By the way: LEC. NOTES START FROM WHERE WE LEFT OFF LAST LECTURE, NOT THE END OF PVS. LEC NOTES!

LEC 2: SPECIAL RELATIVITY & A PLAN FOR THE COURSE   12 Jan

TODAY: $\mathbb{R}^2$ IN CARTESIAN : intro to g_{ij}
ROTATIONS IN $\mathbb{R}^2$
TENSORS
SPECIAL RELATIVITY from SYMMETRY
INTRO TO GR & OUR PLAN

MOVE TO CONF ROOM?
ADDED emails?

LAST TIME: lightning review of LORENTZ TR

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = v/c$$

THEN INTRODUCED STRUCTURE ON $\mathbb{R}^2$ (PLANE):

$g(\cdot,\cdot)$ METRIC: linear, sym. map: takes 2 vectors, gives #

$$g_{ij} V^i W^j = \underline{g_{11} V^1 W^1} + g_{12} V^1 W^2 + \cdots$$

these are just products of ordinary numbers

so $g_{ij} V^i W^j = V^i g_{ij} W^j$, etc.

BUT THIS MEANS I CAN <u>ENGINEER</u> A RELATED MACHINE BY STICKING A VECTOR TO THE METRIC :

$$V_i \equiv g_{ij}V^j = g_{i1}V^1 + g_{i2}V^2$$

one lower index

"DUAL VECTOR" / "1-FORM"

V_i is a machine that eats a vector & spits out a #

$$\text{"}V_i(\underline{w})\text{"} = V_i W^i = \underbrace{g_{ij}V^iW^j}$$

of course, it's all just the <u>same</u> machine that we're using in different ways!

* THAT'S WHY I SAID "ENGINEER" & NOT "INVENT"

INTERPRETATION:

just array of #'s

just as $\underline{V} = V^i \, \underline{e}_{(i)}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\underline{e}_{(1)}$

can think of dual vector as

$\underset{\sim}{V} = V_i \, \underset{\sim}{e}^{(i)}$

$(1 \quad 0)$ for $\underset{\sim}{e}^{(1)}$

array of #'s

SUCH THAT:

$$\underset{\sim}{e}^{(i)}\left(\underline{e}_{(j)}\right) = \delta^i_j$$

"matrix multiplication"

DIFF. GEOMETERS HAVE A FUNNY NOTATION:

$\underline{e}_{(i)} \rightarrow \dfrac{\partial}{\partial x^i}$ — PARTIAL DERIVATIVE (an operator!)

$\underset{\sim}{e}^{(i)} = dx^i$ — DIFFERENTIAL (ONE) FORM (uh... an infinitesimal element?)

thing that gets integrated

evidently: $dx^i \left(\dfrac{\partial}{\partial x_j}\right) = \delta^i_j$

THIS FUNNY NOTATION IS THE FIRST HINT THAT THE UNDERLYING STRUCTURE WE ARE EXPLORING IS A GENERALIZATION OF CALCULUS.

BUT LET'S FOCUS ON SOMETHING MORE PROSAIC BUT EQUALLY ENLIGHTENING:

INDICES

g_{ij} can take a vector → dual vec.

$$g_{ij} V^j \equiv V_i$$

~~RECALL RULE: REPEATED UPPER & LOWER INDICES IMPLY A SUM~~

~~So can view this rule as:~~

$$\cancel{V_i W^i \equiv V^i g_{ij} W^j}$$

WE'D ALSO LIKE A WAY TO BRING INDICES BACK UP AGAIN

INVERSE METRIC: g^{ij} s.t. $g^{ij} g_{jk} = \delta^i_k$

st if I LOWER THEN RAISE AN INDEX, I GET THE SAME THING.

