since when did calculus
PROB LEC 4: have so many indices? 19 JAN '17

- HW LATE – MY BAD
- next wk: more pheno; this wk – calculus (for this wk)
- BOOKS & REFS → $\boxed{\text{Weinberg / Zee / D'inverno / ...}}$

TODAY
- TRANSFORMATION OF Γ ~~HANDOUT~~
- COVARIANT DERIV
- $\Gamma(g)$ (metric connection)
- NEWTONIAN LIMIT & TIME DILATION IN GRAV FIELD

KEY POINTS FROM LAST TIME

- EQUIVALENCE PRINCIPLE:
  GRAVITY = ACCELERATED FRAME

$\Rightarrow$ in free falling frame, "no gravity"
LOCAL, INERTIAL FRAME
WE DENOTED COORDS AS y^α
METRIC IN FREE FALL IS SPECIAL REL: $g_{\mu\nu} = \eta_{\mu\nu}$

WE STARTED TO FLESH OUT THE PHYSICS IN
ANY OTHER FRAME, x^μ

COULD BE SAME FRAME, CURVY COORDS
OR NON-INERTIAL OBSERVER,
eg RIDE OPERATOR OF TOWER OF TERROR

CHANGE IN COORDS

$$y^\alpha \longmapsto x^\mu$$

$$\eta_{\alpha\beta} \longmapsto g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}$$

$$\frac{d^2 y^\alpha}{d\tau^2} = 0 \longmapsto \frac{d^2 x^\rho}{d\tau^2} + \boxed{\Gamma^\rho_{\mu\nu}} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

free fall / LIF

$$\boxed{\frac{\partial x^\rho}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}$$

CHRISTOFFEL SYMBOLS

AFFINE CONNECTION

also:

$$\boxed{\frac{\partial y^\alpha}{\partial x'^\rho} \Gamma^\rho_{\mu\nu} = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}$$

NOTE: $\Gamma^\rho_{\mu\nu}$ are not TENSORS

why? the 2nd derivative – not linear

TENSOR: $T^\rho_{\mu\nu} \rightarrow \underbrace{\left(\frac{\partial x'^\rho}{\partial x^\sigma}\right)}_{\text{"}R\text{"}} \underbrace{\left(\frac{\partial x^\delta}{\partial x'^\mu}\right)}_{\text{"}R^{-1}\text{"}} \underbrace{\left(\frac{\partial x^\gamma}{\partial x'^\nu}\right)}_{\text{"}R^{-1}\text{"}} T^\sigma_{\delta\gamma}$

$$(\Gamma')^{\rho}_{\mu\nu} \equiv \frac{\partial x'^{\rho}}{\partial y^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \frac{\partial y^{\alpha}}{\partial x'^{\nu}}$$

$$\underbrace{\frac{\partial x'^{\rho}}{\partial x^{\sigma}}}_{\text{TRANSFORM "R"}} \underbrace{\frac{\partial x^{\sigma}}{\partial y^{\alpha}}}_{\text{PIECE OF } \Gamma} \qquad \underbrace{\frac{\partial x^{\delta}}{\partial x'^{\mu}}}_{\text{"}R^{-1}\text{"}} \frac{\partial}{\partial x^{\delta}} \left(\overbrace{\frac{\partial x^{\gamma}}{\partial x'^{\nu}}}^{\text{"}R^{-1}\text{"}} \frac{\partial y^{\alpha}}{\partial x^{\gamma}} \right)$$

CHAIN RULE

$$\frac{\partial x^{\delta}}{\partial x'^{\mu}} \left[\underbrace{\frac{\partial^2 x^{\gamma}}{\partial x^{\delta} \partial x'^{\nu}} \frac{\partial y^{\alpha}}{\partial x^{\gamma}}}_{\text{nothing}} + \underbrace{\frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial^2 y^{\alpha}}{\partial x^{\gamma} \partial x^{\delta}}}_{\text{OTHER PIECE OF } \Gamma} \right]$$

$$(\Gamma')^{\rho}_{\mu\nu} = \underbrace{\frac{\partial x'^{\rho}}{\partial x^{\sigma}}}_{R} \underbrace{\frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial x^{\delta} \partial x^{\gamma}}}_{\Gamma^{\sigma}_{\delta\gamma}} \underbrace{\frac{\partial x^{\delta}}{\partial x'^{\mu}}}_{R^{-1}} \underbrace{\frac{\partial x^{\gamma}}{\partial x'^{\nu}}}_{R^{-1}} \Bigg\} \text{ tensorial transf}$$

$$+ \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\mu}} \frac{\partial^2 x^{\gamma}}{\partial x^{\delta} \partial x'^{\nu}} \frac{\partial y^{\alpha}}{\partial x^{\gamma}}$$

junk transf.

