Probability distributions and statistical inference for axial data

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Observations on axes which lack information on the direction of propagation are referred to as axial data. Such data are often encountered in enviromental sciences, e.g. observations on propagations of cracks or on faults in mining walls. Even though such observations are recorded as angles, circular probability models are inappropriate for such data since the constraint that observations lie only in $[0, \pi)$ needs to be enforced. Probability models for such axial data are argued here to have a general structure stemming from that of wrapping a circular distribution on a semi-circle. In particular, we consider the most popular circular model, the von Mises or circular normal distribution, and derive the corresponding axial normal distribution. Certain properties of this distribution are established. Maximum likelihood estimation of its parameters are shown to be surprisingly, in contrast to trigonometric moment estimation, numerically quite appealing. Finally we illustrate our results by several real life axial data sets.

Keywords: axial data, bivariate axial normal conditionals distribution, circular normal distribution, method of trigonometric moments, tests for axial uniformity, wrapped models

1 Introduction

Data relating to the angular position of random lines which do not have a natural orientation associated with them, or in which neither end can be identified as the starting point, are measured in terms of angles, in radians (degrees), with a range of possible values $[0, \pi)([0, 180))$. We will call such data axial data. Examples of axial data abound in environmental and ecological sciences, e.g. data (Fisher, 1993) on face-cleats in walls of coal mines, long-axis orientations of feldspar laths, horizontal axes of outwash pebbles, groove and tool marks, orientations of rock cores, etc.

Considerable discussion may be found in the literature on the related topic of circular data, which has a range of values $[0, 2\pi)$. In particular, circular normal and various wrapped distributions have been proposed to model such data. In many papers, researchers faced with axial data have merely multiplied it by 2, to change the range to $[0, 2\pi)$, and used circular models to fit the data. Minimal introspection confirms that this approach is inappropriate. Also, observations on a circular variable with high concentration, for which the (circular) range of data is below π, may mislead one to adopt an axial distribution as the underlying model. Loosely stated, in contrast to such concentrated circular variables, it is not only "improbable" but it is simply "impossible", for observations on an axial variable to have range beyond π. Essentially, axial data is best viewed as circular data modulo (π).

If we begin with Φ having a circular normal distribution (which might well be deemed appropriate) then the resulting distribution for $\Theta = 2\Phi(mod\ \pi)$ will look superficially similar to a circular normal density but it will not be circular normal. This means that the "doubling approach" may not be that bad but it seems more natural to assume a model such as (5) given below for our data if we believe that it really is circular normal data modulo π. In the present paper we will discuss properties of such axial normal (AN) distributions as in (5). Inference for such models will also be investigated as will be certain multivariate distributions with AN components in their structure.

2 General representation of a p.d.f. for axial data

Assume that the axial random variable $\Theta, \Theta \in [0, \pi)$, has an absolutely continuous distribution. Any probability density function (p.d.f), $f(\theta)$, for an axial random variable Θ should then satisfy the following properties:

$$(i)\ \ f(\theta) \geq 0 \quad (ii)\ \ f(\theta) = f(\theta + \pi) \quad \text{and} \quad (iii)\ \ \int_0^\pi f(\theta) d\theta = 1. \tag{1}$$

Due to the periodicity constraint in (ii) and noting (iii), it also follows that the c.d.f. of an axial random variable must obey the relation $F(\theta+(k+1)\pi)-F(\theta+k\pi) = 1, k = 0, \pm 1, \pm 2, ...$

Remark 1. It follows from condition (ii) above that the usual methods of construction of distributions on restricted range from those on a wider range by truncation, say e.g. an axial distribution on $[0, \pi)$ from the circular normal density on $[0, 2\pi)$, will not, in general, be a valid method here.

As is well known for circular distributions, we observe here that an axial distribution too admits of representation in terms of a Fourier series. Let $f(\theta) \in [0, \infty)$ have period π. Then the Fourier series generated by f is

$$f(\theta) \equiv \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos 2m\theta + b_m \sin 2m\theta)\ , \tag{2}$$

where

$$a_m = \frac{2}{\pi} \int_0^\pi f(\alpha) \cos 2m\alpha\ d\alpha,\ b_m = \frac{2}{\pi} \int_0^\pi f(\alpha) \sin 2m\alpha\ d\alpha, \quad m = 0, 1, 2, \dots\ .$$

In exponential form we have

$$f(\theta) \equiv \sum_{m=-\infty}^{\infty} \gamma_m e^{2im\theta}\ ,$$

where

$$\gamma_m = \frac{1}{m} \int_0^\pi f(\alpha) e^{-2im\alpha} d\alpha, \quad m = 0, \pm 1, \pm 2, \dots\ .$$

It is known that all the convergence theorems for Fourier series of period 2π (i.e. for the circular case) can also be applied to the case of a general period p, and hence to the period π, i.e. to the axial case, by making a suitable change of scale. Letting $f(\theta)$ be a proper p.d.f. defined for the axial variable $\theta \in [0, \pi)$, evaluating the Fourier coefficients a_m and b_m and substituting these in the expression for $f(\theta)$ given in (2), we get the Fourier series representation of the corresponding axial p.d.f.

