P208 ← ?4

LEC 3: INTELLIGENT FALLING 17 Jan '17

BIG PIC: 1. GEOMETRY OF SPACETIME DEFINES HOW STUFF FALLS

2. STUFF SOURCES CURVATURE (distorts geometry)

WE FOCUS (next few weeks) on 1, analogous to "test charge" analysis of electrostatics

THEN 2, EINSTEIN EQ: analog of FULLY COVARIANT ELECTRODYNAMICS
↳ recall: no issue of causality

THEN 1 + 2, GRAVITATIONAL WAVES

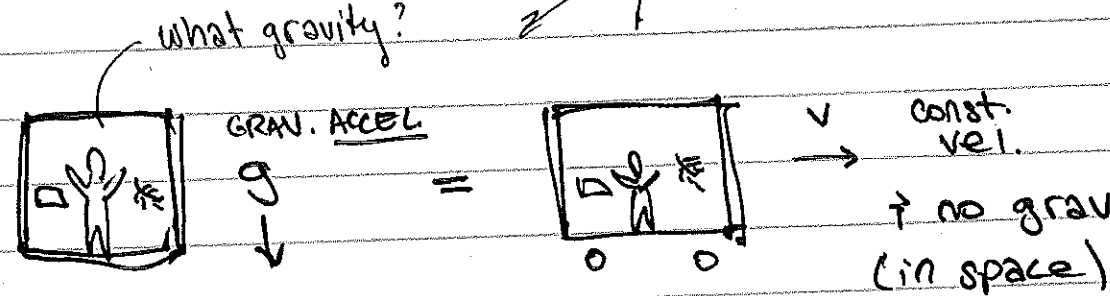
this wk: EQUIVALENCE PRINCIPLE
LOCALLY INERTIAL FRAMES
UGLY FRAMES
↳ CHRISTOFFELS

THE EQUIVALENCE PRINCIPLE

EXPT: grav. accelerates all stuff proportionally to its inertial mass, no matter what the stuff is
↳ ie: indep of additional quantum #'s

HYPOTHESIS: if everything falls the same in a grav. field, maybe gravity is really a modification to "falling"

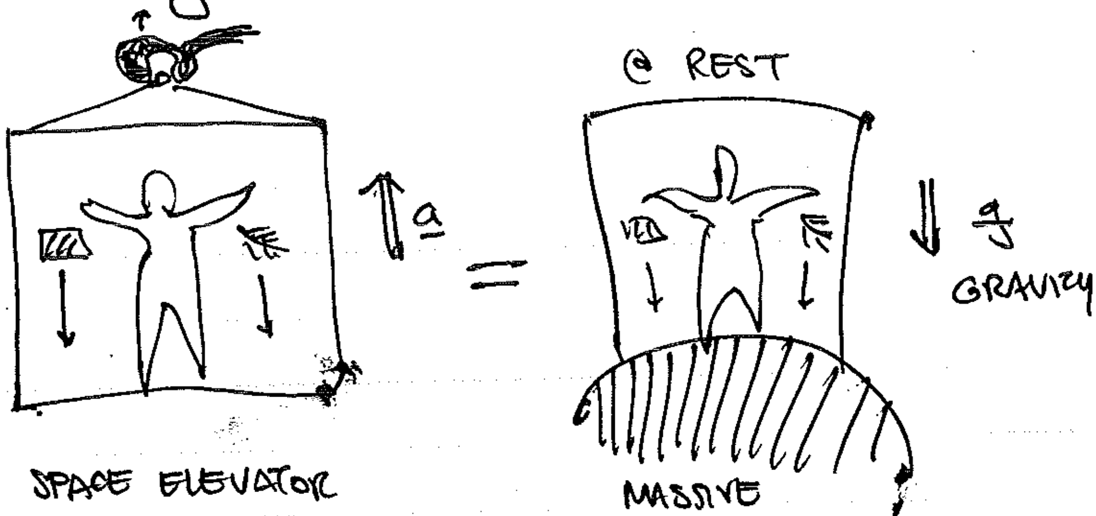
EQUIVALENCE



LOCAL EXPERIMENTS CANNOT DISTINGUISH BETWEEN FREE FALL "IN GRAV. FIELD" (along spacetime geometry)

vs. in a nice, SR inertial frame

if they had a window, they'd see how bad things are alternatively:



SPACE ELEVATOR
no grav, but
mechanical acc.

