integral to an equivalent one:

$$\underbrace{\int d^4x \sqrt{g}}_{\text{INVT.}} \underbrace{V^\mu(x)\, \overbrace{\partial_\mu \phi(x)}^{\text{"CORRECT" DERIVATIVE}}}_{\text{INVT.}}$$

$$= -\int d^4x\, \underbrace{\partial_\mu\left[\sqrt{g}\, V^\mu(x)\right]}_{\equiv \sqrt{g}\, D_\mu V^\mu(x)} \phi(x) + \text{BNDY}$$

b/c we want $d^4x\sqrt{g}$ manifest

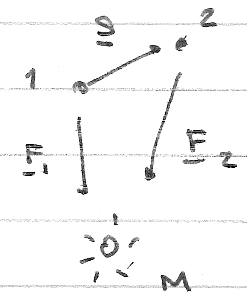
nb. DIMENSIONAL ANALYSIS SHOULD HAVE BEEN YOUR FIRST CHECK!

nb. ALSO A USEFUL TRICK FOR DERIVING VECTOR CALC IDENTITIES IN CURVILINEAR COORDINATES.

talk about black hole forces

TIDAL FORCES

NEWTONIAN:



THE SEPARATION OBEYS $\quad x_1 = (0,0,r)$

$$\ddot{\underline{s}} = \frac{-GM}{r^3} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}}_{\sim \text{moment of inertia}} \underline{s}$$

attractive in ~~some~~ transv. dir,
repulsive in z-dir.

HENCE: GRAVITY OF MOON CAUSES TIDES

even though ⊕ & ☾ ARE FREE FALLING TOWARD EACH OTHER

nb. $1/r^3$ MEANS: EFFECT OF ⊙ is NEGLIGIBLE ON SEPARATION (TIDAL FORCE), EVEN IF IT IS THE DOMINANT GRAV. POTENTIAL.

BY COMPARISON: GR VERSION IS

$$\frac{D^2}{d\tau^2} S^\mu = -R^\mu{}_{\alpha\nu\beta} S^\nu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

SPACETIME SEPARATION

$$\frac{D}{d\tau} = \frac{dx^\lambda}{d\tau} D_\lambda$$

1ST DERIV.

SEPARATION

RECALL:

NEWTONIAN: $\ddot{\underline{x}}_I = -\underline{\nabla}\Phi \longrightarrow \ddot{\underline{S}} \sim \underline{\partial_i \partial_j} S^j$

INDIVIDUAL PARTICLE

2ND DERIVATIVE (HENCE MOMENT OF INERTIA ... MULTIPOLE EXPANSION!)

GR VERSION: WE KNOW $g_{00} \sim 1+2\Phi$ IN NEWTONIAN LIMIT

$$\Gamma \sim \partial g$$
$$R \sim \partial\Gamma \sim \partial^2 g$$

cf. PROBLEM ON EQUIV! WE KNOW LIF FAILS @ $\mathcal{O}(x^2)$, WHERE CURVATURE BECOMES IMPORTANT.

$\Big\{$ 2^{ND} O TERM IN A KIND OF MULTIPOLE EXPANSION

WALD CH 21

MOVING ON: SPACETIMES w/ ROTATIONAL SYM.

$$ds^2 = g_{00}\,dt^2 - g_{rr}\,dr^2 + r^2\,d\Omega^2$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$$

two function's worth of generality

BUT WE ALSO FIXED THE DEFINITION OF "RADIUS" TO GET THIS FORM!

ie the r IN THIS METRIC MAY NOT BE THE "RADIAL DISTANCE" THAT YOU'RE USED TO — we had to redefine it to get rid of $dt\,dr$ cross terms.

