CHAPTER 13

Bagging Binary and Quantile Predictors for Time Series: Further Issues

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Abstract

Bagging (bootstrap aggregating) is a smoothing method to improve predictive ability under the presence of parameter estimation uncertainty and model uncertainty. In Lee and Yang (2006), we examined how (equal-weighted and BMA-weighted) bagging works for one-step-ahead binary prediction with an asymmetric cost function for time series, where we considered simple cases with particular choices of a linlin tick loss function and an algorithm to estimate a linear quantile regression model. In the present chapter, we examine how bagging predictors work with different aggregating (averaging) schemes, for multi-step forecast horizons, with a general class of tick loss functions, with different estimation algorithms, for nonlinear quantile regression models, and for different data frequencies. Bagging quantile predictors are constructed via (weighted) averaging over predictors trained on bootstrapped training samples, and bagging binary predictors are conducted via (majority) voting on predictors trained on the bootstrapped training samples. We find that median bagging and trimmed-mean bagging can alleviate the problem of extreme predictors from bootstrap samples and have better performance than equally weighted bagging predictors; that bagging works better at longer forecast horizons; that bagging works well with highly nonlinear quantile regression models (e.g., artificial neural network), and with general tick loss functions. We also find that the performance of bagging may be affected by using different quantile estimation algorithms (in small samples, even if the estimation is consistent) and by using different frequencies of time series data.

Keywords: Algorithm, bagging, median bagging, binary prediction, frequency, majority voting, multi-step prediction, neural network, quantile prediction, time series

JEL classifications: C3, C5, G0

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1. Introduction

To improve on unstable forecasts, *b*ootstrap *agg*regat*ing* or bagging is introduced by Breiman (1996). In Lee and Yang (2006), we show how bagging, with equal-weight averaging and weighted averaging using Bayesian model averaging (BMA) methods, works for one-step-ahead binary prediction under an asymmetric cost function for time series. In that paper, we considered simple cases with particular choices of a loss function (linlin) and a regression model (linear).

We now consider the following extensions: (a) aggregating the bootstrap forecasts by other combination schemes as considered, e.g., by Stock and Watson (1999) and Timmermann (2006); (b) multi-step forecasts; (c) nonlinear models, such as the neural network quantile model of White (1992); (d) different quantile estimation algorithms, as discussed by Komunjer (2005); (e) a general class of the tick loss functions of Komunjer (2005) and Komunjer and Vuong (2005); and (f) using other macroeconomic and financial time series sample at various frequencies.

According to our experience in Monte Carlo and empirical experiments, some bootstrap predictors may generate extreme values that will seriously worsen the forecasts of equally weighted bagging predictors. To alleviate this problem of extreme forecasts, we consider alternative averaging schemes to generate bagging predictors (an idea borrowed from the forecast combination literature). The first is BMA-weighted bagging, as used in Lee and Yang (2006). The second is trimmed bagging, for which we remove extreme bootstrap forecasts in forming a bagging predictor. However, it will be very hard to decide which bootstrap predictors to keep and which to discard beforehand. In this chapter, we simply trim a certain number of the largest and the smallest bootstrap predictors. We also use the median of the bootstrap predictors as our bagging predictor, which can be considered as an extreme case of trimmed bagging predictors. Hence, we consider the equal-weighted, BMA-weighted, trimmed-mean, and median bagging. Our Monte Carlo and empirical experiments show the following: when the sample size is small and/or predictors lie on the sparse parts of the density, median bagging and trimmed-mean bagging generally give better bagging forecasts than the equal-weighted bagging (which is better than unbagged predictors); when sample size is large and/or the predictor lies on the dense part of the data density, median bagging and trimmed bagging have no obvious advantage over equal-weighted bagging (whose advantage over unbagged predictors is also weak in such a case).

We also explore the performance of bagging predictors for multi-step forecasts (for the conditional quantile) in this chapter. As discussed by Brown and Mariano (1989) and Lin and Granger (1994), there are several ways to generate multi-step forecasts. These methods can be put into two groups: iteration of one-step-ahead forecasts and direct multi-step forecasts. Among iterated multi-step forecasting methods, we can further classify them as the naïve, exact, Monte Carlo, and bootstrap methods. If the true forecast model is linear and known, all these methods should give the same predictions. However, if the true forecast model is nonlinear or unknown, different multi-step forecasting methods give quite different predictions. We use the direct multi-step forecast method for the conditional quantile prediction in our Monte Carlo experiments. It is found that, compared with unbagged predictors, the performance of a bagging predictor tends to improve at longer forecast horizons.

