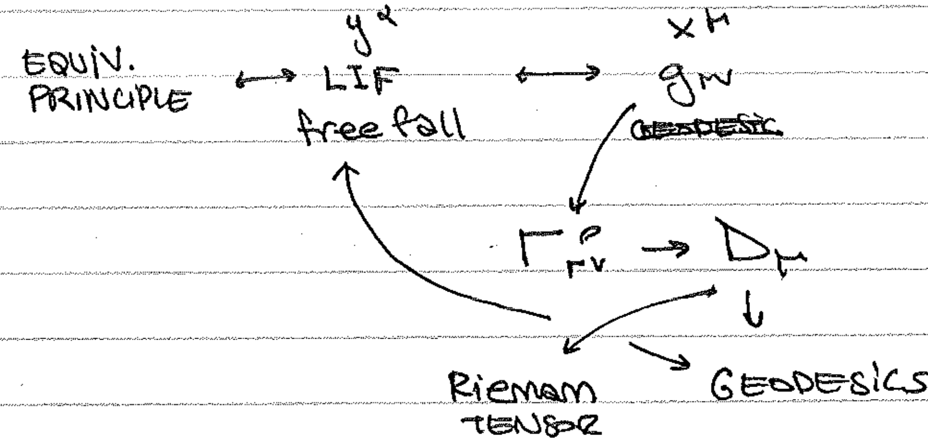
LECTURE 7: 31 Jan '17

★ no class on Thu! LONG HW INSTEAD

SO FAR:



TODAY: another derivation of Riemann

symmetries of $g_{\mu\nu}$

BIGGER PIC: DEVELOPING PIECES TO UNDERSTAND

1. Schwarzschild metric
   → usual examples of GR

2. Einstein's eq.

LAST TIME:

RIEMANN TENSOR

index of V on LHS

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \qquad \text{(TORSION-FREE)}$$

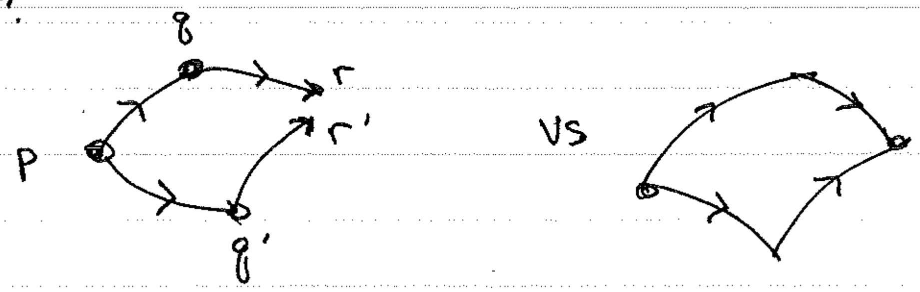
contracts V on RHS

ANTISYM indices of cov. DER.

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)}$$

REMOVES NON-TENSORIAL PART, ENSURING WELL BEHAVED TRANSFORMATION

BUT WE WERE CONFUSED: are we actually comparing vectors @ the same spacetime point?

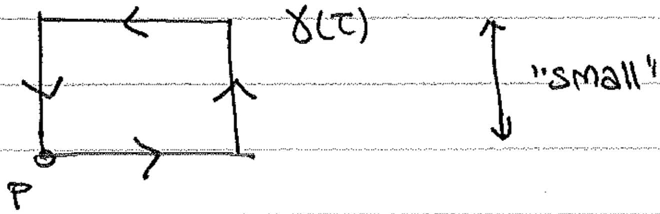


IDEA: LHS is A "SMALL" CORRECTION TO RHS
(full deviation

L.O. term

Zee IX.1

LET'S TAKE A SLIGHTLY DIFF. APPROACH.



Some closed curve, $\gamma(\tau)$
start w/ vector @ P
then push* it around the loop
& ask how it has changed when
it returns to P.

* PUSH: parallel transp. along $\gamma(\tau)$, whether or not γ is a geodesic

$$\Delta V^\rho = V^\rho(\tau_1) - V^\rho(\tau_0)$$

FIRST: to avoid clutter, it is useful to ask the same question w/ rt the one-form $V_\sigma = g_{\sigma\rho} V^\rho$

COMPLETELY EQUIVALENT INFO, EXCEPT INDICES WILL BE s.t. WE IDENTIFY $R^\rho{}_{\sigma\mu\nu}$ @ THE END.

