P208 LEC 5 : 24 Jan 2017

HW2 is up - sorry for delay

THIS LEC: WEINBERG, CHENG, CARROLL

1. NEWTON
2. GRAV TIME DILATION
3. GEODESICS
4. PARALLEL TRANSPORT

LAST WK: why do we fall?
(BATMAN BEGINS)

$\left.\begin{matrix}3.\\4.\end{matrix}\right\}$ how do we fall

next time: curvature

NEWTONIAN LIMIT

what is $g_{\mu\nu}$?

- LOW VELOCITY $d\underline{x}/dt \ll 1$

  write this as

  $$\boxed{\frac{d\underline{x}}{d\tau} \ll \frac{dt}{d\tau}}$$

- STATIONARY FIELD / STATIC

  $$\boxed{\partial_0 g_{\mu\nu} = 0}$$

  time derivative

- WEAK FIELD LIMIT : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$

  components $\ll 1$

FREE FALL OF TEST MASS IN nice frame

gave EQN. OF MOTION IN ANY OTHER FRAME

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\sigma}{\partial \tau} = 0$$

e.g. stationary in static grav field.

AFFINE CONNECTION

$$= \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{\partial x^0}{d\tau}\right)^2$$

$$+ 2\Gamma^\mu_{0i} \left(\frac{\partial x^0}{\partial \tau}\right)\left(\frac{\partial x^i}{\partial \tau}\right)$$

$$+ \Gamma^\mu_{ij} \left(\frac{\partial x^i}{\partial \tau}\right)\left(\frac{\partial x^j}{\partial \tau}\right)$$

SUBDOM BY low-velocity LIMIT.

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu} (\underbrace{\partial_0 g_{0\nu} + \partial_0 g_{\nu 0}}_{} - \partial_\nu g_{00})$$

$\partial_0 g_{..} = 0$

by static field

$$\boxed{\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00}}$$

$\Gamma^0_{00} = 0$; only non-zero for Γ^i_{00}

TO LEADING ORDER IN WEAK FIELD LIMIT

$\underline{\mu=0}$ $$\frac{\partial^2 t}{d\tau^2} = 0 \rightarrow \frac{\partial t}{d\tau} = \text{const.}$$

$\underline{\mu=i}$ $$\frac{\partial^2 x^i}{\partial \tau^2} + \underbrace{\left[-\frac{1}{2}\eta^{ij}\,\partial_j h_{00}(x)\right]}_{+\frac{1}{2}\nabla_i h_{00}(x)} \left(\frac{\partial t}{\partial \tau}\right)^2 = 0$$

$$\frac{\partial}{\partial \tau}\frac{\partial x^i}{\partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial t}{\partial \tau}\frac{\partial x^i}{\partial t}\right)$$

CHANGE VARS & USE $\partial x^i/\partial\tau \ll \partial t/\partial \tau$

$\underline{\text{USE}}$ $\partial/\partial\tau\,(\partial t/\partial\tau) = 0$

time deriv. we want

$$\frac{\partial^2 x^i}{\partial \tau^2} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial \tau}\frac{\partial x^i}{\partial t} = \left(\frac{\partial t}{\partial \tau}\right)^2 \frac{\partial^2 x^i}{\partial t^2}$$

CANCEL $(\partial t/\partial \tau)^2$

$m\ddot{\underline{x}} = -\nabla\Phi \cdot m$

$$\frac{\partial^2 \underline{x}}{\partial t^2} = \frac{-1}{2}\underline{\nabla}\, h_{00} = -\underline{\nabla}\,\Phi$$

newtonian pot.

$\Phi = -GM/r$

$$\Rightarrow h_{00} = 2\Phi + \text{const}$$

shifts in Φ not phys

SO, THE METRIC FOR NEWTONIAN GRAVITY IS

$$\begin{pmatrix} 1+2\Phi & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g_{\mu\nu}$$

[nb: $\mathrm{diag}\big(-(1+2\Phi), 1, 1, 1\big)$ for EAST COAST METRIC]

TIME DILATION

in SR: from different reference frames

in GR: from gravity

SUPPOSE YOU HAVE A CLOCK.
THE MANUFACTURER SPECIFIES THAT (IN THE ABSENCE OF GRAVITY), THE TIME BTWN TICKS IS ΔT.

$$\Delta T^2 = \eta_{\alpha\beta}\, dy^\alpha dy^\beta = g_{\mu\nu}\, dx^\mu dy^\nu$$

FREE FALL / LIF SPACETIME INTERVAL IS dy^α

BY CHANGE OF VARS TO NON-INT. FRAME

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

dx^μ: interval in gen frame

then: $$\left(\frac{\Delta T}{dt}\right)^2 = g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}$$

velocity of clock

so: time btwn ticks in x^μ frame is

$$\frac{dt}{\Delta T} \cdot \Delta T = \frac{\Delta T}{\sqrt{g_{\mu\nu} \, {dx^\mu}/{dt} \, {dx^\nu}/{dt}}}$$