HARTLE 21.9 SCHWARZSCHILD METRIC

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$
$$+ r^2 d\Omega^2$$

looks like $1 - 2\Phi(r)$ $\longrightarrow$ Newtonian limit

dr^2 coefficient? (NOT IN NEWTONIAN LIMIT)

$$\frac{1}{1 - 2GM/r} \approx 1 + \frac{2GM}{r} + \mathcal{O}\left(\frac{GM}{r}\right)^2$$

for $GM/r \ll 1$
(WEAK GRAVITY LIMIT)

DERIVATION: *from EINSTEIN'S EQ (we'll get to it)*

<u>MEANING</u>: GEOMETRY OF <u>EMPTY</u> SPACE OUTSIDE OF A SPHERICALLY SYMMETRIC GRAV. SOURCE

WE CAN CONFIRM THIS FROM NEWTONIAN LIMIT

SCHWARZSCHILD RADIUS

$$\frac{1}{1 - 2GM/r} \rightarrow \infty$$

WHEN $\boxed{r_s = 2GM}$

for a star : $r_s < r_{star}$

ie: this singularity is never in the regime of validity of the metric (EMPTY SPACE OUTSIDE GRAV SOURCE)

$$G = 7 \times 10^{-11} \; m^3/kg\,s^2 = \hbar c / M_{Pl}^2$$

7×10

[PLANCK UNITS : $c = \hbar = G = 1$
ie all masses are relative to PLANCK MASS]

2×10^{30} kg

$$\frac{2GM_\odot}{c^2} = \boxed{3 \text{ km} = r_{s,\odot}}$$

$(3 \times 10^8)^2$

m/s

$$\text{vs. } \boxed{r_\odot = 7 \times 10^5 \text{ km}}$$

in fact: we know $\phi = 0$ there

SIMILARLY: $r = 0$ SINGULARITY is "SAFE"

OF COURSE... WE KNOW THAT IN GR THERE ARE BLACK HOLES, FOR WHICH $r_{BH} \to 0 \ll r_{s,BH}$

↳ is there something bad @ r_s ?

NOT NECESSARILY — YOU CAN HAVE COORDINATE SINGULARITIES THAT ARE NOT PHYSICAL

for SCHWARZSCHILD: $r = r_s$ is COORDINATE
$r = 0$ is "PHYSICAL"

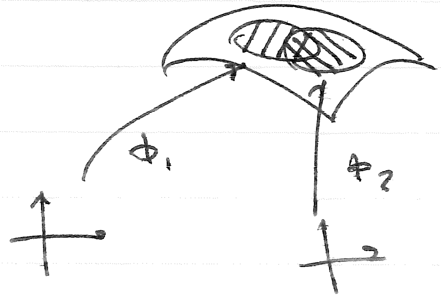
see: CARROLL 5.3

COORD SINGULARITY: $ds^2 = dr^2 + r^2 d\theta^2$

$$g^{\theta\theta} = 1/r^2$$

~~WE'LL DIG INTO THE MEANING OF r_s WHEN IT~~

REMARK: THE FULL MACHINERY OF DIFFERENTIAL GEOMETRY IS BUILT AROUND THE IDEA OF "NICEST COORDS" IN EACH REGION.



s.t. in the overlap, these mappings match & are consistent.

MOST OF THE TIME, PHYSICISTS ARE (unreasonably?) ATTACHED TO HAVING A SINGLE COORDINATE SYSTEM FOR THE WHOLE SPACE.

↳ nb. this is why we talk about magnetic monopoles w/ "DIRAC STRINGS"
↖ COORD ARTIFACT.

HOW TO DIAGNOSE?

↳ $R^{\cdot}{}_{\cdots}$? BUT THESE COMPONENTS ARE COORDINATE DEPENDENT.

NEED SCALAR (COORD INDEP)

↳ IMPORTANT: $\boxed{R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}}$
RICCI TENSOR

$\boxed{R = g^{\mu\nu} R_{\mu\nu}}$ RICCI SCALAR

from HW (last prob — tedious): this is the only INDEPENDENT contraction of Riemann.

WE'LL NEED THESE OBJECTS LATER.

SO YOU CAN TEST: IS R WELL BEHAVED?
(yes)

<u>BUT</u>, there are <u>other</u> SCALARS TO FORM!

$$R_{\mu\nu} R^{\mu\nu}, \quad R^{\alpha\beta\delta\gamma} R_{\alpha\beta\delta\gamma}, \quad R_{\mu\nu\rho\sigma} R^{\rho\sigma\lambda\tau} R_{\lambda\tau}{}^{\mu\nu}, \ldots$$

IN PRINCIPLE, HAVE TO CHECK ALL OF THEM!
if <u>any</u> blow up, then phys. singularity.

if none blow up, maybe you didn't check enough terms ...

$$\text{eg.} \quad R^{\cdots\cdot} R_{\cdots\cdot} = \frac{48 G^2 M^2}{r^6} \quad \text{for SCHWARZSCHILD}$$

$r = 0$ IS A <u>PHYS</u> SING.