3 Constructions of axial distributions

We now discuss a method for constructing axial distributions. As remarked above, it has frequently been assumed that axial data can be appropriately analyzed by assuming that, when multiplied by 2, it will have a circular distribution. It is interesting to investigate the nature of a circular distribution for Φ which will have the property that doubling its associated axial variable will lead to a circular distribution on $[0, 2\pi)$. We thus wish to identify the nature of the distribution for Φ which will be such that $2(\Phi \ mod \ \pi)$ will have a circular distribution.

To avoid confusion let us define $\Theta = \Phi(mod \ \pi)$ and $V = 2\Theta$. We will then have

$$f_\Theta(\theta) = [f_\Phi(\theta) + f_\Phi(\theta + \pi)] \quad I(0 \leq \theta < \pi) \tag{3}$$

and consequently

$$f_V(v) = \frac{1}{2}[f_\Phi(\frac{v}{2}) + f_\Phi(\frac{v}{2} + \pi)] \quad I(0 \leq v < 2\pi)$$

Remarks 2. (a) Observe that since $f_\Phi(.)$ is a circular p.d.f., the construction in (3) implies that $f_\Theta(.)$ satisfies (ii) of (1) and hence qualifies to be an axial p.d.f.

(b) The above representation of $f_\Theta(\theta)$ is clearly seen as "wrapping" of a circular distribution to yield an axial distribution. In case the original circular distribution is a wrapped distribution obtained by wrapping a linear distribution, corresponding to the linear random variable defined on $\mathbf{R}$ or $\mathbf{R}^+$, by the operator mod 2π, this resulting axial distribution can be viewed as a "doubly wrapped" distribution.

4 The axial normal distribution

4.1 *Representations of AN distribution*

The most popular circular distribution is the von Mises or circular normal density of the form

$$f_\Phi(\phi) \propto \exp(a \ sin \ \phi + b \ cos \ \phi) \quad I[0 < \phi < 2\pi) \ . \tag{4}$$

Due to the important role played by the circular normal distribution in the analysis of directional data, we now investigate the nature of a circular distribution for Φ which will have the property that doubling its associated axial variables will lead to a circular normal distribution on $(0, 2\pi]$. Thus the nature of the distribution for Φ should be such that $2(\Phi \; mod \; \pi) \sim CN(a,b)$. This leads to the following

Definition. A random variable Θ will be said to have an axial normal distribution if its density is of the form

$$f_\Theta(\theta) \propto \cosh(a \sin\theta + b\cos\theta) \quad I[0 < \theta < \pi) \tag{5}$$

where $a, b \in \mathbf{R}$. In such a case we will write

$$\Theta \sim AN(a,b) \; .$$

The genesis of such variables via the relation $\Theta = \Phi(mod \; \pi)$ where $\Phi \sim CN(a,b)$ (as in (3)), was remarked on section 1. In both (4) and (5) the admissible range of the parameters is $a \in \mathbf{R}, \; b \in \mathbf{R}$. Note that in (5)

$$\lim_{\theta\to 0^+} f_\Theta(\theta) = \lim_{\theta\to\pi^-} f_\Theta(\theta) = \cosh \; b \; .$$

It may be noted that while the circular normal distributions constitute an exponential family, such is not the case for the axial normal distributions. As a consequence, inference for axial normal data will not be as straightforward as was the case for circular normal data. For example, no data reduction via sufficiency will be possible for the axial normal case (in contrast to the circular normal case where sufficient statistics for a, b based on n i.i.d. observations will be $(\sum_{i=1}^n \cos\phi_i, \; \sum_{i=1}^n \sin\phi_i)$).

An alternative parameterization is available for the circular normal distribution (and indeed it is more commonly encountered in the literature than is the parameterization used in (4)). We will write $\Phi \sim CN(\mu, \kappa)$ if the density of Φ is of the form

$$f_\Phi(\phi) = [2\pi I_0(\kappa)]^{-1} \exp[\kappa\cos(\phi - \mu)] \quad I(0 \leq \phi < 2\pi) \tag{6}$$

for some $\mu \in [0, 2\pi)$ and $\kappa > 0$. In (6), $I_0(\cdot)$ denotes the modified Bessel function of the first kind and of order zero. It is not difficult to determine the relationship between the parameters (μ, κ) in (6) and (a, b) in (4). We have:

$$a = \kappa\sin\mu, \quad b = \kappa\cos\mu \; .$$

One advantage of the (μ, κ) parameterization is that it permits a superficially simple expression for the value of the normalizing constant required to make the density integrate to 1. We can and will borrow this parameterization for use in the axial data setting. Thus we will write $\Theta \sim AN(\mu, \kappa)$ if $\Theta = \Phi(mod \; \pi)$ where $\Phi \sim CN(\mu, \kappa)$. On using (6) in (3), the AN distribution can be directly written as :

$$f_\Theta(\theta) = [2\pi I_0(\kappa)]^{-1}[\exp(\kappa\cos(\theta - \mu)) + \exp(-\kappa\cos(\theta - \mu))] \; I(0 \leq \theta < \pi) \tag{7}$$

An alternative representation of the AN distribution similar to that in (5), in terms of (μ, κ), is given by

$$f_\Theta(\theta) = [\pi I_0(\kappa)]^{-1} \cosh\{\kappa\cos(\theta - \mu)\} \; I(0 \leq \theta < \pi) \tag{8}$$

We will find later (Section 5) that the representations in (7) and (8) are more convenient to deal with for statistical inference.