NOW SPECIALIZE TO THE CASE OF A CLOCK @ REST IN x FRAME : $dx^i/dt = 0$

$$dt = \frac{\Delta T}{\sqrt{g_{00}}} = \frac{\Delta T}{\sqrt{1 + 2\Phi}}$$

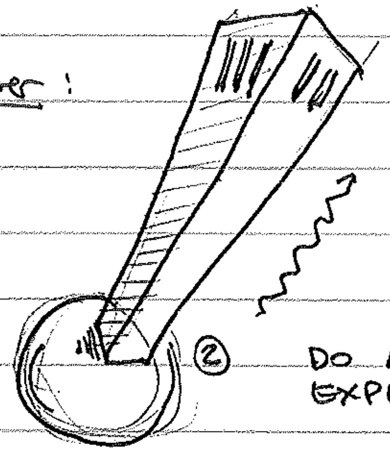
newtonian limit

[of course: could have simply derived this from

$$(\Delta T)^2 = g_{\mu\nu} \, dx^\mu dx^\nu$$
]

EXPERIMENT: (Measure clock @ ① & @ 2 ...
WHAT DO YOU FIND? — no discrepancy!
YOUR measuring device is also affected!

<u>Better</u>:

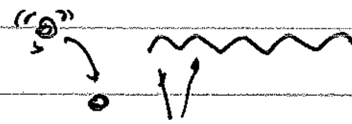


① COMPARE TO TRANSITIONS HERE

photons propagate

② DO ATOMIC TRANSITION EXPERIMENT HERE → fixed freq. of light emitted

(or laser pointer)



crests are ticks

TIME TO PROPAGATE FROM ② → ①

WE CALCULATED THIS IN LEC 3:

USED: $ds^2 = 0 = g_{\mu\nu} dx^\mu dx^\nu$ } SOLVE FOR dt

GOT: $dt = \frac{1}{g_{00}} \left[- g_{i0} dx^i + \sqrt{(g_{i0} g_{j0} - g_{ij} g_{00}) dx^i dx^j} \right]$

integrate this over trajectory (STRAIGHT LINE) to get time to traverse

<u>BUT</u>: $\int dt$ IS JUST SOME CONSTANT FOR EACH CREST

→ <u>time separation</u> between crests unchanged

@ FLOOR: $dt_2 = \Delta\tau / \sqrt{g_{00}(x_2)}$

@ top: $dt_1 = \Delta\tau / \sqrt{g_{00}(x_1)}$

COMPARE OBS FREQ:

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \approx \cancel{\frac{2(\Phi_2-\Phi_1)}{1+\Phi_1+\Phi_2}}$$

$$g_{00} \approx 1 + 2\Phi$$

$$\frac{\Delta\nu}{\nu} = \frac{\nu_2-\nu_1}{\frac{1}{2}(\nu_2+\nu_1)} \approx \sqrt{\frac{2(\Phi_2-\Phi_1)}{1+\Phi_1+\Phi_2}} \approx \boxed{\Phi_2 - \Phi_1}$$

Grav. REDSHIFT

Goal: recovering familiar EQS using variational principle

from CHENG 5.2, 6.1

Geodesics

usually MIN

"PATH OF LIGHT" why?

CURVES OF EXTREMAL LENGTH

require some way to measure distances...

SPACETIME ARCLENGTH:

$$S = \int ds = \int \sqrt{\left(\frac{ds}{d\lambda}\right)^2}\, d\lambda$$

for some TRAJECTORY PARAMETER, λ

could use, say, PROPER TIME $\lambda = \tau$

(clock of someone traversing that path)

$$= \int \underbrace{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}}_{\text{"LAGRANGIAN", } L(x,\dot{x})}\, d\lambda$$

$\dot{x} = dx/dx$

segue to Einstein EQ. LATER

HOW TO FIND EXTREMAL ARCLENGTH?
VARIATIONAL PRINCIPLE

$$\delta s = \delta \int L(x, \dot{x})\, d\lambda = 0$$

EULER-LAGRANGE EQN (recall Mechanics!)

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

RECALL: $\delta L = \frac{\partial L}{\partial x} \delteta x + \underbrace{\frac{\partial L}{\partial \dot{x}} \delta \dot{x}}_{\text{integ by parts}}$

$$= -\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}} \delta x$$

FOR OUR "LAGRANGIAN": $L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$ is cumbersome

TRICK: CONSIDER $\tilde{L} = L^2$

then EULER-LAGR:

$$\frac{d}{d\lambda} \frac{\partial \tilde{L}}{\partial \dot{x}^\mu} - \frac{\partial \tilde{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left(2L \frac{\partial L}{\partial \dot{x}^\mu}\right) - 2L \frac{\partial L}{\partial x^\mu}$$

NOW SUPPOSE λ CHOSEN s.t.

$$\frac{ds}{d\lambda} = L \quad \text{is } \underline{\text{CONSTANT}}$$

i.e. $\boxed{dL/d\lambda = 0}$, then

$$2L \underbrace{\left[\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial \dot{x}^\mu} \right]}_{\text{ORIG. E-L}} = 0$$

SO: GEODESICS OF $\tilde{L}$ = GEODESICS OF L
(extrema of squared arclength also extremize arclength)

arclen.

clock of a comover

$\underline{\text{nb}}$ $\quad ds/d\tau = \text{constant} \equiv 1$

We are moving @ speed of light in space<u>time</u>.

