

# Tests for Standardized Generalized Variances of Multivariate Normal Populations of Possibly Different Dimensions\*

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In many practical problems, one needs to compare variabilities of several multidimensional populations. The concept of standardized generalized variance (SGV) is introduced as an extension of the concept of GV. Considering multivariate normal populations of possibly different dimensions and general covariance matrices, LRTs are derived for SGVs. The criteria turn out to be elegant multivariate analogs to those for tests for variances in the univariate cases. The null and nonnull distributions of the test criteria are deduced in computable forms in terms of Special Functions, e.g., Pincherle's  $H$ -function, by exploiting the theory of calculus of residues (Mathai and Saxena, *Ann. Math. Statist.* 40, 1439-1448). © 1987 Academic Press, Inc.

## 1. INTRODUCTION AND SUMMARY

Let  $X$  be a  $p$ -dimensional random vector with  $\text{Cov}(X) = \Sigma > 0$ . In many applied problems, e.g., the overall risk in portfolio analysis, the overall precision in statistical quality control, the overall variability in agricultural statistics, the overall homogeneity in cluster analysis, etc., a measure of overall scatter becomes necessary. Use of  $\Sigma$  will require specification on each variable individually while  $\text{tr } \Sigma$  will be useless in case the variables are

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standardized. Wilks [14, 15] has proposed the generalized variance (GV),  $|\Sigma|$ , for such a purpose and has shown that it possesses many desirable properties. Intuitively also, since  $|\Sigma|$  is proportional to the volume in  $p$ -dimensions the greater the GV, the greater will be the scatter of the multidimensional points. Further, for elliptically symmetric distributions and, in particular, the multivariate normal distribution, with location parameter  $\mu$ , the higher the value of GV, the flatter will be the probability surface at  $X = \mu$  and the less the concentration there.

A further generalization of overall scatter seems necessary. Consider generalized canonical variable analysis [13]. Let the criterion for optimization be the GV. There may be several types of possible groupings [4, p. 77] which might possibly differ also in their dimensions, i.e., the number of groups. Naturally the smallest dimensional GCV will be the best choice if its GV is the smallest. However, it will not be meaningful here to compare GVs of different dimensions. Similarly, there are many situations in which one might be interested in comparing overall scatter for populations of different dimensions, e.g., portfolio analysis with different numbers of entries, additional or missing information on components of the same item produced by different factories, etc. For such a comparison we propose as a measure, the standardized generalized variance (SGV),  $|\Sigma|^{1/p}$ . This scales down the values (in case the components are measured in the same unit) over populations of different dimensions so as to render them comparable with the univariate case. Further applications and discussions of GV and SGV can be found in [6, 12].

The problem of estimating GV has received much attention, whereas not much is known about tests for GVs. The present paper attempts to bridge that gap. Assuming independent multivariate normal populations, likelihood ratio tests for SGVs are derived. These turn out to be elegant multivariate analogs of the univariate cases. For the case of two populations, the exact null and nonnull distributions of the test statistic are derived in terms of Special Functions. These are presented in a computable form using the theory of the calculus of residues. An example is also given.

## 2. LIKELIHOOD RATIO TESTS FOR SGVs

Let  $X \sim N_p(\mu, \Sigma)$ . Throughout our discussion, unless otherwise stated we will assume  $\Sigma$  to be nonsingular. Denote the population SGV of  $X$ ,  $|\Sigma|^{1/p}$  by  $\Delta^2$  and that of the sample,  $|S/N|^{1/p}$  by  $d^2$ , where  $S$  is the sample sums of products matrix based on a sample of size  $N$ . Also denote  $|S|^{1/p}$  by  $s^2$ . [Note that Anderson [1] defines GV with the divisor  $N - 1$  instead of  $N$ ]. A straightforward derivation of the LRTs through direct differentiation here can be quite frustrating.