4.2 *Properties of AN distribution*

The AN distribution possesses the properties stated as:
Result 1. The AN distribution

(i) reduces to the axial uniform distribution , i.e. uniform distribution on the semi-circle $[0, \pi)$, when $\kappa = 0$
(ii) is symmetric about μ and
(iii) possesses no non-trivial sufficient statistic for its parameters (μ, κ).
(iv) For p even, $E(\cos p\Theta) = E(\cos p\Phi)$, $\quad E(\sin p\Theta) = E(\sin p\Phi)$.
Proof: It is straight forward to prove the properties (i) - (iii). We establish (iv) below.

We define for $p = 1, 2, \ldots$

$$\delta_{c,p}(\Theta) = E(\cos p\Theta), \quad \delta_{s,p}(\Theta) = E(\sin p\Theta). \tag{9}$$

In (9) we permit Θ to have support $[0, \pi)$ or $[0, 2\pi)$.

If $\Phi \sim CN(\mu, \kappa)$ then its trigonometric moments are given by :

$$\delta_{c,p}(\Phi) = A_p(\kappa) \cos \mu \tag{10}$$

$$\delta_{s,p}(\Phi) = A_p(\kappa) \sin \mu \tag{11}$$

where $A_p(\kappa) \equiv I_p(\kappa)/I_0(\kappa), I_s(\kappa)$ being the modified Bessel function of the first kind and order s. Expressions for the moment generating functions of $\sin \Phi$ and $\cos \Phi$ (where $\Phi \sim CN$) are most easily written using our a, b parameterization. Let us define the normalizing constant in terms of a, b as:

$$C(a, b) = \int_0^{2\pi} \exp[a \sin \theta + b \cos \theta] d\theta \ . \tag{12}$$

It follows readily that

$$E(e^{t \sin \Phi}) = C(a + t, b)/C(a, b) \tag{13}$$

and

$$E(e^{t \cos \Phi}) = C(a, b + t)/C(a, b) \ . \tag{14}$$

Turning to consider axial normal variables, recall that $\Theta \sim AN$ iff $\Theta = \Phi(mod\ \pi)$ where $\Phi \sim CN$. If follows that

$$\begin{aligned} E(\cos p\Theta) &= \int_0^{\pi} \cos p\theta f_\Theta(\theta) d\theta \\ &= \int_0^{\pi} \cos p\theta f_\Phi(\theta) d\theta + \int_0^{\pi} \cos p\theta f_\Phi(\theta + \pi) d\theta \\ &= \int_0^{\pi} \cos p\theta f_\Phi(\theta) d\theta + \int_{\pi}^{2\pi} \cos p(\theta - \pi) f_\Phi(\theta) d\theta \\ &= \int_0^{\pi} \cos p\theta f_\Phi(\theta) d\theta + (-1)^p \int_{\pi}^{2\pi} \cos p\theta f_\Phi(\theta) d\theta \end{aligned} \tag{15}$$

Consequently for p even,

$$E(\cos p\Theta) = E(\cos p\Phi) \ . \tag{16}$$

Analogously we may verify that for p even

$$E(\sin p\Theta) = E(\sin p\Phi)\ . \tag{17}$$

Thus the even trigonometric moments of the axial normal distribution coincide with the even trigonometric moments of the circular normal distribution, which proves (iv).

The odd trigonometric moments of Θ and Φ do not coincide. From (15) and an analogous computation for $E(\sin p\Theta)$ we have that, for p odd,

$$\begin{aligned} E(\cos p\Theta) &= \int_0^{\pi} \cos p\phi f_\Phi(\phi) d\phi \\ &\quad - \int_{\pi}^{2\pi} \cos p\phi f_\Phi(\phi) d\phi \end{aligned} \tag{18}$$

and

$$\begin{aligned} E(\sin p\Theta) &= \int_0^{\pi} \sin p\phi f_\Phi(\phi) d\phi \\ &\quad - \int_{\pi}^{2\pi} \sin p\phi f_\Phi(\phi) d\phi\ . \end{aligned} \tag{19}$$

Remarks 3. (a). It may be remarked in passing that (16) and (17) are equivalent to the statement that $2[\Phi(mod\ 2\pi)] \stackrel{d}{=} 2\Theta$, an observation which is actually obvious from the definition of $\Theta(= \Phi(mod\ \pi))$.

(b). For axial data, as is the case for circular data, trigonometric moments are useful informative statistics.

(c). The Fourier series representation for the AN distribution is obtained from the general representation (2) by using $CN(\mu, \kappa)$ as f_Φ. This yields the required a_p and b_p respectively from (16)-(17) or equivalently as in (10)-(11) for p even and from (18)-(19) for p odd.

5 Inference for axial normal data

Suppose that $\theta_1, \theta_2, \ldots, \theta_n$, is a random sample from the AN distribution parameterized either by (a, b) or by (μ, κ), whichever conveniently suits our purpose. We wish to estimate the parameters and test some important simplifying hypotheses in this model.