ASIDE 1

RECALL: COVARIANT DERIVATIVE TELLS US WHAT HAPPENS TO A VECTOR AS I PUSH IT IN A CURVED SPACE

$$D_\lambda V^\rho = \partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma$$

∂_λ: $\partial/\partial x^\lambda$ $\quad$ V^ρ: $V^\rho(x)$

SO ALONG A CURVE $\gamma(\tau)$,

$$\frac{DV^\rho}{d\tau} = \dot\gamma^\lambda \partial_\lambda V^\rho + \dot\gamma^\lambda \Gamma^\rho_{\lambda\sigma} V^\sigma$$

$\dot\gamma = d\gamma/d\tau$, velocity vector

nb: $\dot\gamma^\lambda \partial_\lambda = d/d\tau$

THE VECTOR V^ρ PUSHED ALONG $\gamma(\tau)$ IS GIVEN BY SOLVING THE 1st O ODE

$$\boxed{\frac{dV^\rho}{d\tau} + \Gamma^\rho_{\lambda\sigma} \dot\gamma^\lambda V^\sigma = 0}$$

nb: $\gamma(\tau)$ needn't be a geodesic!
(WE'RE STILL PARALLEL TRANSPORTING)

WE ARGUED THAT FOR LOWER INDICES,

$$D_\lambda V_\sigma = \partial_\lambda V_\sigma - \underline{\Gamma^\rho_{\lambda\sigma} V_\rho}$$

INDICES "HAD TO BE" LIKE THIS

MINUS!
came from $\partial x / \partial x'$ transformation matrix vs $\partial x' / \partial x$.

ASIDE 2 — ALTERNATIVE DERIVATION:

$V^\mu W_\mu = V \cdot W$ is a scalar

"$|V||W| \cos\theta$"

when we parallel transport, relative angle unchanged.

$$\frac{d}{d\tau}(V \cdot W) = \left[-\left(\Gamma^\mu_{\rho\sigma} V^\sigma \right) W_\mu - V^\mu \left(\tilde{\Gamma}^\nu_{\mu\rho} W_\nu \right) \right] \dot{\gamma}^\rho$$

$$\| \quad 0 \longrightarrow \tilde{\Gamma}^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho}$$

(RELABEL DUMMY INDICES)

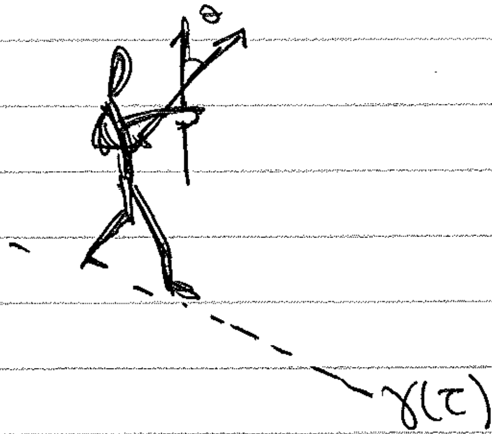
$$-\Gamma^\mu_{\rho\sigma} V^\sigma W_\mu - \underbrace{\tilde{\Gamma}^\nu_{\mu\rho} V^\mu W_\nu}_{\tilde{\Gamma}^\mu_{\rho\sigma} V^\rho W_\mu} = 0$$

s.t. $\tilde{\Gamma}^\mu_{\rho\sigma} = -\Gamma^\mu_{\rho\sigma}$ ✓

ASIDE 3 <u>SIDE-SIDE NOTE</u>

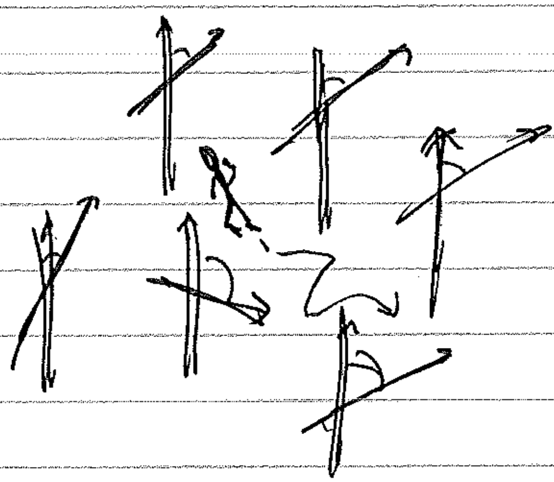
WHEN WE PARALLEL TRANSPORT A VECTOR,
WE EXTEND IT ALONG $\gamma(\tau)$

then $\frac{d}{d\tau}(V \cdot W) = 0$



vector fields

CONTRAST THIS TO $\phi(x) = V(x) \cdot W(x)$



then:

$D_\mu \phi(x) = \partial_\mu \phi(x)$

$\frac{d}{d\tau}\phi = \dot{\gamma}\, \partial_\mu \phi(x)$

so: $\Delta V_\sigma = V_\sigma(\tau_1) - V_\sigma(\tau_0)$

$$= \int_{\tau_0}^{\tau_1} d\tau \boxed{\frac{dV_\sigma}{d\tau}}$$

$$= + \Gamma^{\rho}_{\lambda\sigma} \dot{\gamma}^{\lambda} V_\rho$$

$$= \boxed{\int_{\gamma_0}^{\gamma_1} dx^{\lambda} \, \Gamma^{\rho}_{\lambda\sigma} V_\rho}$$

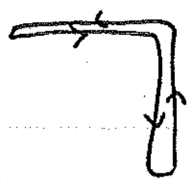
ORIENTED LINE ELEMENT

$\Delta V_\sigma \to 0$ AS  $\to \cdot$

no transport

but <u>how</u>? what does it scale with?

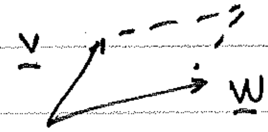
PERIMETER OR AREA?

not this:  has $\Delta V_\sigma = 0$

so $\Delta V_\sigma \sim$ area enclosed

SO: HOW DO WE DESCRIBE AREAS?

↳ CROSS PRODUCTS $\underline{V} \times \underline{W}$



antisymmetric machine that takes 2 vectors & spits out oriented area

HIGH-FALUTIN' LANGUAGE: 2-FORM

$$dx^\mu \wedge dx^\nu \equiv \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right)$$

k-form: antisymmetric map from $V^k \to \mathbb{R}$.

vector space, like T_pM

IT SUFFICES FOR US THAT THE AREA IS GIVEN BY A TENSOR $A^{\mu\nu} = -A^{\nu\mu}$ (really $\delta x^\mu \times \delta x^\nu$)

So: $\Delta V_\sigma \propto A^{\mu\nu}$ AND $\propto V_\sigma$ itself

$$\Delta V_\sigma \equiv R^\rho{}_{\sigma\mu\nu} V_\rho A^{\mu\nu} = \oint dx^\lambda \Gamma^\rho_{\lambda\sigma} V_\rho$$

Define; need to show that this is the Riemann

WRITE : $\oint dx^\lambda \, \Gamma^\rho_{\lambda\sigma} V_\rho$ "=" $\oint dx \, (\Gamma V)$

just drop indices for simplicity

then we can Taylor expand <u>integrand</u> about $x_0 = \gamma(\tau_0)$

$$(\Gamma V) = \underbrace{(\Gamma V)_0}_{\text{vanishes in } \oint} + \partial_\alpha(\Gamma V)|_0 \underbrace{(x - x_0)^\alpha}_{\text{vanishes in } \oint} + \cdots$$

this term: $\partial_\alpha (\Gamma V)_{\lambda\sigma}|_0 \underbrace{\oint dx^\lambda x^\alpha}_{A^{\lambda\alpha}}$

"AREA OF LOOP"

eg. Green's thm / Stokes' thm

(UNITS OF AREA)

nb: by integration by parts

$$A^{\lambda\alpha} = \int d\tau \, \frac{\partial x^\lambda}{\partial \tau} x^\alpha = - \int d\tau \, x^\lambda \frac{\partial x^\alpha}{\partial \tau} = - A^{\alpha\lambda}$$

no boundary

SO: ~~only~~ linear term is ~~important~~ leading piece. (others vanish as area shrinks)

NEED TO SHOW: this is the same RIEMANN TENSOR AS BEFORE

EXPAND TO LINEAR ORDER

$$\Gamma^\rho_{\lambda\sigma} = \Gamma^\rho_{\lambda\sigma}(x_0) + \partial_\alpha \Gamma^\rho_{\lambda\sigma}\big|_{x_0} (x-x_0)^\alpha + \cdots$$

$$V_\rho = V_\rho(x_0) + {}^{dV_\rho}/_{d\tau} \; (\tau_1 - \tau_0)$$

$$—\,||\,— \quad + \Gamma^\beta_{\lambda\rho}\, \dot{\gamma}^\lambda V_\beta (\tau-\tau_0)$$

$$\frac{\partial\gamma^\lambda}{\partial\tau}(\tau-\tau_0) = (x-x_0)^\lambda$$

$$= V_\rho(x_0) + \Gamma^\beta_{\lambda\rho} V_\beta (x-x_0)^\lambda$$

$$\Rightarrow \Delta V_\sigma = \partial_\alpha \left[\Gamma^\rho_{\lambda\sigma} V_\rho\right]\big|_{x_0} \; A^{\lambda\alpha}$$

$$= \left(\Gamma^\rho_{\lambda\sigma} \cancel{(x_0)}\, \Gamma^\beta_{\alpha\rho} V_\beta + \partial_\alpha \Gamma^\rho_{\lambda\sigma} V_\rho\right)_{x_0} A^{\lambda\alpha}$$

