LEC 6: CURVATURE ... & DIGRESSIONS

26 FEB

- PARALLEL TRANSPORT
- $R^{\rho}{}_{\sigma\mu\nu}$, finally ... but briefly
- GEOMETRIC INTERLUDE 1: intrinsic vs extrinsic
- GEOMETRIC INTERLUDE 2: lie derivative

LAST TIME:

"free fall" $\quad d^2y^\mu/d\tau^2 = 0$

$\Updownarrow$

GEODESIC MOTION

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

[OR: $\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0$]

OR: $\underbrace{D/d\lambda}_{\text{DIRECTIONAL COVARIANT DERIVATIVE}}\ \dot{x}^\mu = 0$

- Geodesic: path in spacetime of maximal proper length, s

- PARALLEL TRANSPORTS ITS VELOCITY VECTOR

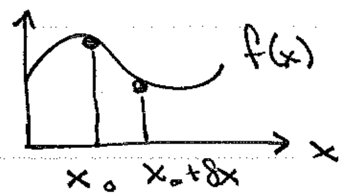
Carroll 3.3

Parallel Transport

Why this is important:

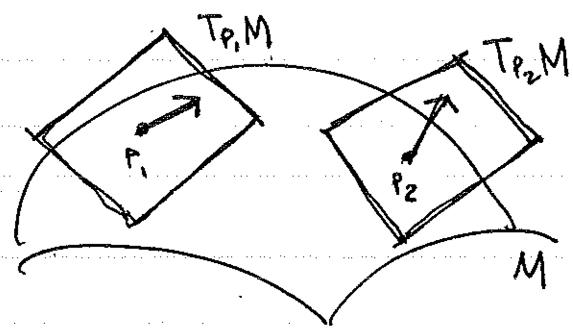
HOW DO WE COMPARE VECTORS (tensors) AT DIFFERENT POSITIONS?

important question – generalizes derivative in calculus



compare 2 numbers $f(x_0)$ & $f(x_0+\delta x)$ as $\delta x \to 0$.

BUT HOW DO YOU COMPARE VECTORS w/ different bases?

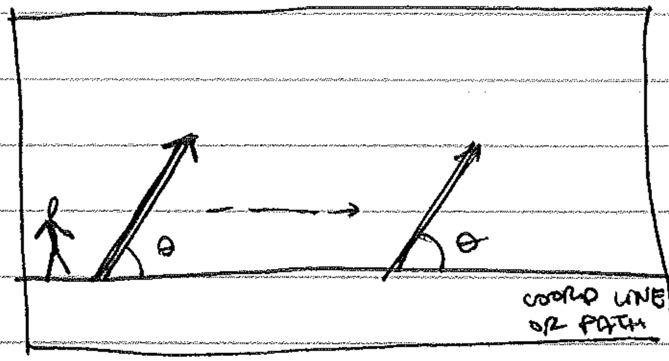


HOW DO THE COORDINATES OF $T_{P_1}M$ COMPARE TO $T_{P_2}M$? HOW DO WE KNOW IF THE VECTOR @ P_1 is THE SAME AS THE ONE @ P_2?

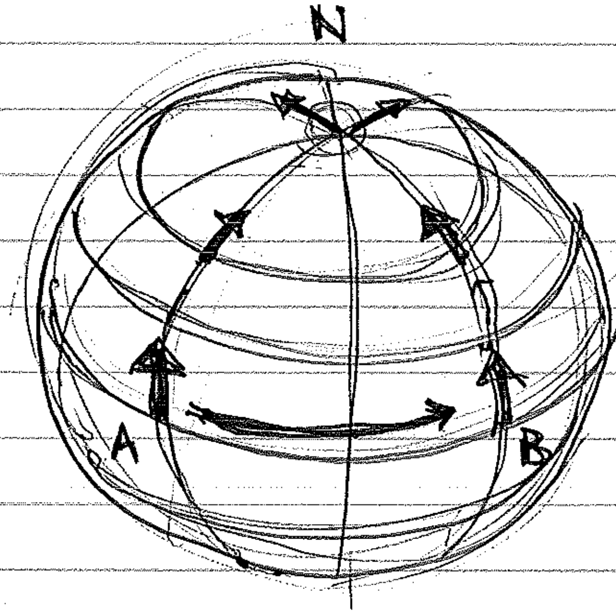
Simple answer: JUST TAKE THE VECTOR, AND WALK IT OVER!