Let  $x_1, \dots, x_N$  be a random sample from  $N_p(\mu, \Sigma)$  and suppose we want to test  $H_0: \Delta^2 = \sigma_0^2$  (specified) against  $H_1: \Delta^2 \neq \sigma_0^2$ . (Note that  $H_0$  is equivalent to the hypothesis that the GV,  $|\Sigma|$ , has the specified value  $\sigma_0^{2p}$ ). Since the  $H_0$  does not constrain  $\mu$ , we have  $\hat{\mu} = \bar{x}$ . To find the MLE of  $\Sigma$  under  $H_0$ , consider

$$\Phi = \ln C + \sum_{i=1}^p \left( \frac{N}{2} \ln \theta_i - \frac{\theta_i}{2} \right) + \frac{\lambda}{2} \left( \ln s^{2p} - \sum_{i=1}^p \ln \theta_i - \ln \sigma_0^{2p} \right),$$

where  $C = (2\pi)^{-Np/2} |\Sigma|^{-N/2}$ ,  $\lambda$  is the Lagrange undetermined multiplier,  $\theta_i$ ,  $i = 1, \dots, p$  are the characteristic roots of  $\Sigma^{-1}S$  and we have used the fact that  $|\Sigma|^{1/p} = \sigma_0^2$  is equivalent to  $\ln s^{2p} - \sum_{i=1}^p \ln \theta_i = \ln \sigma_0^{2p}$ ,  $s^{2p} = |\Sigma|$ . Differentiating  $\Phi$  w.r.t.  $\theta_i$  and equating to zero, we have  $N - \lambda = \theta_i$ ,  $i = 1, \dots, p$ . So,

$$(N - \lambda)^p = s^{2p} / \sigma_0^{2p} \Rightarrow \theta_i = s^2 / \sigma_0^2, \quad i = 1, \dots, p.$$

Hence,  $L_\omega / L_\Omega = C_1 a^{N/2} \exp(-(p/2) a^{1/p}) = f(a)$ , say, where  $C_1 = (e/N)^{np/2}$  and  $a = s^{2p} / \sigma_0^{2p}$ . However,

$$f(a) \uparrow a < N^p \quad \text{and} \quad \downarrow a > N^p.$$

So, we get,

**Result 1.** The LRT for  $H_0: \Delta^2 = \sigma_0^2$  against  $H_1: \Delta^2 \neq \sigma_0^2$  can be equivalently given by

$$\text{Reject } H_0 \quad \text{iff} \quad d^{2p} / \sigma_0^{2p} > a_0 \quad \text{or} \quad < a_1,$$

where  $a_0$  and  $a_1$  are constants to be determined from the specified level of the test.

The following two results can be proven also using the same technique as above.

**Result 2.** The LRT for  $H_0: \Delta_1^2 = \Delta_2^2$  against  $H_1: \Delta_1^2 \neq \Delta_2^2$  can be equivalently given by

$$\text{Reject } H_0 \quad \text{iff} \quad R = d_1^2 / d_2^2 < r_1 \quad \text{or} \quad > r_2,$$

where  $r_1$  and  $r_2$  are constants to be determined from the specified level of the test.

**Result 3.** The LRT for  $H_0: \Delta_i^2$ , all equal against  $H_1$ : at least one of the  $\Delta_i^2$ ,  $i = 1, \dots, k$ , differ is given by

$$\text{Reject } H_0 \quad \text{iff} \quad \eta = \prod_{i=1}^k (d_i^2 / \hat{\sigma}_0^2)^{N_i p / 2} < \eta_0,$$

where  $\hat{\sigma}_0^2 = \sum p_i s_i^2 / \sum p_i N_i$  and where  $\eta_0$  is a constant to be determined from the specified level of the test.

### 3. EXACT NULL AND NONNULL DISTRIBUTIONS OF THE TEST CRITERIA

#### 3.1. Definitions and Discussions

In order to obtain the exact distributions under null and alternative hypotheses, for the test statistics considered in Section 2, the reader is referred to the definitions and discussions in Mathai [7-9] or Sen Gupta [12].

#### 3.2. Exact Distributions of $d^{2p}/\sigma_0^{2p}$ and $R$ for $p_1 = p_2$

Since the sample GV,  $d^{2p}$  and the ratio  $R^p$  (for  $p_1 = p_2 = p$ , say) arise frequently in many multivariate tests, various authors have worked on their exact null and nonnull distributions and are available, e.g., from Mathai [8].