$$\frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial y^{\alpha}}{\partial x^{\gamma}} = \delta^{\sigma}_{\gamma}$$

$$\to \boxed{\frac{\partial x'^{\rho}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial x'^{\mu} \partial x'^{\nu}}}$$

JUNK TERM: $\boxed{\dfrac{\partial x'^{\rho}}{\partial x^{\gamma}} \dfrac{\partial^2 x^{\gamma}}{\partial x'^{\mu}\partial x'^{\nu}}}$

USE: $\dfrac{\partial x'^{\rho}}{\partial x^{\gamma}} \dfrac{\partial x^{\gamma}}{\partial x'^{\nu}} = \delta^{\rho}_{\nu}$

DIFFERENTIATE: $\partial / \partial x'^{\mu}$ BOTH SIDES

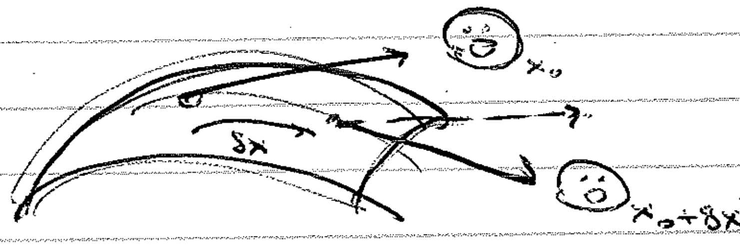
$$\frac{\partial x^{\kappa}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\kappa}} \longrightarrow \underline{\frac{\partial^2 x'^{\rho}}{\partial x'^{\mu}\partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}}} + \boxed{\frac{\partial x'^{\rho}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial x'^{\mu}\partial x'^{\nu}}} = 0$$

$$\boxed{\begin{aligned} \text{s.t. } (\Gamma')^{\rho}_{\mu\nu} &= \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial x^{\delta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \Gamma^{\sigma}_{\delta\gamma} \\ &\quad - \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\rho}}{\partial x^{\kappa}\partial x^{\gamma}} \end{aligned}}$$

ONE USEFUL INTERP/OBS: THE JUNK APPEARS TO BE RELATED TO THE FACT THAT WE'RE DIFFERENTIATING SOMETHING W/ AN INDEX:

$$\frac{\partial}{\partial x^\mu} ☺^\nu(x)$$

BOTH OF THESE TRANSFORM



THIS IS A HINT THAT Γ WILL HAVE SOMETHING TO DO W/ DIFFERENTIATING TENSORS.

TODAY: PURE CALCULUS WAY.

Wemb. 4-6 IN FACT, WE MIGHT AS WELL SEE THIS. CONSIDER A VECTOR V^μ.

transf. law: AS $x \to x'$, the vector V^μ transforms to V'^μ

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

BOTH FUNCTIONS OF x'

$$\frac{\partial V'^\mu}{\partial x'^\lambda} = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial V^\nu}{\partial x^\rho}}_{\text{"GOOD" TRANSFORMATION}} + \boxed{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}} V^\nu \qquad (*)$$

is this an Americanism

looks familiar!

OBSERVE:

$$\Gamma'^\mu_{\lambda\kappa} V'^\kappa = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\kappa}\Gamma^\nu_{\rho\sigma}\frac{\partial x'^\kappa}{\partial x^\eta} V^\eta$$

δ^σ_η

$$- \frac{\partial x^\nu}{\partial x'^\lambda}\frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\eta} V^\eta$$

δ^ρ_η

$$\boxed{- \frac{\partial x^\nu}{\partial x'^\lambda}\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho} V^\rho}$$

CANCELS (*)! (note $\nu \leftrightarrow \rho$ sym.)