⇝ uf. now need to change indices

$$\Delta V_\sigma = R^\rho{}_{\sigma\mu\nu} V_\rho A^{\mu\nu}$$

1st term $\lambda, \alpha, \beta \to \mu, \nu, \rho$

$$\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\beta}_{\alpha\rho} \longmapsto \Gamma^{\cdot}_{\mu\sigma}\Gamma^{\rho}_{\nu\cdot}$$

$\uparrow$ ΔV_σ stays

2nd term : $\lambda, \alpha \to \mu, \nu$

$$\partial_\alpha \Gamma^{\rho}_{\lambda\sigma} \longmapsto \partial_\nu \Gamma^{\rho}_{\mu\sigma}$$

$$\Rightarrow \Delta V_\sigma = \left(\partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\nu\lambda}\right) V_\rho A^{\mu\nu}$$

ANTISYMMETRIC

so you get $(-)(\mu \leftrightarrow \nu)$

maybe factor of 1/2

maybe overall sign...

BUT PHYSICAL CONTENT IS CLEAR, yes?

$$\Delta V_\sigma = \underline{\underline{R^{\rho}{}_{\sigma\mu\nu}}}\, V_\rho A^{\mu\nu}$$

different deriv. of R...

Carroll 3.8

Something different

GEODESIC EQ: VELOCITY VECTOR ALONG GEODESIC (free fall) IS PARALLEL TRANSPORTED

$$V^\mu = \frac{\partial x^\mu}{\partial \tau}$$

$$(V \cdot D)\, V^\mu = 0$$

DIRECTIONAL COVARIANT DERIVATIVE

$$\frac{D}{\partial \tau} = \frac{\partial x^\mu}{\partial \tau} D_\mu = V \cdot D$$

<u>PHYSICS</u>: 4-momentum: $p^\mu = m V^\mu$

(massless: τ no good, use $p^\mu = dx^\mu / d\lambda$)

s.t. GEODESIC: $\boxed{(p \cdot D)\, p^\mu = 0}$

btw: $(p \cdot D)\, p_\mu = 0$

$\Rightarrow$ ~~$(p\cdot\partial)p$~~ since (HW) $D g_{..} = 0$

<u>then</u>: $(p \cdot D)\, p_\mu = (p \cdot \partial)\, p_\mu - \Gamma^{\cdot}_{\cdot\mu}\, p^{\circ} p_{\cdot}$

CONT'D: $(p \cdot D)\, p_\mu = m \frac{\partial x^\bullet}{\partial \tau} \partial_\bullet p_\mu + \cdots$

$= \underbrace{m\, dp_\mu / d\tau}_{\text{change in } p_\mu \text{ along geodesic}} + \cdots$

CONNECTION TERM

$$\Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = \frac{1}{2} g^{\sigma\nu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right) p^\lambda p_\sigma$$

$$= \frac{1}{2} \Big(\underbrace{\qquad\qquad}_{\text{antisym in } \lambda \leftrightarrow \nu} \Big) \underbrace{p^\lambda p^\nu}_{\text{sym in } \lambda \leftrightarrow \nu}$$

$$= \frac{1}{2} \left(\partial_\mu g_{\nu\lambda}\right) p^\lambda p^\nu$$

So: $\underbrace{\partial_3 g_{\nu\lambda} = 0}_{\text{isometry}} \Rightarrow \underbrace{\frac{dp_3}{d\tau} = 0}_{\text{conservation law}}$

(SYM OF METRIC)

ISOM: $M \to M$ s.t. $g_{\mu\nu}$ UNCHANGED

Noether's thm in curved space

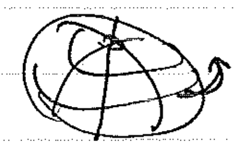
WANT TO MAKE THIS MORE TRANSPARENT
(in ugly coords, not obvious that
$g_{\mu\nu}$ indep of a direction)

also: can have more
isometries than coordinates
eg ROT, BOOST, TRANSL. in Minkowski

LET K be a vector pointing
in the direction of an isometry

eg $K = \partial/\partial x = (0, 1, 0, 0)$

or $K = \partial/\partial\phi$



"K generates isometry"