#### 3.3. Exact Distributions of $R$ for $p_1 \neq p_2$

In the case of unequal dimensions, the distribution of  $R$  is not available. We obtain the distribution in terms of the  $H$ -function and present it in a computable form through the use of the calculus of residues. Now,

$$E(R^h) = C (\Delta_1^2 / \Delta_2^2)^h \prod_{i=1}^{p_1} \Gamma \left\{ \frac{1}{2} \left( N_1 + \frac{2h}{p_1} - i \right) \right\} \prod_{i=1}^{p_2} \Gamma \left\{ \frac{1}{2} \left( N_2 - \frac{2h}{p_2} - i \right) \right\}, \quad (3.31)$$

where

$$C = \left[ \prod_{i=1}^{p_1} \Gamma \{ (N_1 - i)/2 \} \prod_{i=1}^{p_2} \Gamma \{ (N_2 - i)/2 \} \right].$$

Using the inverse Mellin transform and the  $H$ -function, the density of  $R$ ,  $g_2(r)$ , can be written as

$$\begin{aligned} g_2(r) &= (2\pi i)^{-1} r^{-1} \int_L E(R^h) r^{-h} dh \\ &= Cr^{-1} \cdot H_{p_2, p_1}^{p_1, p_2} \left[ \frac{r}{\delta^2} \left| \begin{matrix} (a_1, 1/p_2), \dots, (a_{p_2}, 1/p_2) \\ (b_1, 1/p_1), \dots, (b_{p_1}, 1/p_1) \end{matrix} \right. \right], \quad 0 < r < \infty, \end{aligned} \quad (3.32)$$

where  $a_j = 1 - (N_2 - j)/2$  and  $b_j = (N_1 - j)/2$ .

We can use the Gauss-Legendre multiplication formula to express  $g_2(r)$

in terms of the  $G$ -function also. However, use of the  $H$ -function here is a more direct and convenient approach.

In order to present (3.3.2) in a computable form, note the discussions in Mathai [7]. The notations in the remainder of this section correspond also to those in Mathai [7].

We now determine the poles and their corresponding order for only the first product of gammas in (3.3.1). It will thus be convenient, for computational purposes, to choose  $p_1 < p_2$ .

For a fixed  $i$ , the poles of  $\Gamma[(N_1 - i)/2 + h/p_1] = \Gamma_i$  are given by the equation

$$-s = p_1 \{(N_1 - i)/2 + v\}, \quad v = 0, 1, 2, \dots$$

Note that the poles of  $\Gamma_i$  and  $\Gamma_j$  coincide only when  $i$  and  $j$  are both even or both odd. From Mathai [7], we note

$$H(z) = \sum_{j=1}^m \sum_{s_{j1}^{(j)} \dots s_{jm}^{(j)}} R_j. \quad (3.3.3)$$

LEMMA 3.3.1. *The poles, with their corresponding orders, are given by*

Case A.  $p_1$  odd:

$$\{v_{1010 \dots 101}^{(11)}\} = \{0, 1, 2, \dots\} = \{v\}. \quad (3.3.4)$$

Poles are  $p_1((N_1 - 1)/2 + v)$ , repeated  $(p_1 + 1)/2$  times;

$$\{v_{\underbrace{00 \dots 0}_{j-1} 101 \dots 101}^{(jj)}\} = \{0\}.$$

A pole is  $p_1(N_1 - j)/2$ , repeated  $(p_1 + 1)/2 - (j - 1)/2$  times,  $j = 3, 5, \dots, p_1$ ;

$$\{v_{0101 \dots 010}^{(22)}\} = \{0, 1, 2, \dots\} = \{v\}.$$

Poles are  $p_1((N_1 - 2)/2 + v)$ , repeated  $(p_1 - 1)/2$  times;

$$\{v_{\underbrace{0 \dots 0}_{j-1} 10 \dots 010}^{(jj)}\} = \{0\}.$$

A pole is  $p_1(N_1 - j)/2$ , repeated  $(p_1 - 1)/2 - (j - 2)/2$  times,  $j = 4, 6, \dots, p_1 - 1$ .