flat space:

"let me just push this over and keep θ constant... angle btwn vector and my path"



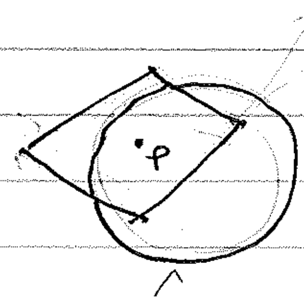
BUT ON S^2: transport from AN vs. ABN gives DIFFERENT VECTORS @ N



So: "PARALLEL TRANSPORT & COMPARE" ← nonlocal
is A WAY TO DIAGNOSE CURVATURE

COMPARE TO EQUIVALENCE PRINCIPLE

$T_pM = \mathbb{R}^{3,1}$



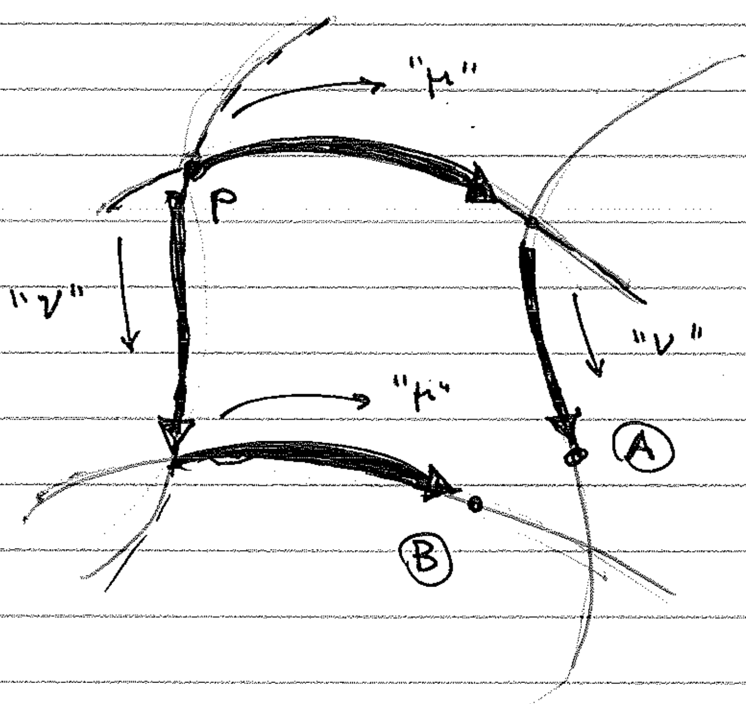
LOCALLY MINKOWSKI

By the way: this strategy of COMPARING VECTORS after // transport is <u>BASIS - INDEPENDENT</u>.

<u>recall</u>: how do I know if space is curved, or if i'm just using curvilinear coordinates?

"funny metric" is <u>not</u> a robust diagnosis for physical curvature

HERE'S THE STRATEGY: flow infinitesimally along different geodesics



compare

SO TAKE A VECTOR V @ x_0^μ ; PUSH IT in a^μ

← FLOW ALONG GEODESIC IN a^μ DIR.

$$V^\rho \xrightarrow{a} \underbrace{V^\rho + a^\mu D_\mu V^\rho}$$

this is V^ρ parallel transported to (x_0+a)

THEN PUSH IT IN ANOTHER DIRECTION $(+b^\mu)$

$$\boxed{\xrightarrow{a,b} V^\rho + a^\mu D_\mu V^\rho + b^\nu D_\nu V^\rho + a^\mu b^\nu \underline{D_\nu D_\mu V^\rho}}$$

IF WE DID THIS IN THE OTHER ORDER, WE'D HAVE GOTTEN

$$\boxed{V^\rho \xrightarrow{b,a} V^\rho + a^\mu D_\mu V^\rho + b^\nu D_\nu V^\rho + a^\mu b^\nu \underline{D_\mu D_\nu V^\rho}}$$