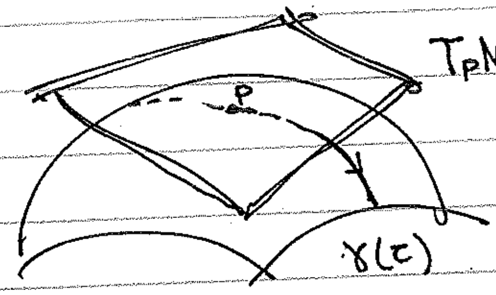
MASSIVE
PLANET
looks PROSAIC, BUT WAY
MORE IMPRESSIVE WHEN
IT'S LIGHT BENDING!!
← new prediction

SO: maybe gravity is just falling.
~~through~~

AS PHYSICISTS, WE USE COORDINATE SYSTEMS AS A CRUTCH ⟶ WANT TO FIGURE OUT HOW COORDINATE SYSTEMS "FALL" THROUGH SPACETIME.

GEOMETRIC PICTURE

I HAVE ONE PICTURE THAT I WILL DRAW OVER & OVER AGAIN.



T_pM = tangent space @ p

M = SPACETIME

curved ... whatever that means (will figure out soon)

A "TEST PARTICLE" (test observer) WILL FREE FALL ALONG A PATH $\gamma(\tau)$ THROUGH SPACETIME.

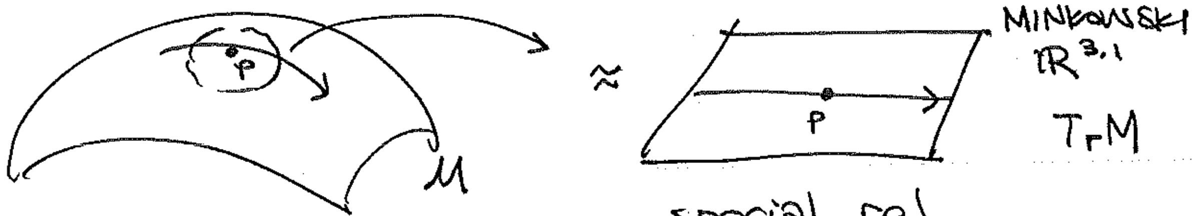
trajectory

one goal of ours is to develop the machinery where we can take a "curvy geometry" and figure out what the free fall trajectory is.

LOCALLY (in space & time — ie, near an event) THE TEST PARTICLE SEES T_pM.

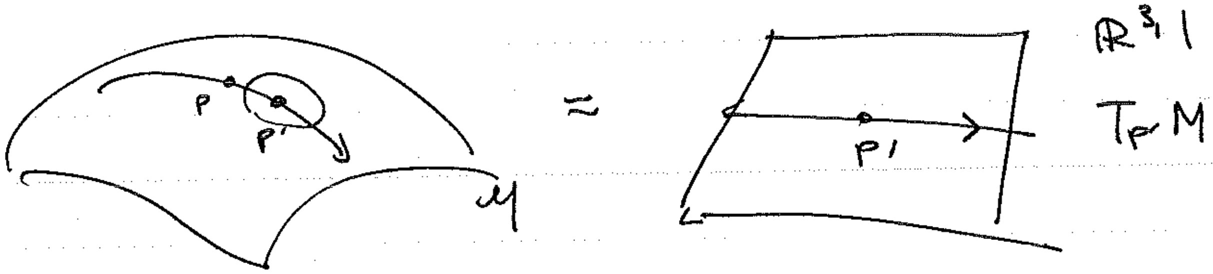
T_pM IS FLAT = MINKOWSKI

in that ~~snapshot~~ little region around p



special rel.
... for a moment.

SIMILARLY FOR NEIGHBORING POINTS



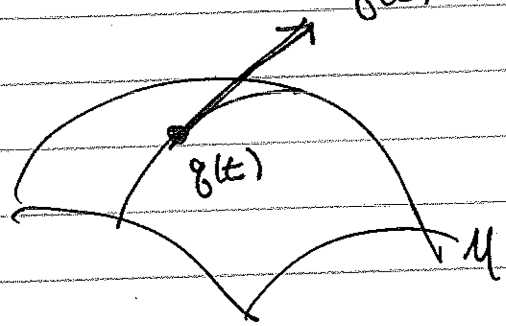
BUT THESE TANGENT PLANES ARE <u>DIFFERENT</u>. EACH IS SEPARATELY <u>isomorphic</u>, BUT THEY ARE PATCHED TOGETHER BY THE <u>GEOMETRY</u> OF THE CURVED SPACE.