5.1 *Estimation of parameters*

5.1.1 Method of maximum likelihood

Maximum likelihood estimation when feasible, typically provides fairly good estimates. In order to implement maximum likelihood estimation, we will need an explicit expression for the normalizing constant in the AN density. The common density of the Θ_i's, using (5) with the (a, b) parameterization, is

$$f_\Theta(\theta; a, b) = \frac{\cosh(a \sin\theta + b\cos\theta)}{\pi I_0(\sqrt{a^2 + b^2})} \tag{20}$$

and so the log likelihood function of a sample will be

$$\begin{aligned} \ell(a,b) &= \sum_{i=1}^{n} \log(\cosh(a\sin\theta_i + b\cos\theta_i)) \\ &\quad -n\log\pi - n\log I_0(\sqrt{a^2+b^2})\ . \end{aligned} \tag{21}$$

The corresponding likelihood equations, obtained by setting $\frac{\partial}{\partial a}\ell(a,b) = 0$ and $\frac{\partial}{\partial b}\ell(a,b) = 0$, are:

$$\frac{1}{n}\sum_{i=1}^{n} \tanh(a\sin\theta_i + b\cos\theta_i)\cdot\sin\theta_i = \frac{I_0'(\sqrt{a^2+b^2})}{I_0(\sqrt{a^2+b^2})}\frac{a}{\sqrt{a^2+b^2}} \tag{22}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \tanh(a\sin\theta_i + b\cos\theta_i)\cdot\cos\theta_i = \frac{I_0'(\sqrt{a^2+b^2})}{I_0(\sqrt{a^2+b^2})}\frac{b}{\sqrt{a^2+b^2}} \tag{23}$$

The system of likelihood equations in (22) - (23) seem complicated to solve. We present now an alternative system of likelihood equations based on the representation (7), which lends itself amenable to solution conveniently. Based on (7), the likelihood equations obtained by setting $\frac{\partial}{\partial\mu}\ell(\mu,\kappa) = 0$ and $\frac{\partial}{\partial\kappa}\ell(\mu,\kappa) = 0$, are given respectively by

$$f(\mu,\kappa) = \sum_{i=1}^{n} \sin(\theta_i - \mu) - 2\sum_{i=1}^{n}[\sin(\theta_i-\mu)/\{1+\exp(2\kappa\cos(\theta_i-\mu))\}] = 0 \tag{24}$$

$$g(\mu,\kappa) = \sum_{i=1}^{n} \cos(\theta_i - \mu) - 2\sum_{i=1}^{n}[\cos(\theta_i-\mu))\{1+\exp(2\kappa\cos(\theta_i-\mu))\}] + nA(\kappa) = 0 \tag{25}$$

where $A(\kappa) = I_1(\kappa)/I_0(\kappa) \equiv I_0'(\kappa)/I_0(\kappa)$. A convenient yet reasonably good approximation for $A(\kappa)$ is given by,

$$A(\kappa) = \begin{cases} \frac{\kappa}{2}\{1 - \frac{\kappa^2}{8} + \frac{\kappa^4}{48} - \frac{11\kappa^6}{3072}\}, & \kappa \le 1 \\ 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3}, & \kappa > 1 \end{cases} \tag{26}$$

Equations (24) and (25) need to be solved iteratively for μ and κ. The Newton-Raphson method of iteration for multiple parameters can be used here. This requires all the partial derivatives of both functions f and g w.r.t. μ and κ both. These are given by

$$\begin{aligned} f_\kappa(\mu,\kappa) &= \sum_{i=1}^{n}(-\cos(\mu-\theta_i) + \frac{2\cos(\mu-\theta_i)}{1+e^{2\kappa\cos(\mu-\theta_i)}} + \frac{4e^{2\kappa\cos(\mu-\theta_i)}\kappa(\sin(\mu-\theta_i))^2}{(1+e^{2\kappa\cos(\mu-\theta_i)})^2}) \\ g_\mu(\mu,\kappa) &= \sum_{i=1}^{n}(-\sin(\mu-\theta_i) + \frac{2\sin(\mu-\theta_i)}{1+e^{2\kappa\cos(\mu-\theta_i)}} - \frac{4e^{2\kappa\cos(\mu-\theta_i)}\kappa\sin(\mu-\theta_i)\cos(\mu-\theta_i)}{(1+e^{2\kappa\cos(\mu-\theta_i)})^2}) \\ f_\kappa(\mu,\kappa) &= \sum_{i=1}^{n} -(\frac{4e^{2\kappa\cos(\mu-\theta_i)}\cos(\mu-\theta_i)\sin(\mu-\theta_i)}{(1+e^{2\kappa\cos(\mu-\theta_i)})^2}) \\ g_\kappa(\mu,\kappa) &= \sum_{i=1}^{n}(\frac{4e^{2\kappa\cos(\mu-\theta_i)}(\cos(\mu-\theta_i))^2}{(1+e^{2\kappa\cos(\mu-\theta_i)})^2}) - (nA'(\kappa)) \end{aligned}$$

For this we need $A'(\kappa)$. Again, a convenient form for it also may be used, as given by

$$A'(\kappa) = 1 - A(\kappa)/\kappa - A^2(\kappa).$$

The initial values of μ and κ required for the iteration above, may be taken as the sample mean direction $\bar{\theta}$ and $\hat{\kappa} = A^{-1}(\bar{R}), R^2 = C^2 + S^2$, the MLE of κ for a CN distribution, respectively. We have implemented this approach for the two examples presented later.