FINALLY: the METRIC is related to measurement of length.

$$ds^2 = g_{ij}\,dx^i dx^j$$

s.t. $g_{ij} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ gives Pythagorean thm

nb: infinitesimal length; QUADRATIC FORM APPROXIMATION.

in $\mathbb{R}^2$; transformations that preserve length are rotations.

$$R^i{}_j = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}$$

why ${}^i{}_j$ vs ${}_{ij}$, ${}_i{}^j$, or ^{ij} ?

$$\underline{\underline{R}}\,\underline{V} = R^i{}_j V^j = (\underline{V}')^i$$

$$\underline{W}^T \underline{\underline{R}}^T = W_j R^j{}_i = (W')_i \;\text{"}{=}\text{"}\; (\underline{\underline{R}}\,\underline{W})^T$$

$$\underline{W}^T \underbrace{\underline{\underline{R}}^T \underline{\underline{R}}}_{=1} \underline{V} = \underline{W}^T \underline{V} \Rightarrow \cancel{R^j{}_i R^i{}_k = \delta^i{}_k}$$

$$\cancel{R^i{}_j R^j{}_k = \delta^i{}_k}$$

<u>SIDEBAR</u> (not for lecture

HOW TO SEE INDEX STRUCTURE OF $\underline{R}^T$?

COMES FROM REP. THY, WHICH GIVES

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \Lambda^T = \eta \Lambda^{-1} \eta^{-1}$$

we know $(\Lambda^{-1})^i{}_j$ b/c $(\Lambda^{-1})^i{}_j (\Lambda)^j{}_k = \mathbb{1}^i{}_k$

THUS Λ^T HAS A LOWER THEN UPPER INDEX

so: $\left((R^T)R\right)^i{}_k = \underbrace{(R^T)^i{}_j}\, R^j{}_k$

$$\equiv R_j{}^i$$

So, eg: $(R^T R)^1{}_2 = R_j{}^1 R^j{}_2 = R_1{}^1 R^1{}_2 + R_2{}^1 R^1{}_2$

$$= c_\theta\, s_\theta + (-s_\theta)\, c_\theta$$

$$\begin{pmatrix} R^1{}_1 = c_\theta & R^1{}_2 = s_\theta \\ R^2{}_1 = -s_\theta & R^2{}_2 = c_\theta \end{pmatrix}$$

& $R^i{}_j = R_i{}^j$ as numbers

<u>EVIDENTLY</u>: <u>LOWER INDEX</u> : $W_i \mapsto (R^T)^j{}_i W_j = R_i{}^j W_j$

<u>UPPER INDEX</u> : $V^i \mapsto R^i{}_j V^j$

THAT'S THE PHYSICIST'S DEFINITION OF TENSORS.

$$T_i^{\ jk}$$

↑ well: mathematically, it's a multilinear map, takes 1 vec, 2 dual vecs → #.

Physically: THIS IS AN OBJECT THAT TRANSFORMS ACCORDING TO

$$\underbrace{R_i^{\ l}}_{R^T} R^j_{\ m} R^k_{\ n} T_l^{\ mn}$$

eg. Metric: $g_{ij} \rightarrow R_i^{\ k} R_j^{\ l} g_{kl}$

$$g_{11} = R_1^{\ 1} R_1^{\ 1} g_{11} + R_1^{\ 2} R_1^{\ 2} g_{22}$$
$$= c_\theta^2 + s_\theta^2$$
$$= 1 \quad \checkmark$$

examples: moment of inertia tensor in mechanics.

GENERALIZES "transformation of a matrix"

~~WE WILL~~

IMPORTANT: THINGS W/ NO INDICES DO NOT TRANSFORM
↳ invarient

rediscover Lorentz from SYMMETRY PRINCIPLES

from $\mathbb{R}^2 \mapsto \mathbb{R}^{1,1}$, MINKOWSKI SPACE

$ds^2 = dx^2 + dy^2$ $\qquad$ $\boxed{ds^2 = dt^2 - dx^2}$

overall sign a convention
WILL CAUSE ~~HEADACHES~~.

$$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

WANT: ANALOG OF ROTATION:
transformations that leave inner product invariant

$\mathbb{R}^2$: $ds^2 =$ const $\longrightarrow$ circle
$\mathbb{R}^{1,1}$: $ds^2 =$ const $\longrightarrow$ hyperbola

SO GUESS: $\begin{pmatrix} t' \\ x' \end{pmatrix} \rightarrow \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$

η is transf. param
ANALOG OF θ

$$\cosh^2 - \sinh^2 = 1$$

hyperbolic

CHECK: $V = \begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} ct + sx \\ st + cx \end{pmatrix}$