Case B.  $p_1$  even:

$$\{v_{1010 \dots 10}^{(11)}\} = \{0, 1, 2, \dots\} = \{v\}. \quad (3.3.5)$$

Poles are  $p_1((N_1 - 1)/2 + v)$ , repeated  $p_1/2$  times;

$$\{v_{0 \dots 010 \dots 10}^{(j)}\} = \{v\}.$$

A pole is  $p_1(N_1 - j)/2$ , repeated  $p_1/2 - (j - 1)/2$  times,  $j = 3, 5, \dots, p_1 - 1$ ;

$$\{v_{0101 \dots 01}^{(22)}\} = \{0, 1, 2, \dots\} = \{v\}.$$

Poles are  $p_1((N_1 - 2)/2 + v)$ , repeated  $p_1/2$  times;

$$\{v_{00 \dots 1 \dots 01}^{(j)}\} = \{0\}.$$

A pole is  $p_1(N_1 - j)/2$ , repeated  $p_1/2 - (j - 2)/2$  times,  $j = 4, 6, \dots, p_1$ .

For  $i \neq j$ ,  $\{v^{(ij)}\}$  is vacuous unless  $i$  and  $j$  are both odd or both even (W omit the subscripts of  $v^{(ij)}$ , since it is clear what they are.)

$$\{v^{(1l)}\} = \left\{v + \frac{l-1}{2}\right\}, \quad v = 0, 1, 2, \dots; \quad l > 1 \text{ is odd.} \quad (3.3.6)$$

Poles are identified with those of  $v^{(11)}$ ;

$$\{v^{(2l)}\} = \left\{v + \frac{l-2}{2}\right\}, \quad v = 0, 1, 2, \dots; \quad l > 2 \text{ is even.}$$

Poles are identified with those of  $v^{(22)}$ ;

$$\{v^{(l'l)}\} = \frac{l-l'}{2}, \quad l > l' > 2; \quad l, l' \text{ both odd or both even.}$$

Poles are identified with those of  $v^{(ll)}$ .

*Proof.* The above results follow from the following observations. Consider Case A. Let  $i = 1$  and  $j \leq p_1$  be any odd number. Then, poles of  $\Gamma_1$  and  $\Gamma_j$  coincide, as

$$p_1\left(\frac{N_1 - 1}{2} + v\right) = p_1\left(\frac{N_1 - j}{2} + \lambda\right) \Rightarrow (v, \lambda) = \left\{\left(0, \frac{j-1}{2}\right), \left(1, \frac{j+1}{2}\right), \dots\right\}$$

But this set excludes the poles coming from  $\lambda \in \{0, 1, \dots, (j-1)/2 - 1\} = E_j$ . Consider  $j, j'$  both odd,  $3 \leq j < j' \leq p_1$ . Then

$$p_1\left(\frac{N_1 - j}{2} + \lambda\right) = p_1\left(\frac{N_1 - j'}{2} + \lambda'\right) \quad \text{for} \quad \lambda = \frac{j-1}{2} - 1 + \frac{j'-j}{2}. \quad (3.3.7)$$

Thus considering the "excluded sets"  $E_j$ 's we note that the smallest element, i.e., 0, is repeated in all succeeding  $E_{j'}$  through the relation (3.3.7). This

establishes (3.3.4). A similar argument holds for (3.3.5); (3.3.6) follows from the definition of the corresponding sets.

**THEOREM 3.3.1.** *The probability density function of  $R$  is given by, for  $p_1$  odd,*

$$\begin{aligned} g_2(r) = C \cdot p^{-1} & \left[ \sum_{v=0}^{\infty} (r/\delta^2)^{p_1((n_1-1)/2+v)} \sum_{u=0}^{(p_1-1)/2-1} f(r/\delta^2; u, a_1, A_0, B_0) \right. \\ & + \sum_j^* (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{(p_1+1)/2-(j-1)/2-1} f(r/\delta^2; u, a_j, A_0, B_0) \\ & + \sum_{v=0}^{\infty} (r/\delta^2)^{p_1((N_1-2)/2+v)} \sum_{u=0}^{(p_1-1)/2-1} f(r/\delta^2; u, b_1, A_0, B_0) \\ & \left. + \sum_j^{**} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{(p_1-1)/2-(j-2)/2-1} f(r/\delta^2; u, b_j, A_0, B_0) \right], \end{aligned}$$