THE DIFFERENCE OF PUSHING a, b vs. b, a is :

$$a^\mu b^\nu [D_\mu, D_\nu] V^\rho$$

← $D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho$

NOW USE $D_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\lambda} V^\lambda$

IN ORDER TO DO THIS, NEED ONE GENERALIZATION:

$$D_\sigma T^\alpha{}_\beta = \partial_\sigma T^\alpha{}_\beta$$
$$+ \Gamma^\alpha_{\sigma\lambda} T^\lambda{}_\beta \quad \text{DEAL W/ UPPER } \alpha \text{ INDEX}$$
$$- \Gamma^\lambda_{\sigma\beta} T^\alpha{}_\lambda \quad \text{DEAL W/ LOWER } \beta \text{ INDEX}$$

minus sign for lower index!

You are "fixing" each transformation independently

$$T^\alpha{}_\beta \longrightarrow \left(\frac{\partial x'}{\partial x}\right)^\alpha{}_\rho \left(\frac{\partial x}{\partial x'}\right)^\sigma{}_\beta T^\rho{}_\sigma$$

BAD TRANSF COMES FROM
DERIVATIVE HITTING THESE TRANSF MATRICES
BUT DERIVATIVE HITS THEM "ONE AT A TIME"
BY LEIBNIZ RULE

$$\partial T' = \left[\partial\left(\frac{\partial x'}{\partial x}\right)\right] \frac{\partial x}{\partial x'} T + \frac{\partial x'}{\partial x}\left[\partial\left(\frac{\partial x}{\partial x'}\right)\right] T + \partial \cdots$$

SO WE CORRECT THEM "ONE @ A TIME"

USING THIS:

$$[D_\mu, D_\nu] V^\rho = \partial_\mu (D_\nu V^\rho) - \Gamma^{\lambda}_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma - [\mu \leftrightarrow \nu]$$

go from outside-in s.t. you're only acting on tensorial indices

$$(\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} (\partial_\mu V^\sigma)$$

$$= \partial_\mu \left(\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma} V^\sigma \right)$$

$$- \Gamma^\lambda_{\mu\nu} \left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma \right)$$

$$+ \Gamma^\rho_{\mu\sigma} \left(\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\lambda} V^\lambda \right)$$

$$- [\mu \leftrightarrow \nu]$$

ASSUMED Γ IS SYM. (IT MAY HAVE TORSION)

UNDERLINED: CANCELS WHEN $-[\mu \leftrightarrow \nu]$ ADDED IN.

WIGGLY: SYMMETRIC IN $\mu \leftrightarrow \nu$ SO ALSO CANCELS.

relabel s.t. we can combine

$$[D_\mu, D_\nu] V^\rho = \underbrace{\left(\partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right)}_{\equiv R^\rho{}_{\sigma\mu\nu}} V^\sigma$$

WE CALL THIS THE <u>RIEMANN TENSOR</u>
it's actually a tensor

$$[D_\mu, D_\nu] V^\rho = \underline{R^\rho{}_{\sigma\mu\nu}} V^\sigma \quad \left(- \; T^\lambda{}_{\mu\nu} D_\lambda V^\rho \right)$$

TORSION: $T^\lambda{}_{[\mu\nu]}$
can be set to 0

- PART OF $[D_\mu, D_\nu] V^\rho$ $\propto V$ itself

- OBSERVE R is antisym in $\mu \leftrightarrow \nu$ manifestly from its definition

on more general tensor:

$$[D_\mu, D_\nu] Q^\alpha{}_\beta = R^\alpha{}_{\lambda\mu\nu} Q^\lambda{}_\beta - R^\lambda{}_{\beta\mu\nu} Q^\alpha{}_\lambda$$