5.1.2 Method of trigonometric moments

As an alternative to maximum likelihood estimation, we may consider some form of the method of moments estimation. Since, as observed in Remark 3(a), $2\Theta \overset{d}{=} 2\Phi(mod\ 2\pi)$ and since sine and cosine functions have period 2π, it follows then that $\sin 2\Theta \overset{d}{=} \sin 2\Phi$ and $\cos 2\Theta \overset{d}{=} \cos 2\Phi$. Consequently the following two moment equations could be used to estimate (a, b) :

$$\frac{1}{n}\sum_{i=1}^{n} \sin(2\theta_i) = E(\sin 2\Phi) \tag{26}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \cos(2\theta_i) = E(\cos 2\Phi) \tag{27}$$

On differentiating $f_\Theta(\theta)$ w.r.t. μ, we have

$$(\frac{\partial}{\partial\mu}) \int f_\Theta(\theta)d\theta = -E\sin(\theta-\mu) + 2\int_0^\pi \sin(\theta-\mu)f_\Phi(\theta)d\theta = 0$$

$$E\sin(\theta-\mu) = 2\int_0^\pi \sin(\theta-\mu)f_\Phi(\theta)d\theta \tag{29}$$

Similarly differentiating $f_\Theta(\theta)$ w.r.t. κ, we have

$$E\cos(\theta-\mu) = 2\int_0^\pi \cos(\theta-\mu)f_\Phi(\theta)d\theta - A(\kappa) \tag{30}$$

So

$$\begin{aligned}
&\int_0^\pi \sin(\theta-\mu)\exp\{\kappa\cos(\theta-\mu)\}d\theta \\
= \quad & \frac{1}{\kappa}\int_{-\mu}^{\pi-\mu} \kappa\sin\theta\exp\{\kappa\cos\theta\}d\theta = \frac{1}{\kappa}[-\exp\{\kappa\cos\theta\}]|_{-\mu}^{\pi-\mu} \\
= \quad & \frac{1}{\kappa}[\exp\{\kappa\cos(-\mu)\} - \exp\{\kappa\cos(\pi-\mu)\}] \\
= \quad & \frac{1}{\kappa}[\exp\{\kappa\cos\mu\} - \exp\{\kappa(-\cos\mu)\}] \\
= \quad & \frac{1}{\kappa}[e^{\kappa\cos\mu} - e^{-\kappa\cos\mu}] \equiv U(\mu,\kappa), \text{ say.}
\end{aligned}$$

So,

$$E\sin(\theta-\mu)=\frac{1}{\pi}\frac{1}{I_0(\kappa)}U(\mu,\kappa)$$

$$\Rightarrow \sum_{i=1}^{n}\sin(\theta_i-\hat{\mu})=\frac{n}{\pi I_0(\hat{\kappa})}U(\hat{\mu},\hat{\kappa})$$

$$\Rightarrow [S\cos\hat{\mu}-C\ \sin\hat{\mu}][\pi I_0(\hat{\kappa})]=nU(\hat{\mu},\hat{\kappa}), \tag{31}$$

where, $S=\sum_{i=1}^{n}\sin\theta_i,\ C=\sum_{i=1}^{n}\cos\theta_i$. Also, define

$$N(\kappa,\mu)\equiv\frac{1}{\pi}\int_0^{\pi}\exp[\kappa\cos(\theta-\mu)]d\theta$$

Note that $N(\kappa,0)=I_0(\kappa)$ and $N'(\kappa,0)=I_1(\kappa)$. $N(\cdot)$ and $N'(\cdot)$ are to be obtained by numerical integration using, e.g. Gaussian quadrature. Then

$$\int_0^{\pi}\cos(\theta-\mu)\exp[\kappa\cos(\theta-\mu)]d\theta=\pi N'(\kappa,\mu)$$

So, using (30) we get,

$$E\cos(\theta-\mu)=\frac{1}{\pi}\frac{1}{I_0(\kappa)}\pi N'(\kappa,\mu)-A(\kappa)$$

$$\Rightarrow [(C\cos\hat{\mu}+S\sin\hat{\mu}][I_0(\hat{\kappa})]=n[N'(\hat{\kappa},\hat{\mu})-I_1(\hat{\kappa})] \tag{32}$$

Thus it is left only to solve (31) and (32) iteratively for $\mu(0\le\mu<\pi)$ and $\kappa(>0)$.

The method of trigonometric moments for axial data are thus seen to be more complicated, unlike for the linear and circular data, than the method of maximum likelihood.

5.2 *Testing of hypotheses*

We consider in this section two important problems related to axial data and our model (4). We may also wish to test whether the AN model for the Θ_i's is preferable to an assumption that each $2\Theta_i$ has a CN distribution - this hypotheses, however, will be dealt with elsewhere. As with circular data, test for isotropy or equivalently for uniformity is the first and fundamental test required for axial data. The second important testing problem is that of testing for a specified direction of symmetry. Both these problems pose interesting situations. In the first problem, the nuisance parameter μ appears only under the alternative, which is a somewhat non-standard situation. In the second problem, no usual reduction, e.g. through sufficiency, invariance or ancillarity can be achieved, since the nuisance parameter κ is a non location-scale parameter. Also, note that there does not exist any non-trivial sufficient statistic for (μ,κ) appearing in model (1). Below we present statistical tests for the above two testing problems.