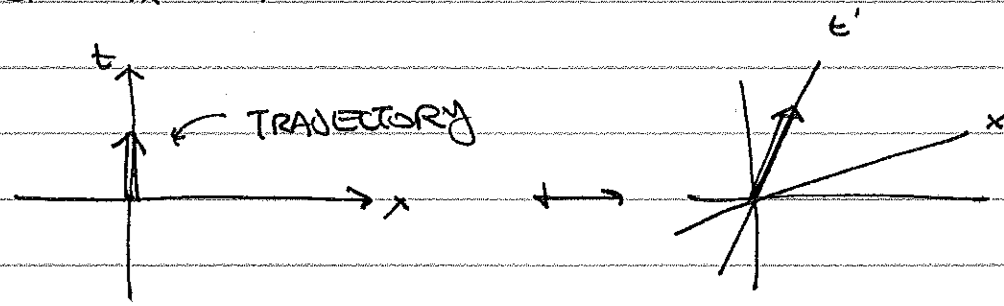
then $V^2 = V \cdot V = t^2 - x^2$

$$\begin{aligned} &\mapsto (ct + sx)^2 - (st + cx)^2 \\ &= c^2t^2 + 2ctsx + s^2x^2 \\ &\quad - s^2t^2 - 2\,\text{''} - c^2x^2 \\ &= t^2 - x^2 \ \checkmark \qquad \text{using } c^2 - s^2 = 1 \end{aligned}$$

WHAT IS η?

say, of a particle, clock, whatever

IN REST FRAME, $\Delta x / \Delta t = 0$



$$\begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \Delta t \cosh \eta \\ \Delta t \sinh \eta \end{pmatrix} \qquad \text{since } (0,0) \to (0,0)$$

$$\downarrow$$

$$\boxed{\frac{\Delta x'}{\Delta t'} = \tanh \eta}$$

η IS RAPIDITY

↑ VELOCITY, β

so: $$1 - \underbrace{\tanh^2 R}_{\beta^2} = \frac{1}{\cosh^2 R}$$

$$\Rightarrow \cosh^2 R = \frac{1}{1-\beta^2} \equiv \gamma$$

$$\Rightarrow \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh R = \frac{1}{\sqrt{1-\beta^2}} & \sinh R \\ \sinh R & \cosh R = \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}$$

↑ analog of rot

$$\sinh R = \underbrace{\tanh R}_{\pm\beta} \underbrace{\cosh R}_{\gamma}$$

$$\Lambda = \begin{pmatrix} \gamma & \pm\gamma\beta \\ \pm\gamma\beta & \gamma \end{pmatrix} \checkmark$$

<u>DERIVED</u> LORENTZ TRANSFORMATION

SUMMARY in SR

$g_{\mu\nu} = \mathrm{diag}(1, -1)$

$g^{\mu\nu} = \text{———}\,''\,\text{———}$ } not nec equal in gen! $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$

$V^{\mu} \leftarrow$ 4-vector

$W_{\nu} \leftarrow$ dual vec, 1-form, ...

$g_{\mu\nu} V^{\nu} = V_{\mu}$; $g^{\mu\nu} W_{\nu} = W^{\mu}$

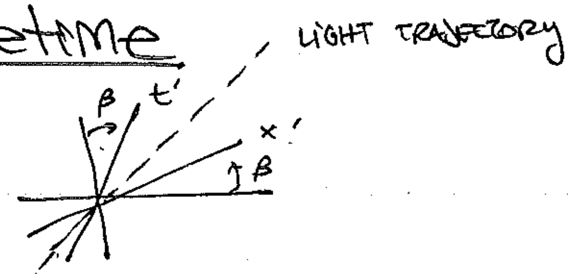
LORENTZ TRANSF: gen. of $\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$

$\hookrightarrow$ BUT 4×4 ; MAY INCLUDE ROTATIONS

<u>SO, FINALLY</u>: let me give an intro to our course

GR: physics of <u>spacetime</u>

$\boxed{SR}$ → <u>causality</u> (HW2)



you never cross this
in a physical boost

SO: CAUSALITY PRESERVED
; "action @ a distance" ← "SPOOKY" OR OTHERWISE

WHAT ABOUT $V_{EM} \sim \frac{\alpha}{r}$?
(coulomb)

→ MAXWELL'S EQNS ARE <u>LORENTZ INVT</u>
(<u>know</u> about $c = 1$) ; COULOMB POT "PROPAG" @ c

WHAT ABOUT <u>GRAVITY</u>? $V_N \sim \frac{G}{r}$?