BY THE WAY

- "obvious" that tangent space is Minkowski, same way that you can use sheets of paper to ~~cover~~ wrap a 3D object.

- VECTORS (& DUAL VEC ... & tensors) live on tangent spaces (& their generalizations; eg cotangent space for dual vectors)

$\dot{q}(t) \in T_{q(t)}M$ eg: GEOMETRIC MECHANICS



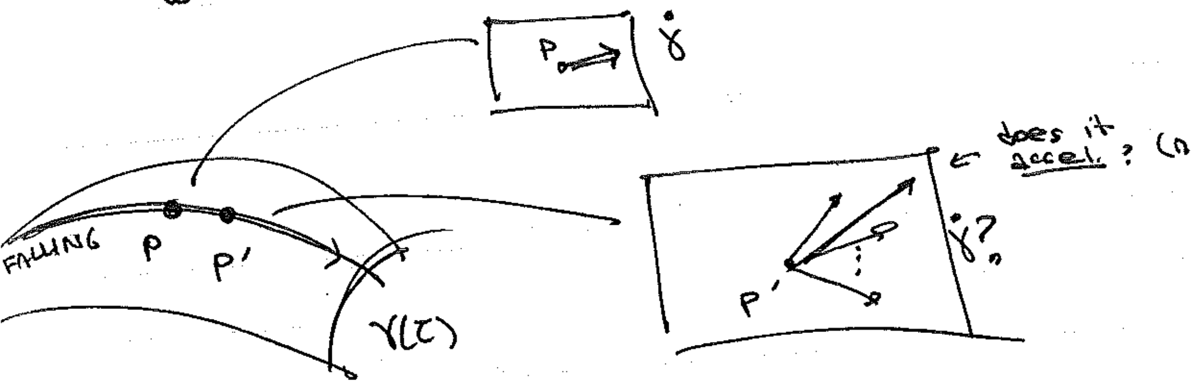
$\dot{q}(t)$ does not live on M

$\dot{q}(t+\varepsilon)$ lives in a different space, still.

btw: in this sense "position vector" is a very weird (WRONG) notion! POSITION is A POINT $q(t) \in M$.

DIFF OBJECTS: x^μ ∈ M ; $\partial/\partial x^\mu$ BASIS FOR T_pM ; dx^μ BASIS FOR T^*_pM

...RENTIAL
...ETRY IS ALL ABOUT PATCHING THESE
...BY TANGENT SPACES TOGETHER
where all the calculus appears to live



...R GOAL IS TO DEVELOP MACHINERY OF THIS ...APPING, THEN USE IT TO ASSESS CURVATURE.

...S YOU CAN IMAGINE, COORDINATES ...LAY A CENTRAL ROLE.

...n fact: NEED A WAY TO DISTINGUISH "COORD CURVATURE" FROM ACTUAL CURVATURE

↳ weird metric $\not\Rightarrow$ curvature

Weinb. 3-2

PRINCIPLE OF EQUIVALENCE @ WORK

SUPPOSE WE LIVE IN SOME GRAV. BG
eg near black hole, in expanding univ, ...

EQUIV. $\Rightarrow \exists$ freely falling coord system
s.t. OBSERVERS "ATTACHED"
TO THIS SYS (co-falling)
don't realize they're in
a grav. field.

or TARDIS

nb I think this is why the starship
enterprise doesn't jerk around when
it goes to WARP DRIVE :
GEOMETRY CHANGES, IT JUST KEEPS FALLING

LET y^α BE THESE COORDINATES
INSIDE TARDIS :

POSITION OF TEST PARTICLE

$$\frac{d^2 y^\alpha}{d\tau^2} = 0$$

PROPER TIME : time in the y system

PROPER TIME : $d\tau^2 = dt^2 - d\underline{x}^2$
$$= dx^\mu dx^\nu \underbrace{g_{\mu\nu}\big|_{FLAT}}$$

$$\eta_{\mu\nu} \equiv (+ - - -)$$

NB: $-\eta_{\mu\nu}^{EAST} = \eta_{\mu\nu}$

1. when $d\underline{x} = 0$, ie no spatial displacement, $d\tau = dt$; so this is indeed the time meas. by obs @ rest in the frame.