5.2.1 Tests for axial uniformity

The null hypothesis of axial isotropy translates to the multiparameter hypothesis H_0: a=0 and b=0, in terms of the parameters a and b defining the AN distribution in the form given in (4). The optimal locally most mean powerful unbiased (LMMPU) test can be derived here following the construction given in SenGupta and Vermeire (1986). However, on using property (i) and expression (4), this multi-parameter hypothesis can be stated equivalently in terms of the much simpler one-parameter hypothesis $H_0 : \kappa = 0$. Note that H_0 specifies the value of the parameter κ on the boundary. Further, the "nuisance" parameter μ is absent under H_0 and is present only under the alternative. Thus, the construction of an exact optimal test seems to pose problems - though, see e.g. SenGupta (1991) and SenGupta and Pal (2001) for methods to deal with such situations. However, a likelihood ratio test may be conveniently invoked here, since as we have seen in Secn. 5.1.1, the (unconstrained) MLEs of μ and κ can be computed without much difficulty. Under H_0, no parameter appears and hence (maximum likelihood) estimation is not called for. The critical region becomes:

$$\omega : \quad T_1 \equiv -n \log I_0(\hat{\kappa}) + \sum_{i=1}^{n} \log \cosh[\hat{\kappa} \cos(\theta_i - \hat{\mu})] > K_1,$$

where $\hat{\kappa}$ and $\hat{\mu}$ are the m.l.e.s of μ and κ respectively, given in Section 5.1.1 above, and K_1 is the cut-off point to be determined from the specified size of the test. T_1 is easy to compute. But the exact distribution of T_1 is non-trivial to derive. Further, care needs to be taken in deriving even its large-sample distribution. The standard result on the asymptotic χ^2 distribution of the -2 log likelihood ratio statistics need not hold, since here we are faced with the non-regular situation of testing for the value of the parameter (κ) on the boundary, and not for an interior point, of the parameter space. However, the exact cut-off points can be conveniently determined through simulation since the generation of the necessary random variables under H_0, i.e. from a Uniform distribution on $[0, \pi)$, can be trivially done.

5.2.2 Tests for specified direction of symmetry

For testing $H_0 : \mu = \mu_0$, the likelihood ratio test may be adopted. The unconstrained m.l.e.s $\hat{\kappa}$ and $\hat{\mu}$ of μ and κ are available again from (24) and (25) of Section 5.1.1. The m.l.e. $\hat{\kappa}_0$ of κ under H_0 is obtained by solving (24) with μ simply replaced by μ_0. The test then has the critical region:

$$\omega : \; T_2 \equiv \sum_{i=1}^{n} [\log \cosh[\hat{\kappa}_0 \cos(\theta_i - \mu_0) - n \log I_0(\hat{\kappa}_0)] - [\log \cosh[\hat{\kappa} \cos(\theta_i - \hat{\mu}) - n \log I_0(\hat{\kappa})] \; < \; K_2,$$

where the cut-off point K_2 is to be determind from the specified size condition. The exact distribution of T_2 is complicated even under H_0 and the cut-off points are to be obtained by simulation. However, in large samples, $-2T_2$ follows a χ^2 distribution with 1 d.f., and this yields K_2 easily.

6 Bivariate axial distributions

Related sets of axial data are frequently encountered. It is thus of interest to consider bivariate distributions (eventually multivariate distributions) with marginals of the AN type. Alternatively, if we think that conditional densities might be more easily visualized than marginal densities, we might consider distributions with conditional densities of the AN type. Arguments in favor of such conditional specification may be found in Arnold, Castillo and Sarabia (1999).

6.1 Distributions with AN marginals and conditionals

Consider bivariate distributions of two axial random variables Θ and Λ with support on $[0, \pi) \times [0, \pi)$. Joint densities of Θ and Λ which are desired to have AN marginals can be constructed in the usual way as done for bivariate distributions on R^2. That is, we may appeal to the well-known methods by Morgerstern, Gumbel or Frechet.

Now consider the construction of bivariate axial distributions for which it is desired to have AN conditionals. Consider bivariate axial densities of the form:

$$\begin{aligned} f_{\Theta,\Lambda}(\theta, \lambda) \quad \propto \quad & \cosh(a_{11} \sin\theta \sin\lambda \quad + \quad a_{12} \sin\theta \cos\lambda \\ & + \; a_{21} \cos\theta \sin\lambda \; + \; a_{22} \cos\theta \cos\lambda) \; I(0 \leq \theta < \pi, 0 \leq \lambda < \pi) \; . \end{aligned} \tag{33}$$

Here $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbf{R}$. It is obvious that densities of the form (33) have axial normal conditionals. Thus for each $\lambda \in [0, \pi)$,

$$\Theta | (\Lambda = \lambda) \sim AN(a_{11} \sin\lambda + a_{12} \cos\lambda, a_{21} \sin\lambda + a_{22} \cos\lambda) \; .$$

Analogously for each θ, Λ given $\Theta = \theta$ has an AN distribution. One negative feature of the model (33) is that it does not include any case with independent marginals. The model does include the axial uniform (AU) distribution as a special case: $a_{11} = a_{12} = a_{21} = a_{22} = 0$.

6.2 A variant form of the AN conditionals model

For axial data we have noted that it is often plausible to require that the density satisfy:

$$\lim_{\theta \to 0^+} f(\theta) = \lim_{\theta \to \pi^-} f(\theta) \; .$$

If we are willing to forego this requirement, we may consider an extended $3-$ parameter form of axial normal family with densities of the form

$$f(\theta; a, b, c) \propto \cosh(a \sin\theta + b \cos\theta + c) \;\; I(0 \leq \theta < \pi) \; . \tag{34}$$

If Θ has density given by (34), we can write $\Theta \sim AN^*(a, b, c)$, where the star is used to differentiate this extended family from the usual AN family. A flexible family of bivariate densities with conditionals in this extended AN family (34), is given by:

$$f(\theta, \lambda; A) \propto \cosh((1, \sin\theta, \cos\theta) A (1, \sin\lambda, \cos\lambda)') \;\; I(\theta, \lambda \in [0, \pi)) \tag{35}$$

where A is a 3×3 matrix of parameters. It is obvious that the conditional distributions for (35) are all in the AN^* family (34). The model (35) of course includes the AN-conditionals model (33) and also includes a model in which Θ and Λ have independent AN^* distributions and, as a further specialization, a model with independent AN marginals.