Zee 1.7

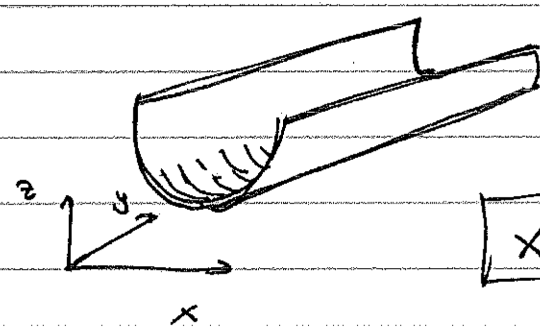
A Geometric Interlude

I. EXTRINSIC VS INTRINSIC CURVATURE

SPACETIME

EMBEDDINGS : when you think of CURVED SPACE AS A SUBSPACE IN A FLAT, LARGER SPACE (surface)

HEURISTIC EXAMPLE



$$\boxed{x^2+z^2=R^2}$$

CURVATURE HAS SOMETHING TO DO WITH SECOND DERIVATIVES; eg near $x=0$

$$z = f(x,y) = \sqrt{R^2-x^2} \approx R - \frac{1}{2}\frac{x^2}{R} \cdots$$

$$z = R + \frac{1}{2}(x\ y)\underbrace{\begin{pmatrix} -1/R & 0 \\ 0 & 0 \end{pmatrix}}_{\text{HESSIAN}}\begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

HESSIAN HAS 2 INVARIANTS

$\det H = 0$ ← intrinsic curvature

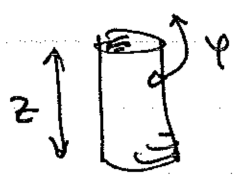
$\mathrm{tr}\, H = -1/R$ ← extrinsic curvature

this is a <u>flat</u> piece of paper just curled up in 3RD DIM

LET'S GO TO CYLINDRICAL COORDINATES

$x^1 = \varphi$

$x^2 = z$



THE CYLINDER IS EMBEDDED IN $\mathbb{R}^3$ (<u>flat</u>) WHICH HAS A NATURAL <u>AMBIENT</u> BASIS

~~OUR CYLINDRICAL COORDINATES HAVE BASIS DIRECTIONS~~

THE CYLINDER IS GIVEN BY POINTS $\in \mathbb{R}^3$ s.t.

$$\underline{X} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix} \qquad \forall\ \varphi, z$$

THE TANGENT VECTORS (BASIS OF) @ A GIVEN POINT ARE

$$\underline{e}_i \equiv \partial_i \underline{X} = \partial \underline{X} / \partial x^i$$

$$\underline{e}_1 = \begin{pmatrix} -R \sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} \qquad \underline{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

DEPENDS ON φ
B/C TANGENT PLANES ROTATE
AS YOU MOVE ALONG CYLINDER!

DISTANCE BTWN NEARBY POINTS:

$$d\underline{X} = \partial_i \underline{X}\, dx^i$$

DOTTING $\mathbb{R}^3$ INDICES

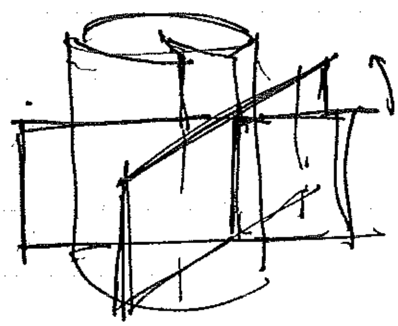
$$ds^2 = d\underline{X} \cdot d\underline{X} = \left(\partial_i \underline{X} \cdot \partial_j \underline{X}\right) dx^i dx^j$$

$$= \underbrace{\underline{e}_i \cdot \underline{e}_j}_{g_{ij}} dx^i dx^j$$

{ recall: $g_{\mu\nu} = (\partial_\mu y^\alpha)(\partial_\nu y^\beta)\eta$

HOW DO $\underline{e}_i$ CHANGE AS WE MOVE?

$$\partial_j \underline{e}_i = \partial_j \partial_i \underline{X}$$



CHANGE IN $\underline{e}_i$ IS NOT LIMITED TO TANGENT DIRECTIONS!