A GOOD DIAGNOSTIC:
FIGURE OUT GEODESICS & SEE IF ANYTHING FISHY HAPPENS.

eventually we'll look @ particles falling into black hole ... they fall right past r_s. CAN EVEN CHECK: for VERY LARGE BH, tidal forces btwn particles IS MODEST @ r_s! $(\sim 1/r_s^3)$

GEODESIC EQ

$$\frac{D}{d\tau}\dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = 0$$

$\dot{x}^\mu$: $dx^\mu/d\tau$

have to calc a bunch of these!!

τ: AFFINE PARAM.

eg $$\ddot{t} + \underbrace{\frac{2GM}{r(r-2GM)}}_{2\Gamma^t_{tr} = \Gamma^t_{tr} + \Gamma^t_{rt}} \dot{r}\dot{t} = 0$$

$$\ddot{r} + (4 \text{ terms}) = 0$$
$$\ddot{\theta} + (2 \text{ terms}) = 0$$
$$\ddot{\phi} + (2 \text{ terms}) = 0$$

this is a mess!

BUT EVEN IN ~~CLASSICAL~~ NEWTONIAN PHYSICS, THIS IS NOT HOW WE LIKE TO SOLVE PROBLEMS.

↳ USE SYMMETRY ⟷ CONSERVED QUANTITIES.

CHENG
P.129
HARTLE
9.3

SYMMETRY $\leftrightarrow$ CONSERVATION LAW
$\downarrow$
conserved quantity
(integral of motion)

dynamics is 2^{ND} O, $\ddot{x}$
cons quantity in terms of 1^{st} O $g(\dot{x})^2$

RECALL: GEODESICS GIVEN BY EXTREMIZING "LAGRANGIAN"

$$L(x,\dot{x}) = (ds/d\tau)^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

EULER-LAGRANGE:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}$$

if L is x^μ-INDEPENDENT,
THEN $\partial L/\partial \dot{x}$ is CONSERVED

eg for SCHWARZSCHILD, $g_{\mu\nu}$ is t-INDEP.

so: $\underbrace{2 g_{\mu 0}\dot{x}^\mu}_{\partial L/\partial \dot{x}^0}$ is A CONS. QUANTITY.

BUT $2g_{\mu 0}\dot{x}^{\mu}$ is COORDINATE DEPENDENT !

(want to be able to say that something is conserved, the value is #, and that's that.

So DEF. KILLING VECTOR in t-DIRECTION

$$K_{(t)}^{\mu} = (1,0,0,0)$$

t not an index

s.t.

$$\boxed{\gamma_{(t)} = g_{\mu\nu}\dot{x}^{\mu}K_{(t)}^{\nu}}$$

is conserved.

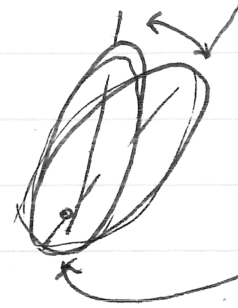
WE'LL FORMALIZE THIS ON THU ? IN THE HW.

FOR NOW, LET'S JUST APPLY IT.

PRECESSION OF PERIHELION OF ☿ (mercury)

NEWTONIAN $1/r^2$ FORCE LAW:
PLANETS HAVE ELLIPTICAL, CLOSED ORBITS

+ PERTURBATIONS FROM OTHER PLANETS:
no longer closed orbit



PRECESSION of PERIHELION
↑ axis, I guess
↑ POINT CLOSEST TO SUN

OBSERVED PRECESSION PER 100 yrs:

↓ ARC SEC

5600" + 574" + 48"

↑ ROT. OF EARTH

↑ PLANETS
(VENUS: 277"
JUPITER: 153"
⊕ : 90")

↑ ??