6.3 Other bivariate axial distributions

An alternative approach to the construction of bivariate axial models is to begin with a suitable model for a bivariate circular variable (Φ, Ψ) (see e.g., SenGupta, 2004) and then define a related bivariate axial variable (Θ, Λ) by

$$\Theta \;=\; \Phi(mod\ \pi), \quad \Lambda \;=\; \Psi(mod\ \pi)\ . \tag{36}$$

For example, we could begin with a bivariate circular normal conditionals distribution of the form discussed extensively in Arnold and SenGupta (2004),

$$f_{\Phi,\Psi}(\phi,\psi) = \exp\{(1, \sin\phi, \cos\phi)A(1, \sin\psi, \cos\psi)'\} \quad I(\phi, \psi \in [0, 2\pi))\ . \tag{37}$$

If we define (Θ, Λ) using (36) we will have

$$f_{\Theta,\Lambda}(\theta,\lambda) \;=\; f_{\Phi,\Psi}(\theta,\lambda) + f_{\Phi,\Psi}(\theta,\lambda+\pi) + f_{\Phi,\Psi}(\theta+\pi,\lambda) + f_{\Phi,\Psi}(\theta+\pi,\lambda+\pi)\ . \tag{38}$$

Such a distribution, though it may be a reasonable model, fails to have AN marginals or conditionals.

7 Examples

We now illustrate our approach with two real-life examples. The first one refers to measurements of long-axis orientations of 60 feldspar laths in basalt. The data are listed in degrees in Appendix B5 ("164" there is a typo to be read as 60) of Fisher (1993). The raw plot of the data is exhibited in Figure 1. The initial values of μ and κ were obtained (as explained earlier) as 80.65 and 1.10 respectively. These led to the maximum likelihood estimates as $(\hat{\mu}, \hat{\kappa})$ = (108.9637, 1.0838). The low value of κ may be indicative of axial uniformity, a hypothesis which was deemed to be of interest here. A formal test for uniformity may be conducted using the likelihood ratio test derived in Section 5.2.1. The fitted AN distribution is shown in Figure 3.

The second example refers to 63 measurements of median directions of face-cleats taken at 20-metre intervals along a tunnel in Wallsend Borehole Colliery, NSW, Australia. Changes in the (median) directions of face-cleats are indicators of possible hazardous mining conditions ahead. The data are listed in degrees in Appendix B22 of Fisher (1993). The raw plot of the data is exhibited in Figure 2. The initial value of μ was obtained as 90.28 and that of κ was taken as 1. These led to the maximum likelihood estimates as $(\hat{\mu}, \hat{\kappa})$ = (89.9859, 10.9050). Since a change in the direction of face-cleat is important here, an initial test may be formulated for the discrepancy of the underlying direction of symmetry from a specified value. This boils down to a test for μ which may be conducted using the likelihood ratio test given in Section 5.2.2. The fitted AN distribution is shown in Figure 4.

The raw plots of the two data sets and the initial values of μ and of κ were obtained using DDSTAP (SenGupta, 1998), a statistical package for the analysis of directional data.

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Biographical sketches

Prof. Barry C. Arnold is Distinguished Professor of Statistics at the University of California, Riverside. He received his Ph.D. from Stanford University in 1965 and subsequently was a Professor of Mathematics and Statistics at Iowa State University for 14 years before moving to Riverside. His recent research has focused on multivariate models (especially those involving conditional specification and hidden truncation) and order statistics. He is a Fellow of AAAS, ASA and IMS and an elected member of ISI.

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Figure 1. Raw plot of Feldspar data