SO AS A RESULT (dropping primes)

$$D_\mu V^\nu \equiv \left(\frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma} V^\sigma \right)$$

is COVARIANT (is a tensor)

$$D_\mu V^\nu \longrightarrow \boxed{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} D_\rho V^\sigma \equiv D'_\mu V'^\nu}$$

transforms the way its indices "want to"

$$D_\mu = \left(\frac{\partial}{\partial x^\mu} + \Gamma^{\circ}_{\mu \odot} \right)$$

will generalize to more complicated tensors

Q: D_μ on ϕ ?

no indices

NOW LET'S STEP BACK A BIT & GO BACK TO OUR "TOWER OF TERROR" EXAMPLE

nice coords : y^α ← LOCAL, INERTIAL FRAME

gen coords : x^μ ← e.g. RIDE OPERATOR

WE ARGUED THAT @ A GIVEN POINT IN SPACETIME (= an event), CAN COOK UP y^α COORDS AS A FUNCTION OF x^μ, given $\underbrace{\Gamma^{\cdot}_{\cdot\cdot} \text{ \& } g_{\cdot\cdot}}$ ← proper time

"curvature"

evidently gravity lives in these objects

THESE OBJECTS CAN BE RELATED

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

DIFFERENTIATE $g_{\mu\nu}$ (know this brings out Γ's!)

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} = \Bigg(\underbrace{\frac{\partial^2 y^\alpha}{\partial x^\rho \partial x^\mu}}_{\frac{\partial y^\alpha}{\partial x^\sigma}\Gamma^\sigma_{\rho\mu}}\frac{\partial y^\beta}{\partial x^\nu} + \frac{\partial y^\alpha}{\partial x^\mu}\underbrace{\frac{\partial^2 y^\beta}{\partial x^\rho \partial x^\nu}}_{\frac{\partial y^\beta}{\partial x^\sigma}\Gamma^\sigma_{\rho\nu}}\Bigg)\eta_{\alpha\beta}$$

* : warning - I'm not obviously allowed to do this!

SO WE HAVE FOUND:

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} = \underbrace{\Gamma^\sigma_{\mu\rho}}_{\text{sym}} \underbrace{g_{\sigma\nu}}_{\text{sym}} + \Gamma^\sigma_{\rho\nu} g_{\sigma\mu}$$

great! now we want to massage into
$\Gamma = f(g)$

BUT THIS IS HARD - CANNOT SIMPLY GROUP the Γ's — they have different indices!

HAVE TO BE MORE CLEVER:

WRITING $\partial_\mu \leftrightarrow \partial/\partial x^\mu$

same tensor struct!

$$\begin{aligned} \partial_\lambda g_{\mu\nu} &= g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} + \boxed{g_{\sigma\mu}\Gamma^\sigma_{\lambda\nu}} \\ + \partial_\mu g_{\lambda\nu} &= g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} + g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} \\ - \partial_\nu g_{\mu\lambda} &= -g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} - \boxed{g_{\sigma\mu}\Gamma^\sigma_{\lambda\nu}} \\ &= \boxed{2 g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda}} \end{aligned}$$

mult w/ $\frac{1}{2} g^{\nu\sigma}$ on both sides:

$$\boxed{\Gamma^\sigma_{\mu\lambda} = \frac{1}{2} g^{\sigma\nu}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right)}$$

metric connection

Wein. 3-4

Newtonian Limit

EOM: $\frac{d^2x^\mu}{\partial\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{\partial x^\rho}{\partial\tau}\frac{\partial x^\sigma}{\partial\tau} = 0$

NEWTON / NON-REL LIMIT → low "velocity" $\partial x/\partial t \ll 1$

→ WEAK, STATIONARY FIELD

so: $\boxed{\frac{\partial x}{\partial\tau} \ll \frac{\partial t}{\partial\tau}}$

$$\text{EOM} \rightarrow 0 = \boxed{\frac{\partial^2 x^\mu}{\partial\tau^2} + \Gamma^\mu_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2}$$

t; $i = 1,2,3$

$$\left.\begin{aligned} &+2\Gamma^\mu_{0i}\frac{\partial x^0}{\partial\tau}\frac{\partial x^i}{\partial\tau} \\ &+\Gamma^\mu_{ij}\frac{\partial x^i}{\partial\tau}\frac{\partial x^j}{\partial\tau}\end{aligned}\right\} \text{SUBDOM.}$$

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(\underline{\partial_0 g_{0\nu} + \partial_0 g_{0\nu}} - \partial_\nu g_{00}\right)$$

NEWTONIAN GRAVITY LIVES IN HERE.