τ is an example of an "<u>AFFINE PARAMETER</u>"
for now: "nice" way to parameterize a trajectory.

check:

$$\frac{\partial}{\partial \dot{x}^\mu}\left(g_{\sigma\nu}\dot{x}^\sigma \dot{x}^\nu\right)$$

$$= g_{\sigma\nu}\left(\delta^\sigma_\mu \dot{x}^\nu + \delta^\nu_\mu \dot{x}^\sigma\right)$$

$$\frac{\partial \tilde{L}}{\partial \dot{x}^\mu} = 2 g_{\mu\nu}\dot{x}^\nu$$

$$\frac{\partial \tilde{L}}{\partial x^\mu} = \frac{\partial g_{\sigma\nu}}{\partial x^\mu}\dot{x}^\sigma \dot{x}^\nu$$

S.t. EULER-LAGRANGE:

$$\underbrace{\frac{d}{d\lambda}\left(2 g_{\mu\nu}\dot{x}^\nu\right)}_{} - \frac{\partial g_{\sigma\nu}}{\partial x^\mu}\dot{x}^\sigma \dot{x}^\nu = 0$$

$g_{\mu\nu}(x(\lambda))$

$$2\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\dot{x}^\sigma \dot{x}^\nu + 2 g_{\mu\nu}\ddot{x}^\nu$$

$$= 2 g_{\mu\nu}\ddot{x}^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\dot{x}^\sigma \dot{x}^\nu - \frac{\partial g_{\sigma\nu}}{\partial x^\mu}\dot{x}^\sigma \dot{x}^\nu$$

$$+ \frac{\partial g_{\mu\sigma}}{\partial x^\nu}\underbrace{\dot{x}^\sigma \dot{x}^\nu}_{}$$

pull out sym. part of $(\partial g_{\mu\nu}/\partial x^\sigma)$

symmetric in $\sigma \leftrightarrow \nu$

<u>MASSAGING TERMS</u>: MULT BY $\frac{1}{2} g^{\lambda\mu}$

$$\frac{d\ddot{x}^{\lambda}}{d\tau^2} + \underbrace{\frac{1}{2} g^{\lambda\mu} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} \right]}_{\equiv \Gamma^{\lambda}_{\sigma\nu} \;!!} \dot{x}^{\sigma} \dot{x}^{\nu} = 0$$

WE HAVE REDISCOVERED OUR "FREE FALL" EQUATION

FREE FALLING INERTIAL FRAMES FALL ALONG ~~OBEY~~ GEODESICS.

OUR "FREE FALL EQ" IS CALLED THE GEODESIC EQN.

<u>NOW</u>: given $g_{\mu\nu}(x)$
can solve to get trajectories
of test particles in
grav. fields.

CARROLL 3.3

PARALLEL TRANSPORT

geodesics give us special trajectories. NOW LET'S MOVE STUFF ALONG THOSE TRAJECTORIES.

vectors.

RECALL : LIF $y \longrightarrow$ any frame $x(y)$,
w/ $g_{\mu\nu}(x) = \partial y/\partial x \, \partial y/\partial x \, \eta$

$\Gamma^{\cdot}_{\cdot\cdot} \longrightarrow$ geodesics

D_μ COVARIANT DERIVATIVES
"nice" derivatives $\longrightarrow$ (HW)

act nicely on objects w/ indices

GIVEN A PATH $x^\mu(\lambda)$, CAN DEFINE DIRECTIONAL COVARIANT DERIVATIVE

$$\boxed{\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu}$$

THEN <u>PARALLEL TRANSPORT</u> OF A VECTOR ALONG A TRAJECTORY IS DEFINED BY

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

IN PARTICULAR: CONSIDER THE <u>TANGENT VECTOR</u> TO A PATH $x^\mu(\lambda)$: $\boxed{dx^\mu(\lambda)/d\lambda}$

PARALLEL TRANSPORT OF TANGENT VECTOR:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 = \underbrace{\frac{d^2x^\mu}{dx^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}}_{\text{GEODESIC EQN}}$$

<u>SO</u>: GEODESICS <u>PARALLEL TRANSPORT</u> THEIR <u>VELOCITY</u> VECTORS

nb. PARAMETERIZATION MATTERS

$\lambda' = a\lambda + b$ works

but $\lambda'' = c\lambda^3$ does not

from geodesic eqn

<u>next</u>: <u>CURVATURE</u>