→ ~~need~~ analog of maxwell

<u>OBSERVATION</u>: PRINCIPLE OF EQUIVALENCE
everything falls the same
in a grav. field ... doesn't
matter what "material"
↳ QUANTUM #'S !
(CHARGE, ISOSPIN, ...)

* SO MAYBE: GRAVITY is NOT LIKE MAXWELL;
MAYBE IT'S TELLING US SOMETHING
ABOUT "FALLING"

trajectories in spacetime

⇒ GRAVITY is A BENDING/WARPING
OF SPACETIME

MASS/ENERGY ⟶ $g_{\mu\nu}(x)$ (SPACETIME)

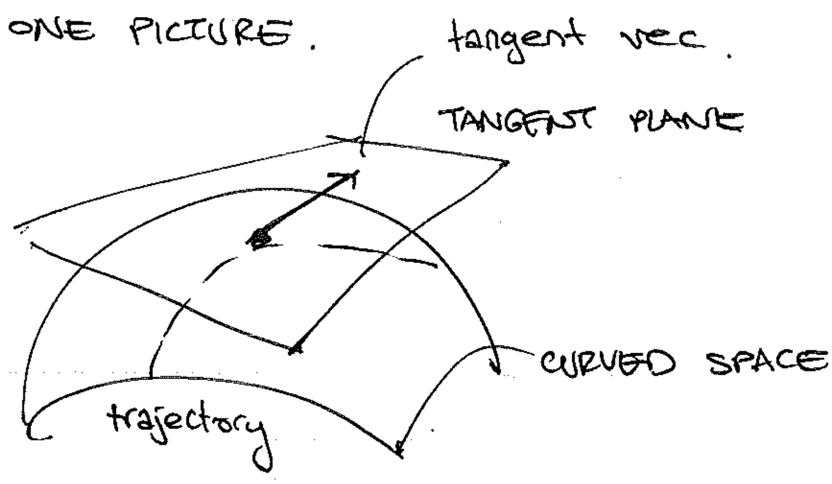
BUT HOW TO TELL IF $g_{\mu\nu}$
REPRESENTS CURVED SPACE
OR CURVILINEAR COORDS?
⇒ will need tensor/geometric
tools.

have to learn to understand curved
space(time)s.

nb: MORE ADVANCED: ACTUALLY, MAYBE MAXWELL
IS MORE LIKE GRAVITY!

Yang Mills thy as FIBER BUNDLE

I HAVE ONE PICTURE.



WE LIVE @ A POINT IN SPACETIME,
ON SOME TRAJECTORY

LOCALLY, WE APPROX. SPACETIME AS FLAT - TANGENT PLANE

Minkowski

in fact, all we know is local... have to build up notion of nonlocality to diagnose curvature

→ COMPARING NEARBY TANGENT PLANES.

Diff. Geometry is the tool to do this ② & give coord-ind ways to describe curvature.

② → then we can understand "FREE FALL" AS "TAKE THE SHORTEST PATH"

THIS WORKS FOR BIG, MASSIVE GRAV. SOURCES; BUT IS NOT YET THE WHOLE PICTURE.

Analog of $V_{EM} \sim {}^{2}/_{r}$

CURVED SPACE $\longrightarrow$ moves matter/ENERGY

SOURCES

in fact: GRAV. WAVES are based on this

ONE OF OUR GOALS.

So we need EINSTEIN EQ

ANALOG OF MAXWELL.

$$G_{\mu\nu} = T_{\mu\nu}$$

SPACETIME ENERGY

OUTLINE OF COURSE

1. SR $\rightarrow$ CURVATURE
2. PHYSICS IN CURVED SPACE $\rightarrow$ eg BLACK HOLES
3. EINSTEIN'S EQ
4. GRAV. WAVES