<u>CONSERVED QUANTITY</u> : $K \cdot p$

$(p \cdot D)p = 0$

$$0 = \frac{d}{d\tau}(K \cdot p) = p^\mu D_\mu (K \cdot p)$$

$$= p^\mu (D_\mu K_\nu) p^\nu + p^\mu K_\nu (D_\mu p^\nu)$$

$$= p^\mu p^\nu D_\mu K_\nu$$

$$= p^\mu p^\nu \underbrace{(D_\mu K_\nu + D_\nu K_\mu)}_{D_{(\mu} K_{\nu)}} \quad \text{(sym. part)}$$

KILLING'S EQ: $D_{(\mu}K_{\nu)} = 0 \Rightarrow p\cdot D\,(K\cdot p) = 0$

if K satisfies this
(sym part of cov. DER)

then $K\cdot p$ is conserved
along worldline.

say: K_μ is a Killing vector Field

film about Khmer Rouge
& ongoing TV series about
murder of a grad student :)

& its existence → conserved quantity.

since $g_{\mu\nu}$ is unchanging in K dir.,
no grav. force in that direction
s.t momentum in that dir. is cons.

Weinb. 6.5 & 11.1

HW: $\boxed{D_\mu D_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu}$

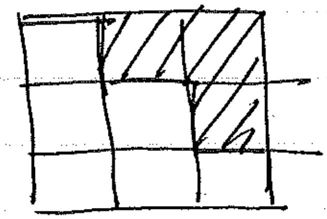
NO SYSTEMATIC WAY TO WRITE ALL KILLING FIELDS.

given K & DK, have 2nd & higher derivatives
RELATED TO LOWER DERIVATIVES

$$K^\mu(x) = f(x)^\mu{}_\nu \underbrace{K^\nu(x_0)} + g(x)^{\mu\nu}{}_\rho \underbrace{D_\nu K^\rho(x_0)}$$

in d-dim. $\quad d$ DOF $\qquad \frac{1}{2}d(d-1)$

since $D_\mu K_\nu + D_\nu K_\mu = 0$



sym piece vanishes
left w/ antisym
part of DK

so: up to $d + \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1)$
KILLING VECTORS

A SPACE W/ FULL SET OF KILLING VECTORS IS <u>MAXIMALLY SYMMETRIC</u>.

eg <u>MINKOWSKI</u> : 4 translations (HOMOGENEOUS)
6 "ROTATIONS" (ISOTROPIC)

= 10 Killing

$\frac{1}{2}4(4+1) = 10$ ✓

~~Schutz~~
Cheng 7.1

SPHERICALLY SYMMETRIC SPACES

FLAT SPACETIME:

$$ds^2 = dt^2 - dr^2 - r^2 \underbrace{(d\theta^2 + \sin^2\theta \, d\varphi^2)}_{d\Omega^2}$$

HOW TO GENERALIZE WHILE MAINTAINING SPHR. SYM ?
no preferred spatial dir.

↳ x^i & dx^i must appear in dot products

$$ds^2 = A \, d\underline{x}^2 + B(\underline{x} \cdot d\underline{x})^2 + C \, dt(\underline{x} \cdot d\underline{x}) + D \, dt^2$$

functions of t & $\underline{x}^2$.

$$= A\left[dr^2 + r^2 d\Omega^2\right] + Br^2 dr^2 + Cr \, dr \, dt + D \, dt^2$$

$\bar{B} = A + Br^2$ s.t. $\bar{B} \, dr^2$

change coords to remove $dt\,dr$ term

$$\bar{t} = t + f(r)$$

$$dt^2 = d\bar{t}^2 - (f'dr)^2 \underbrace{- 2f'\,dr\,dt}$$

$$ds^2 = \ldots\ C_r\,dr\,dt \quad \underbrace{+ D\,dt^2}$$

REABSORB

$$+ D\,d\bar{t}^2 - D(f')^2 dr^2$$

$$- 2Df'\,dr\,dt$$

choose: $\boxed{C_r = 2Df'}$

$$ds^2 = \underline{\underline{Ar^2 d\Omega^2}} + \bar{B}\,dr^2 + D\,d\bar{t}^2$$

$$- D(f')^2 dr^2$$

$$\equiv \tilde{B}\,dr^2$$

$$\bar{r}^2 = Ar^2 \longrightarrow \tilde{\tilde{B}}\,d\bar{r}^2$$

DROPPING ALL ORNAMENTATION (all hiding arbitrary functions, anyway!)

$$ds^2 = g_{00}\, dt^2 + g_{rr}\, dr^2 + r^2\, d\Omega^2$$

TWO FUNCTIONS WORTH
OF DOF.