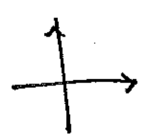
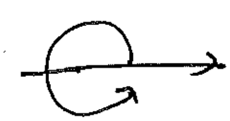
2. $d\tau^2$ is invariant.. I CAN MEASURE IT IN ANY FRAME BY $(dx')^\mu (dx')^\nu g_{\mu\nu}$; DOESN'T MATTER IF IN x' FRAME, THE TEST PARTICLE IS MOVING.

So: in free falling frame, test particle obeys

$$\boxed{\frac{d^2 y^\alpha}{d\tau^2} = 0}$$

↖ cartesian coords.

WHAT IF WE'RE STILL IN THE TARDIS (free fall), BUT I REPLACE

 w/ 

ie: ANOTHER COORD SYSTEM X^μ THAT MAY BE CURVILINEAR?

OF COURSE : no need to be "in TARDIS"! ← like TIDAL lo

CAN HAVE ACCEL FRAME ("sees GRAV FORCE")

SO x^μ COULD BE COORD OF SOMEONE NOT ON ELEVATOR

HOW DO WE CHANGE COORDINATES?

→ calculus

$$0 = \frac{d}{d\tau}\left(\frac{dy^\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau}\right)$$

$$= \frac{\partial^2 y^\alpha}{\partial\tau\,\partial x^\mu}\frac{\partial x^\mu}{\partial\tau} + \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial\tau^2}$$

$$= \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} + \text{---}''\text{---}$$

NOW MULTIPLY BY $\frac{\partial x^\rho}{\partial y^\alpha} = \left[\left(\partial y/\partial x\right)^{-1}\right]^\rho{}_\alpha$

$$\hookrightarrow \frac{\partial x^\rho}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(x^\rho) = \delta^\rho_\mu$$

DON'T EVER SHOW THIS TO A REAL MATHEMAT.

$$0 = \frac{\partial^2 x^\rho}{\partial\tau^2} + \underbrace{\frac{\partial x^\rho}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau}$$

$\equiv \Gamma^\rho_{\mu\nu}$, AFFINE CONNECTION

or CHRISTOFFEL SYMBOL

- *NOT A TENSOR*, BUT CONVENIENT TO LET IT HAVE INDICES ...
  (will combine w/ ∂ to give tensorial quantities)

WHAT ABOUT METRIC? IN y^α COORDS,

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \leftarrow ds^2 = d\tau^2 = (dy^0)^2 - (d\vec{y})^2$$

TAKE INVARIANT & "do calculus" to change vars:

$$d\tau^2 = \eta_{\mu\nu}\, dy^\mu dy^\nu \quad \leftarrow - \text{ SIGN IF EAST COAST}$$

$$= \eta_{\alpha\beta} \underbrace{\left(\frac{\partial y^\alpha}{\partial x^\mu} dx^\mu\right)}_{dy^\mu} \underbrace{\left(\frac{\partial y^\beta}{\partial x^\nu} dx^\nu\right)}_{dy^\nu}$$

~~FREE~~ TO RELABEL DUMMY INDICES

$$= \underbrace{\eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}}\; dx^\mu dx^\nu$$

$$\equiv g_{\mu\nu}(x) \qquad \text{metric in } x \text{ COORDS}$$

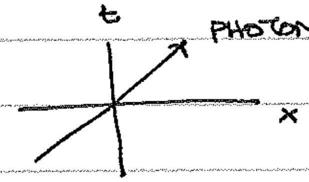
X-DEP IN GEN!

SPACE STILL FLAT (inside TARDIS)

BUT COORDS CURVED.

What about PHOTONS?

$d\tau^2 = 0$



so $dy/d\tau$ means nothing to a photon

PHOTONS NEVER AGE

CAN PICK DIFFERENT PARAMETER.