STATIC, so $\partial_0 = \partial/\partial x^0 = \partial/\partial t$

PARTS SHOULD VANISH

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu \boxed{g_{00}}$$

WEAK FIELD LIMIT

SPACETIME IS ALMOST MINKOWSKI

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{CONST.}} + h_{\mu\nu}$$

may have x-dep.

LEADING ORDER:

$$\Gamma^{\mu}_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00}$$

PLUG BACK INTO EOM:

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = \frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} \left(\frac{\partial t}{\partial \tau}\right)^2$$

mostly minus metric

$$\boxed{\frac{\partial^2 x^i}{\partial \tau^2} = -\frac{1}{2} \underline{\nabla}_i h_{00} \left(\frac{\partial t}{\partial \tau}\right)^2}$$

ORDINARY GRADIENT

$$\frac{\partial^2 t}{\partial \tau^2} = 0 \qquad \text{b/c } \frac{1}{2}\eta^{00} \underbrace{\partial_0 h_{00}}_{=0}$$

$$\rightarrow \boxed{\partial t / \partial \tau = \text{CONST}}$$

$$\frac{\partial}{\partial \tau}\frac{\partial \underline{x}}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial \tau}\left(\frac{\partial t}{\partial \tau}\frac{\partial \underline{x}}{\partial t}\right)$$

but: $\partial/\partial\tau\,(\partial t/\partial\tau) = 0$
by EOM

$$= \cancel{\left(\frac{\partial t}{\partial \tau}\right)^2}\frac{\partial^2 \underline{x}}{\partial t^2} = -\frac{1}{2}\underline{\nabla} h_{00}\cancel{\left(\frac{\partial t}{\partial \tau}\right)^2}$$

SPATIAL
EOM

$$\Rightarrow \frac{\partial^2 \underline{x}}{\partial t^2} = -\frac{1}{2}\underline{\nabla} h_{00} \stackrel{?}{=} -\nabla \Phi$$

acceleration

const shift in Φ
unphysical

$$\Rightarrow \boxed{h_{00} = 2\Phi + \text{const}}$$

$$\boxed{\begin{aligned} g_{00} &= (1 + 2\Phi) \\ g_{ij} &= -\delta_{ij} \end{aligned}}$$

exercise:
WHAT HAPPENS
TO SIGNS
IN EAST
COAST METRIC?

//

vs. from SR comes!

TIME DILATION IN A GRAV FIELD   not nec in free fall
eg on a table

CLOCK IN SAME FRAME
IN A CURVED SPACETIME
in a grav. field

in NICE FRAME (free fall / loc. INERT.)

$$\Delta T^2 = \eta_{\alpha\beta}\, dy^\alpha dy^\beta = g_{\mu\nu}\, dx^\mu dx^\nu$$

TIME BTWN TICKS
IN ABS. OF GRAV.
AS OBS BY NICE FRAME (manufacturer spec.)

$$\left(\eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}\right)$$

$$\left(\frac{\Delta T}{dt}\right)^2 = g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}$$

velocity of clock

then time btwn ticks in x frame
is

$$\frac{dt}{\Delta T} \cdot (\Delta T) = \frac{1}{\sqrt{g_{\mu\nu}\, dx^\mu/dt\, dx^\nu/dt}}$$