USE OUR SYMMETRIES:

$$ds^2 = (1 - r_s/r)\,dt^2 - (1 - r_s/r)^{-1}\,dr^2 + \boxed{r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2}$$

simplify our lives:

SKIP A FEW STEPS & USE CONS OF ANG'R MOMENTUM

NEWTONIAN INTUITION $\rightarrow$ STAY IN PLANE

SO CAN FIX: $\sin\theta = 1$

$\qquad\qquad\qquad d\theta = 0$

s.t. $r^2\,d\Omega^2 = r^2\,d\phi^2$

NOW EFFECTIVELY 3D PROBLEM.

$g_{\mu\nu}$ INDEP OF t, ϕ

KILLING VECTORS:

$$K_{(t)} = (1,0,0,0)$$
$$K_{(\phi)} = (0,0,0,1)$$

CONSERVED QUANTITIES:

$$g_{\mu\nu}\dot{x}^\mu K^\nu_{(t)} = (1 - r_s/r)\,\dot{t} = E/m \equiv k$$

$$g_{\mu\nu}\dot{x}^\mu K^\nu_{(\phi)} = r^2\dot{\phi} = l/m$$

$\uparrow$

ANGULAR MOMENTUM

SANITY CHECK:

WHY IS $E = m(1 - r_s/r)\dot{t}$?

OBSERVER W/ 4-VELOCITY u^μ

IN OBS. FRAME, $u = (1,0,0,0)$

S.T. ENERGY OF A PARTICLE W/

MOMENTUM $p^\mu = m\dot{x}^\mu$ IS $u \cdot p = \underset{\text{OBS FRAME}}{p^0}$

SO $E\big|_{\text{our frame}} = m\, g_{\mu\nu} \underset{(1,0,0,0)}{u^\mu} \dot{x}^\nu = m g_{00} \dot{t}$

$L = (ds/d\tau)^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

$$= \underbrace{(1-r_s/r)\dot{t}^2}_{\frac{K^2}{(1-r_s/r)}} - (1-r_s/r)^{-1}\dot{r}^2 \underbrace{- r^2\dot{\phi}^2}_{-\frac{\ell^2}{m^2 r^2}} = \underset{\substack{\text{speed of light} \\ \text{through spacetime}}}{c^2} = 1$$

mult by $\frac{1}{2}m(1-r_s/r)$

$$\frac{1}{2}mK^2 - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(1-r_s/r)\frac{\ell^2}{mr^2} = \frac{1}{2}m(1-r_s/r)$$

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}(1-r_s/r)\frac{\ell^2}{mr^2} - \frac{\overset{2GM}{r_s} m}{2r} = \underbrace{\frac{1}{2}m(K^2-1)}_{\mathcal{E}}$$

COMPARE TO :

$$E_{tot} = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2} m r^2 \dot{\theta}^2}_{\frac{1}{2}\frac{\ell^2}{m r^2}} + V(r)$$

GR: $$\mathcal{E} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{m r^2} - \underbrace{\frac{GM\ell^2}{m r^3}}_{\text{GR!}} - \underbrace{\frac{GMm}{r}}_{\text{NEWTON}}$$

HW: SOLVE $r(\phi)$ PERTURBATIVELY
[CHENG P. 133]

RESULT :

$$r = \frac{\overbrace{(\ell^2/GMm^2)}^{\text{const.} = \alpha}}{1 + \underbrace{e}_{\text{eccentricity}} \cos(\underbrace{(1-\varepsilon)}_{\varepsilon = 3r_s/2\alpha}\phi)}$$

s.t. r RETURNS TO r_{min} @ $\phi = 2\pi/1-\varepsilon \approx 2\pi + \underline{3\pi\, r_s/\alpha}$

$$\delta\phi = \frac{3\pi r_s}{\alpha} = \frac{3\pi r_s}{(1+e)\, r_{min}}$$

PER REV

$3\pi r_s \leftarrow$ 3 km for ☉

$r_{min} \leftarrow 5 \times 10^7$ KM

$(1+e) \uparrow$ 0.2 for ☿

RESULT : $\delta\phi = 5 \times 10^{-7}$ rad / revolution

$\frac{180}{\pi} \times 60 \times 60$

DEG/RAD MIN/DEG SEC/MIN

$= 0.103''$ / rev

$\times \frac{100 \text{ yrs}}{0.241 \text{ yrs}}$

PERIOD OF MERCURY

$= \boxed{43'' / \text{century}}$