$$\frac{\underline{e}_1 \times \underline{e}_2}{|\underline{e}_1 \times \underline{e}_2|}$$

$$\partial_j \underline{e}_i \equiv \Gamma^{\ell}_{ji}\, \underline{e}_\ell + K_{ji}\, \underline{n}$$

normal direction

FULL BASIS FOR $\mathbb{R}^3$

AFFINE CONNECTION
tells us about how tangent plane basis @ 1 point changes as you go to neighboring point; but projected along T_pM.

something to do w/ CURVATURE from EMBEDDING

A Geometric Interlude II

LIE DERIVATIVE

element of TM vs. T_pM

SUPPOSE YOU HAVE A VECTOR FIELD

that is:

$$W^\mu(x) \leftarrow \text{some vec @ each tangent space}$$

USUAL NOTION OF DERIVATIVE:

COMPARE $W^\mu(x)$ TO $W^\mu(x')$:

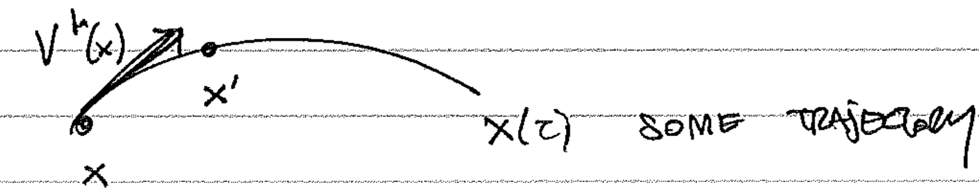
$$\uparrow x' = x + \delta x$$

BUT WE SAW THAT THIS FAILS BECAUSE THE TRANSFORMATION OF "THE INDEX" ISN'T ACCOUNTED FOR. $\rightsquigarrow$ LED TO COVARIANT DERIV.

THERE IS ANOTHER TYPE OF DERIVATIVE THAT IS APPROPRIATE TO DISCUSS NOW:

LIE DERIVATIVE

SUPPOSE $\delta x = V^\mu(x(\tau))\, d\tau$



$$\Rightarrow W^\mu(x') = W^\mu(x) + \underline{d\tau\, V^\nu \partial_\nu W}$$

THEN ~~COMPARE $W^\mu(x)$ TO~~ DEFINE $\underbrace{W'^\mu(x')}$

$d\tau$

$$= W^\nu(x)\frac{\partial x'^\mu}{\partial x^\nu} = W^\mu + W^\nu \partial_\nu V^\mu(x(t))$$

"change of coords"

<u>LIE</u>: COMPARE $W^\mu(x') - W'^\mu(x')$

@ same positions

$$= d\tau\left(V^\nu \partial_\nu W - W^\nu \partial_\nu V^\mu\right)$$

$$= d\tau\left(V^\nu D_\nu W - W^\nu D_\nu V^\mu\right)$$

since "bad" 2ND DER TERMS CANCEL

$$\equiv d\tau\,[V, W] \quad \text{or} \quad d\tau\,\mathcal{L}_V W$$

LOOKS LIKE ONE IN RIEMANN TENSOR DEF!

→ EMPHASIZES THE UTILITY OF THE PICTURE THAT VECTORS ARE DIFFERENTIAL OPERATORS, $V = V^\mu \partial_\mu$.

- $\mathcal{L}_V$ is a bona fide DERIVATIVE

- in fact: $[V,W] = \underbrace{[V,W]^\mu \partial_\mu}$

ie: $\mathcal{L}_V$ acts on vector to spit out another vector.

generalizes for tensors → of COVARIANT DERIV, WHICH ADDS AN INDEX (makes things "more" tensor-y)

- further: [vec, vec] = vec is familiar in QUANTUM MECHANICS ; this is a commutator.

the set of commutators is an ALGEBRA ; gives group theoretical structure of manifold.

- ALSO IMPORTANT WHEN DEFINING COORDINATE GRIDS. GIVEN VECTOR FIELDS V, W, ... , CAN "TRACE" THEM TO GIVE INTEGRAL CURVES. THESE CAN BE COORDINATES ONLY IF V, W, ... ARE INVOLUTIVE: $[X_i, X_j] = c_{ij}{}^k X_k$ (FROBENIUS THM)

CAR PARKING