and for  $p_1$  even,

$$\begin{aligned} g_2(r) = C \cdot p^{-1} & \left[ \sum_{v=0}^{\infty} (r/\delta^2)^{p_1((N_1-1)/2+v)} \sum_{u=0}^{p_1/2-1} f(r/\delta^2; u, a'_1, A_0, B_0) \right. \\ & + \sum_j^* (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{(p_1/2)-(j-1)/2-1} f(r/\delta^2; u, a'_j, A_0, B_0) \\ & + \sum_{v=0}^{\infty} (r/\delta^2)^{p_1((N_1-2)/2+v)} \sum_{u=0}^{p_1/2-1} f(r/\delta^2; u, b_1, A_0, B_0) \\ & \left. + \sum_j^{**} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{p_1/2-(j-2)/2-1} f(r/\delta^2; u, b_j, A_0, B_0) \right], \end{aligned}$$

where

$$\begin{aligned} f(r; u, d, A_0, B_0) &= \frac{1}{(u_0-1)!} \binom{u_0-1}{u} \\ &\quad \times (-\log r)^{u_0-1-u} \left[ \sum_{\gamma_1=0}^{u-1} \binom{u-1}{\gamma_1} A_0^{(u-1-\gamma_1)} \right. \\ &\quad \times \sum_{\gamma_2=0}^{\gamma_1-1} \binom{\gamma_1-1}{\gamma_2} A_0^{(\gamma_1-1-\gamma_2)} \dots \left. \right] B_0 \\ B_0 &= (s-d)^{u_0} \prod_1^{p_1} \Gamma \left\{ \frac{1}{2} \left( N_1 + \frac{2h}{p_1} - i \right) \right\} \quad \text{at } s=d, \end{aligned}$$

where  $d$  is a pole of order  $u_0$  (the upper limit  $+1$ , for  $u$  in the summation in the theorem) of the product of the gamma functions defined by  $B_0$ ;

$$A_0^{(t)} = \frac{\delta^{t+1}}{\delta s^{t+1}} \log B_0, \quad t \geq 0;$$

$C$  is the constant defined in (3.3.1) and  $\delta^2 = \Delta_1^2/\Delta_2^2$ .  $\sum_j^*$  and  $\sum_j^{**}$  denote the summations over all  $j \in \{3, 5, \dots, p_1^*\}$  and  $j \in \{4, 6, \dots, p_1^{**}\}$ , respectively;  $p_1^* = p_1$  if  $p_1$  is odd and  $p_1^* = p_1 - 1$  if  $p_1$  is even;  $p_1^{**} = p_1$  if  $p_1$  is even and  $p_1 - 1$  if  $p_1$  is odd.

*Proof.* The proof follows by noting (3.3.3) and combining Lemma 3.3.1 above with Lemma 1 of Mathai [7].

Finally, a convenient computational form of the p.d.f. of  $R$  is obtained from the following theorem proved as Theorem 1 in Mathai [7].

**THEOREM M.**  $H(z)$  is given in (3.3.3), where

$$\begin{aligned} R_j = & \frac{(j_1 + \dots + j_m) Z^{(b_j + v_{j_1 \dots j_m}^{(j)})/\beta_j}}{(j_1 + \dots + j_m)!} \sum_{r=0}^{j_1 + \dots + j_m - 1} \binom{j_1 + \dots + j_m - 1}{r} \\ & \times (-\log z)^{j_1 + \dots + j_m - 1 - r} \left[ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} C_j^{(r-1-r_1)} \right. \\ & \left. \times \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} C_j^{(r_1-1-r_2)} \dots \right] D_j, \end{aligned}$$

where the  $C_j$ 's and  $D_j$ 's are defined in (4.23) and (4.24) of Mathai [7].