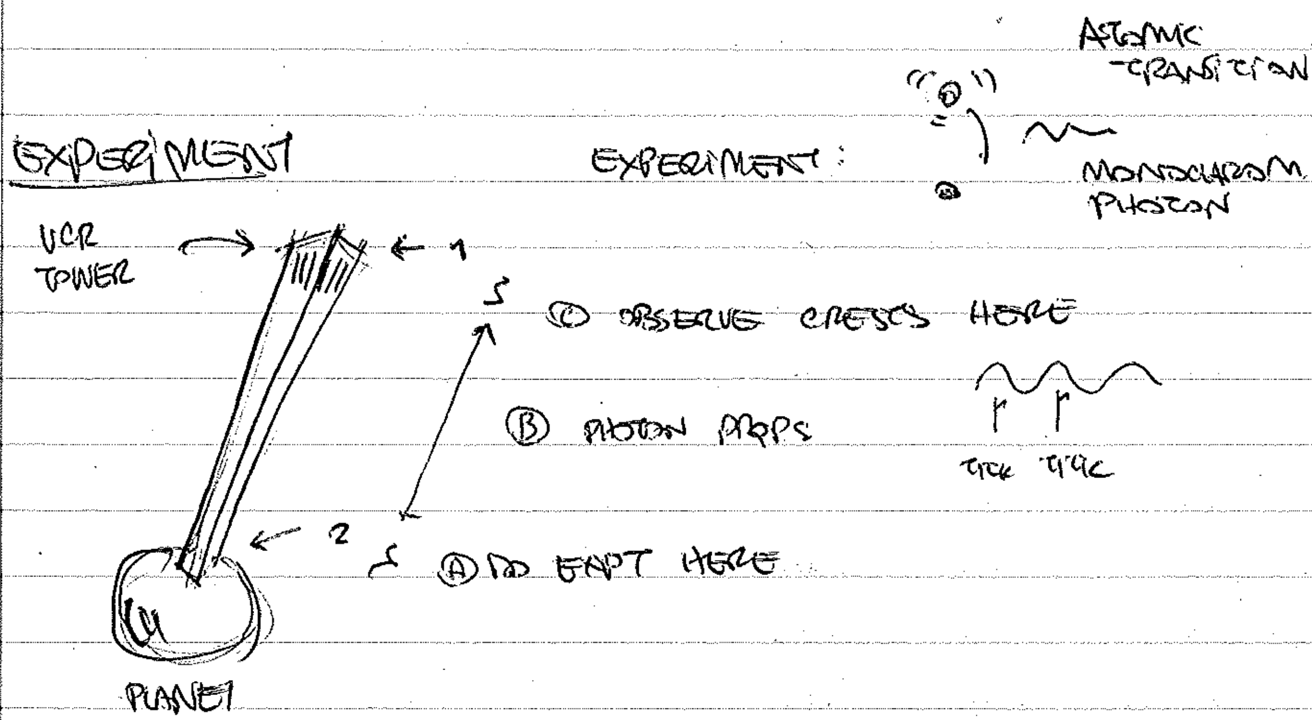
$$\xrightarrow[\text{IN } x \text{ FRAME}]{\text{CLOCK @ REST}} \frac{1}{\sqrt{g_{00}}} \approx \frac{1}{\sqrt{1+2\Phi}}$$

newtonian lim

SO DO EXPT. MEASURE CLOCKS NEAR BH & FAR AWAY. WHAT DO YOU FIND?

↳ no measured discrepancy from manufacturer's description, ΔT!

how do you measure a clock?
w/ another clock ... which is also dilated.



time for photon to go from ground to top is something we "calculated"

<u>SOLVE</u> : $0 = g_{\mu\nu}\, dx^\mu dx^\nu$

<u>got</u>: $dt = \frac{1}{g_{00}}\left[-g_{i0}\, dx^i + \sqrt{(g_{i0}g_{j0} - g_{ij}g_{00})\, dx^i dx^j} \right]$

integrate this
to get time for
a photon to traverse
height of tower.

(ASSUME
SPH. SYM GRAV FIELD, eg. NEWTONIAN,
SO TRAJECTORY IS STRAIGHT RADIAL)

<u>BUT</u> : $\int_{\text{FLOOR}}^{\text{TOP}} dt$ IS JUST SOME CONSTANT
FOR EACH PHOTON / <u>CREST</u>.

SO TIME <u>SEPARATION</u> BTWN
CRESTS IS UNAFFECTED BY
PROPAGATION FROM $2 \longrightarrow 1$.

@ the floor: $dt_2 = \Delta T / \sqrt{g_{00}(x_2)}$

@ the top: $dt_1 = \Delta T / \sqrt{g_{00}(x_1)}$

then the ratio of obs. frequencies are:

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}$$

emitted @ 2

emitted @ 1

BOTH OBS @ 1

$g_{00} = 1 + \phi$

Newtonian limit: weak field, $\phi \ll 1$

$$\frac{\nu_2 - \nu_1}{\frac{1}{2}(\nu_2 + \nu_1)} = \sqrt{\frac{2(\phi_2 - \phi_1)}{1 + \phi_1 + \phi_2}}$$

$\phi_i = \phi(x_i)$

$$\approx \boxed{\phi_2 - \phi_1}$$

shift in weak field limit.