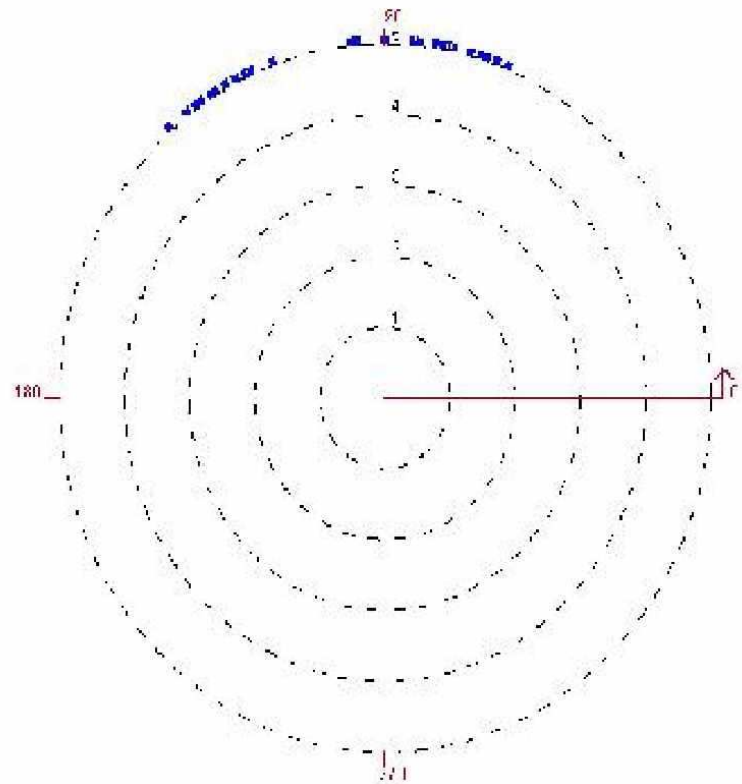
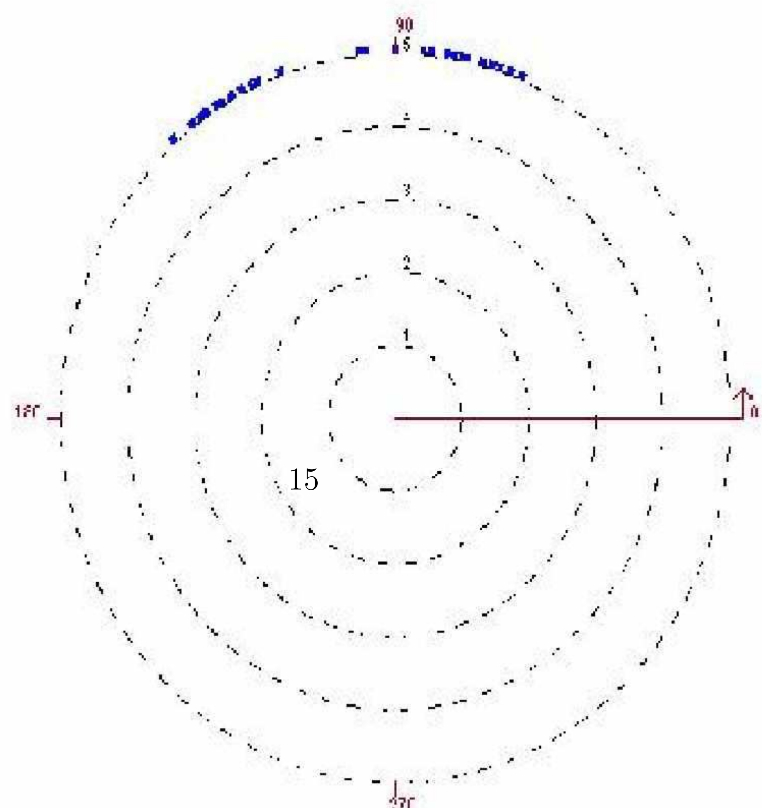


Figure 2. Raw plot of Face-cleat data



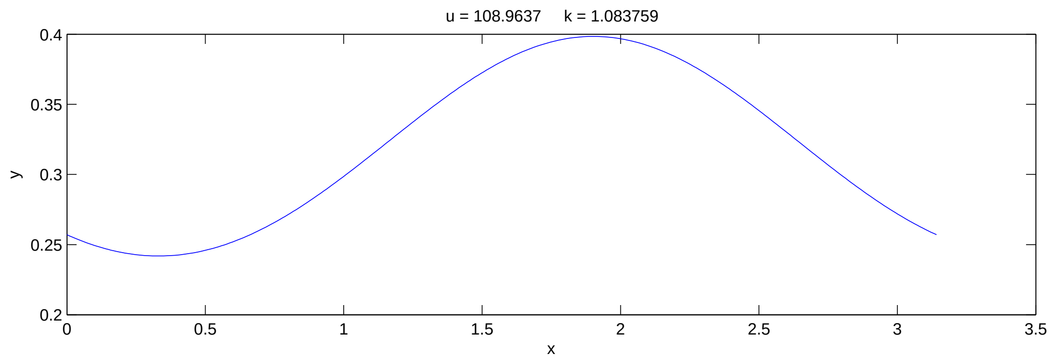


Figure 3. Fitted p.d.f of AN(1.9018,1.0838) for Feldspar data

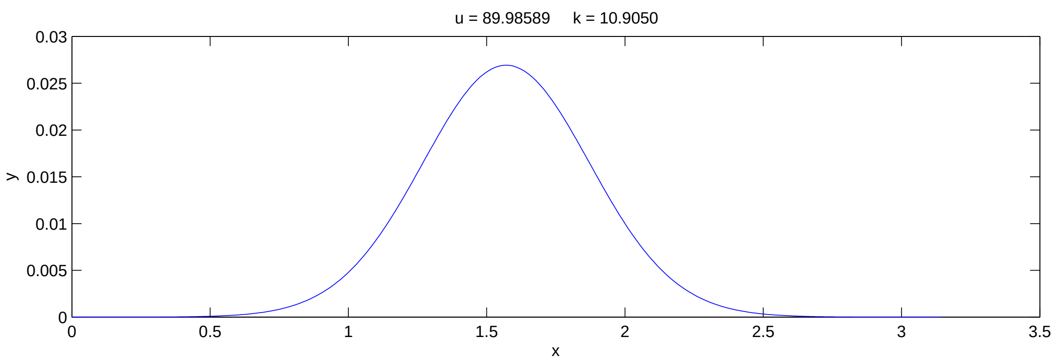


Figure 4. Fitted p.d.f of AN(1.5706,10.9050) for Face−cleat data