We note that the sets  $\{v_{j_1 \dots j_m}^{(jh)}\}$  are not needed for  $h < j$  in (4.23) and (4.24) of Mathai. Thus, Lemma 4.3.1 gives us all the desired sets needed to use Theorem M above, which expresses  $H(z)$  in terms of the convenient computable functions, e.g., the psi and the generalized zeta functions. Examples of the computation of  $H(z)$  are given in Section 5 of Mathai [7]. Also computational procedure and computer programs for calculating the percentage points of the distribution of  $R$  can be obtained in a manner similar to Mathai and Katiyar [10]. The null distribution is obtained by putting  $\delta^2 = \Delta_1^2/\Delta_2^2 = 1$  and the nonnull distribution by substituting the specified value, under the given alternative,  $\delta^2 = \Delta_1^2/\Delta_2^2$  in Theorem 3.3.1. It is known that for  $p_i = 1$  or  $2$ ,  $X_j = p_j n_j u_j^2 / \Delta_j^2$ , where  $n_j = N_j - 1$  and  $n_j u_j^2 = N_j d_j^2$ ,  $j = 1, 2$  is distributed as a  $\chi^2$  with d.f.  $p_j(n_j - p_j + 1)$ . Hence if  $p_1 = 1$  or  $2$ ,  $p_1$  not necessarily equal to  $p_2$ , the exact distribution of  $R$ , under both the null and alternative hypotheses are obtained as central  $F_{\xi_1, \xi_2}$  distributions, with obvious multipliers, having d.f. given by  $\xi_i = p_i(n_i - p_i + 1)$ ,  $i = 1, 2$ .



### 3.4. Exact Distribution of $\eta$

We consider a Bartlett type modification for  $\eta$ . Let  $X_i = p_i n_i u_i^2 / \sigma_0^2$ , where  $n_i = N_i - 1$  and  $n_i u_i^2 = N_i d_i^2$ ,  $i = 1, \dots, k$ . As in the univariate case, we propose the modified test statistic

$$\begin{aligned}\eta_B^2 &= \prod_{i=1}^k (u_i^2)^{n_i p_i / \sum n_i p_i} \left/ \left( \sum_{i=1}^k n_i p_i u_i^2 \right) \left/ \sum_{i=1}^k n_i p_i \right. \right) \\ &= C_b^{-1} \prod X_i^{b_i} / \sum X_i,\end{aligned}\quad (3.4.1)$$

where

$$b_i = (n_i p_i / \sum n_i p_i), \quad C_b = \prod b_i^{b_i}.$$

For  $p_i = 1$  or  $2$ ,  $X_i \sim \chi_{p_i(n_i - p_i + 1)}^2$ . Using this result and the representation (3.4.1) we get

**THEOREM 3.4.1.** For  $p_i = 1$  or  $2$ ,  $p_i$  not necessarily equal to  $p_j$ ,  $i \neq j$ ,  $i, j = 1, \dots, k$ , the exact density of  $\eta_B^2$  is given by

$$\begin{aligned}f(t) &= \left[ \Gamma(m) / \prod_1^k \Gamma(m a_j) \right] \left( \prod_1^k b_j^{a_j} \right)^m \left( \prod_1^k b_j^{-1/2} \right) \\ &\quad \times [(2\pi)^{(k-1)/2} / \Gamma((k-1)/2)] \\ &\quad \times t^{m-1} (-\log t)^{(k-3)/2} \eta_{m, a, b}(t), \quad 0 < t < 1,\end{aligned}$$

where

$$\begin{aligned}m &= \sum_{i=1}^k p_i(n_i - p_i + 1)/2, \quad a_j = p_j(n_j - p_j + 1)/2m, \quad j = 1, \dots, k, \\ b_j, \quad j &= 1, \dots, k, \text{ are defined in (3.4.1),}\end{aligned}$$

and  $\eta_{m, a, b}$  is defined in Theorem 2 of Chao and Glaser [2].

Percentage points and approximations to the above distribution are obtained from Dyer and Keating [3]. For  $p_i \geq 3$ , the distributions of  $\eta^2$  or  $\eta_B^2$  seem to be complicated.

## 4. EXAMPLE

Based on different varieties of rice, Goodman [5] had proposed a grouping according to their sample GVs. This was also found to be consistent

with geographical and other agro-economic considerations. However, the need for a statistical basis for such grouping is felt. Here, one may require that the population GV's be same for two varieties to belong to the same group. For 45 observations each on  $X = (\text{ear length, ear breadth})$ , for the two varieties Cateto Sulino and Avanti Piching Ihu, we have,  $R = d_1^2/d_2^2 = (0.8686/0.0961)^{1/2} = 3.01$ . Under  $H_0$ ,  $\Delta_1^2 = \Delta_2^2$ ,  $R \sim F_{2(45)-2, 2(45)-2}$  so that using equal tails,  $H_0$  is rejected at 0.01 level of significance. Hence the two varieties should belong to different classifications as also concluded by Goodman using just the magnitudes of the GV's for the purpose.

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