A CONVENIENT ONE IS y^0 (time in y coords)

$\overset{\shortparallel}{\sigma}$ ↑ the one w/ minus sign

then: FREE FALL OF PHOTON:

$$\frac{d^2 y^\alpha}{d(y^0)^2} = 0$$

$$\frac{d\tau^2}{d\sigma^2} = 0 = \eta_{\alpha\beta}\frac{dy^\alpha}{d\sigma}\frac{dy^\beta}{d\sigma}$$

$$\Rightarrow \frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0$$

$$0 = g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}$$

PHOTON, CONT'D: but $\sigma = y^0$ ... WHAT IF I ONLY CARE ABOUT X COORDS?

eg: in x coordinates, HOW LONG DOES IT TAKE A PHOTON TO TRAVEL A DISTANCE $d\underline{x}$?

$$0 = +g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \qquad (- \text{ for EAST COAST})$$

$$= g_{00}\, dt^2 + 2 g_{i0}\, dx^i dt + g_{ij}\, dx^i dx^j$$

i, j : SPATIAL INDICES

x^i : SPATIAL DIR, BUT NOT NEC CARTESIAN

SANITY CHECK: MINUS SIGNS?

SOLVE THIS FOR dt $\quad (t = x^0)$

$$dt = \frac{-g_{i0}\, dx^i - \sqrt{(g_{i0} g_{j0} - g_{ij} g_{00})\, dx^i dx^j}}{g_{00}}$$

integrate this to get finite time it takes for light to traverse some trajectory ie ? $y^\alpha(x)$?

WE CAN GO THE OTHER WAY, TOO.

GIVEN x^μ, WHAT ARE <u>LOCAL INERTIAL COORDS.</u> y^α ?

nb: of course, ~~both are free falling~~ &

HOLDS FOR CHANGE OF COOR W/IN TARDIS, OR GEN. COOR IN REL ACC. FRAME

~~describe flat space ... but change of coord & curvature are (in many ways) identical, so THIS EXERCISE IS WORTHWHILE~~

$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\rho_{\mu\nu}\left(\frac{\partial y^\alpha}{\partial x^\rho}\right)$$

mult by this on both sides of Γ def.

2^{ND} θ PDE

SOLVE ABOUT SOME POINT $X_{(0)}$

$$y^\alpha(x) = a^\alpha + b^\alpha{}_\mu (x^\mu - x^\mu_{(0)})$$

$$+ \frac{1}{2} b^\alpha{}_\rho \Gamma^\rho_{\mu\nu} (x^\mu - x^\mu_{(0)})(x^\nu - x^\nu_{(0)}) + \cdots$$

$$a^\alpha = y^\alpha(x_{(0)}) \qquad b^\alpha{}_\rho = \left.\frac{\partial y^\alpha}{\partial x_\rho}\right|_{x_0}$$

FROM B.C.

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

(*) $\Rightarrow g_{\mu\nu}(x_{(0)}) = \eta_{\alpha\beta}\, b^\alpha{}_\mu\, b^\beta{}_\nu$ ← eq for b's.

So: given $\Gamma^\rho_{\mu\nu}(x_{(0)})$ & $g_{\mu\nu}(x_{(0)})$
CAN RECONSTRUCT y^α to $\mathcal{O}((x-x_{(0)})^2)$
"locally"

a^α is ARBIT., & (*) HAS A REDUNDANCY:
if $b^\alpha{}_\mu$ SOLVES (*),
so DOES $\Lambda^\alpha{}_\beta b^\beta{}_\mu$.

if y^α is an inertial frame,
so is $\Lambda^\alpha{}_\beta y^\beta + c^\alpha$

- WE HAVEN'T SAID ANYTHING ABOUT INTEGRABILITY OF Γ & g TO SOLVE FOR y^α AT ALL x.

so: conversion btwn $y^\alpha \longleftrightarrow x^\mu$

'FREE FALLING' LIF, MINKOW. — 'stationary' I FEEL GRAV.

is encoded in $\Gamma^\rho_{\mu\nu}$ ← gravity lives here? ALSO COORD CURV...

& proper time is encoded in $g_{\mu\nu}$

THESE ARE RELATED
(st GRAVITY "LIVES" IN $g_{\mu\nu}$)

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

$$\Gamma^\rho_{\mu\nu} = \frac{\partial x^\rho}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}$$

$$\frac{\partial y^\beta}{\partial x^\rho}\Gamma^\rho_{\mu\nu} = \delta^\beta_\alpha \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\rho}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial^2 y^\beta}{\partial x^\nu \partial x^\rho}\eta_{\alpha\beta}$$

ALWAYS CHECK THAT INDICES MATCH! (like DIM-ANALYSIS of tensors)

$$= \Gamma^\sigma_{\mu\rho}\frac{\partial y^\alpha}{\partial x^\sigma}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\sigma_{\nu\rho}\frac{\partial y^\beta}{\partial x^\sigma}\frac{\partial y^\alpha}{\partial x^\mu}\eta_{\alpha\beta}$$

$$= \boxed{\Gamma^\sigma_{\mu\rho}\, g_{\sigma\nu} + \Gamma^\sigma_{\nu\rho}\, g_{\sigma\mu}}$$

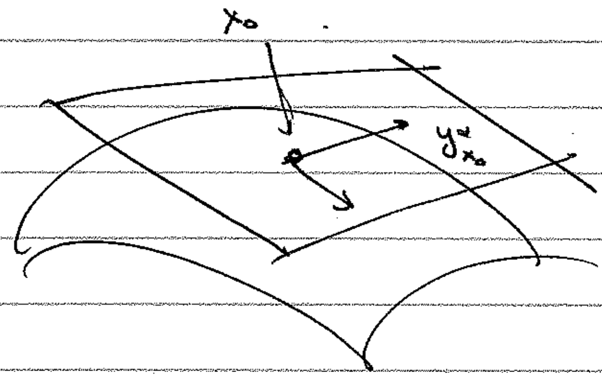
WANT TO SOLVE FOR Γ.

BUT FIRST: IMPORTANT ASIDE – we're taking spacetime derivatives of $g_{\mu\nu}$

$$g_{\mu\nu}(x_0) = \frac{\partial y^{\alpha}_{x_0}(x)}{\partial x^{\mu}} \frac{\partial y^{\beta}_{x_0}(x)}{\partial x^{\nu}} \eta_{\alpha\beta} \Big|_{x=x_0}$$

$$y^{\alpha}_{x_0}(x)$$

the coords we set up @ x_0



$$y^{\alpha}_{x_0}(x) = a^{\circ} + b^{\alpha}_{\mu}(x^{\mu} - x_0) + \cdots$$

<u>SIMILARLY</u> : $$\left.\frac{\partial^2 y^\alpha_{x_0}(x)}{\partial x^\mu \partial x^\nu}\right|_{x=x_0} = \Gamma^\rho_{\mu\nu}(x_0) \left.\frac{\partial y^\alpha_{x_0}(x)}{\partial x^\rho}\right|_{x_0}$$

SO WHEN WE DIFFERENTIATE

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \frac{\partial}{\partial (x_0)^\sigma} g_{\mu\nu}(x_0) ,$$

THERE ARE 2 KINDS OF "CHANGES IN $g_{\mu\nu}$"

1. change in argument of functions

$$y^\alpha_{x_0}(\underset{\Uparrow}{x})$$

2. change in where "we set up shop" locally

$$y^\alpha_{\underset{\Uparrow}{x_0}}(x)$$

so in $\frac{\partial g_{\mu\nu}(x_{(0)})}{\partial x^\rho_{(0)}}$, worry about

$$\left(\frac{\partial^2 y^\alpha_{x_0}(x)}{\partial (x_0)^\rho \partial x^\mu} \right) \text{ terms}$$

JUSTIFYING OUR $\frac{\partial}{\partial x^\rho} g_{\mu\nu}$ EXPRESSION REQUIRES SOME INPUT:

UNJUSTIFIED ANSATZ: LOCALLY INERTIAL COORS y_{x_0} (NICE COORDINATES) ARE CHOSEN s.t. THE FIRST DERIVATIVES OF THE METRIC VANISH @ x_0

(for now WILL HAVE PHYS. MOTIV. w/rt. rates of nearby clocks

meaning: $x_0' = x_0 + \delta x_0$

ARGUMENTS

$$g_{\mu\nu}(x_0') = \frac{\partial y^\alpha_{x_0'}}{\partial x^\mu} \frac{\partial y^\beta_{x_0'}}{\partial x^\nu} \eta_{\alpha\beta} \Big|_{x = x_0'}$$

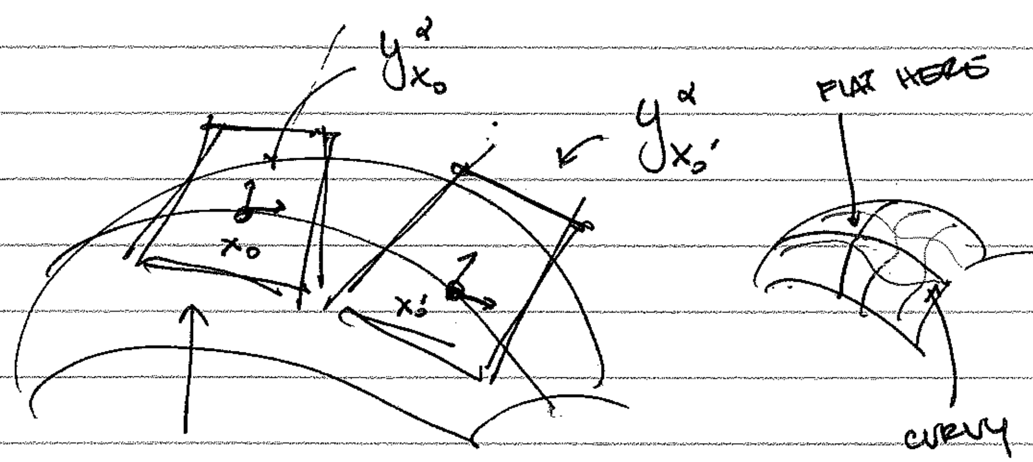
BASE

is also

$$= \frac{\partial y^\alpha_{x_0}}{\partial x^\mu} \frac{\partial y^\beta_{x_0}}{\partial x^\nu} g^{(x_0)}_{\alpha\beta}(x_0') \Big|_{x = x_0}$$

1^{st} line: FLAT @ x_0' B/C LIF SET UP @ x_0'!

2^{nd} line: not flat @ x_0' B/C LIF SET UP @ $x_0 \neq x_0'$



EXTEND THIS COORD SYSTEM TO x_0', AWAY FROM WHERE IT IS FLAT

$$\frac{\partial g_{\mu\nu}(x_0')}{\partial (x_0')^\rho} = \frac{\partial}{\partial (x_0')^\rho}\left[g^{(x_0)}_{\alpha\beta} \frac{\partial y^\alpha_{x_0}}{\partial x^\mu} \frac{\partial y^\beta_{x_0}}{\partial x^\nu} \right]_{x=x_0'}$$

$$= \frac{\partial}{\partial (x_0')^\rho}\left[\eta_{\alpha\beta} \frac{\partial y^\alpha_{x_0'}}{\partial x^\mu} \frac{\partial y^\beta_{x_0'}}{\partial x^\nu} \right]_{x=x_0'}$$

$g^{(x_0')}_{\alpha\beta}$

DROP ORNAMENTS

$$\frac{\partial g_{\mu\nu}(x)}{\partial x^\rho} = \eta_{\alpha\beta} \frac{\partial}{\partial x^\rho}\left[\frac{\partial y^\alpha_{x_0}}{\partial x^\mu} \frac{\partial y^\beta_{x_0}}{\partial x^\nu} \right]$$

$$= \Gamma^\sigma_{\mu\rho} g_{\sigma\nu} + \Gamma^\sigma_{\nu\rho} g_{\sigma\mu}$$

from p. 16

(JUSTIFYING OUR quick RESULT)

@ cost of <u>UNJUSTIFIED ANSATZ</u>.

RECALL: WANT TO "SOLVE" $\partial g = \Gamma g + \Gamma g$

BUT CONTRACTIONS ARE HARD

BACK TO TASK : $\Gamma(g)$

playing w/ indices

$$\left.\begin{aligned} &\partial_\lambda g_{\mu\nu} \\ +&\partial_\mu g_{\lambda\nu} \\ -&\partial_\nu g_{\mu\lambda} \end{aligned}\right\} = \begin{aligned} &g_{\kappa\nu}\Gamma^\kappa_{\lambda\mu} + g_{\kappa\mu}\Gamma^\kappa_{\lambda\nu} \\ &+ g_{\kappa\nu}\Gamma^\kappa_{\mu\lambda} + g_{\kappa\lambda}\Gamma^\kappa_{\mu\nu} \\ &- g_{\kappa\lambda}\Gamma^\kappa_{\nu\mu} - g_{\kappa\mu}\Gamma^\kappa_{\nu\lambda} \end{aligned}$$

$\partial_\mu = \partial/\partial x^\mu$

$$= 2 g_{\kappa\nu}\Gamma^\kappa_{\lambda\mu}$$

USING SYMM. w/rt these

HIT WITH INVERSE METRIC : $g^{\nu\sigma}$

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right)$$

2 terms w/ same sign
REQ. BY $\lambda \leftrightarrow \mu$
SYMMETRY