Lee and Yang (2006) attributed a part of success of bagging predictors to small-sample estimation uncertainties. Therefore, a question that may arise is whether the good performance of bagging predictors critically depends on the algorithms employed in estimation. Lee and Yang (2006) used the interior point algorithm for quantile estimation as suggested by Portnoy and Koenker (1997). To examine how other algorithms may work for bagging, we also use the minimax algorithm of Komunjer (2005) in this chapter. The interior point algorithm for quantile estimation can be used for a linear quantile regression model under the standard linlin tick loss function, while the minimax algorithm allows flexible functional forms for quantile regressions, such as a neural network model.

We use the minimax algorithm to estimate linear and nonlinear quantile regression models under a general class of tick functions, namely, the tick-exponential family defined by Komunjer (2005). Our simulation results show that the bagging works (i.e., better than unbagged predictors) for quantiles almost equally well for the different tick functions in the tick-exponential family in small samples. Komunjer (2005) shows that QMLE under the tick-exponential family is consistent.

With the flexibility provided by the minimax algorithm, we check the performance of bagging predictors on highly nonlinear quantile regression models – artificial neural network models. When the sample size is limited, it is usually hard to choose the number of hidden nodes and the number of inputs (lags) and to estimate the large number of parameters in neural network models. Therefore, a neural network model can generate poor predictions with a small sample. In such cases, bagging can do an excellent job of improving forecasting performance, as shown in our empirical experiments.

We finally investigate whether the performance of bagging can be affected by the frequency of the data.

The plan of this chapter is as follows. Section 2 gives a brief introduction to bagging predictors. Section 3 explains different ways to aggregating bootstrap predictors. In Section 4, we examine how bagging works for multi-step predictions of conditional quantiles. In Section 5, we examine how bagging works for quantile prediction under the different tick loss functions of the tick-exponential family. In Section 6, we consider whether the performance of bagging predictor will be affected by different estimation algorithms. In Section 7, we examine bagging predictors on (nonlinear) neural network quantile regression models. Section 8 examines the effect of different data frequencies on the bagging performance of bagging. In Section 9, we discuss a potential extension with pretesting for bagging. Section 10 summarizes what we have learned in this chapter and provides concluding comments.

2. What is bagging?

A bagging predictor is a combined predictor formed over a set of training sets to smooth out the "instability" caused by parameter estimation uncertainty and model uncertainty. A predictor is said to be "unstable" if a small change in the training set will lead to a significant change in the predictor (Breiman, 1996). In this section, we will show how a bagging predictor may improve the predicting performance of its underlying predictor. Let

$$\mathcal{D}_t \equiv \left\{(Y_s, \mathbf{X}_{s-1})\right\}_{s=t-R+1}^{t} \quad (t = R, \ldots, T)$$

be a training set at time t and let $\varphi(\mathbf{X}_t, \mathcal{D}_t)$ be a forecast of Y_{t+1} or of the binary variable $G_{t+1} \equiv \mathbf{1}(Y_{t+1} \geqslant 0)$ using this training set $\mathcal{D}_t$ and the explanatory variable vector $\mathbf{X}_t$. The optimal forecast $\varphi(\mathbf{X}_t, \mathcal{D}_t)$ for Y_{t+1} will be the conditional mean of Y_{t+1} given $\mathbf{X}_t$ if we have the squared error loss function or the conditional quantile of Y_{t+1} on $\mathbf{X}_t$ if the loss is a tick function. Below, we also consider the binary forecast for $G_{t+1} \equiv \mathbf{1}(Y_{t+1} \geqslant 0)$.

Suppose each training set $\mathcal{D}_t$ consists of R observations generated from the underlying probability distribution $\mathbf{P}$. The forecast $\{\varphi(\mathbf{X}_t, \mathcal{D}_t)\}_{t=R}^{T}$ can be improved if more training sets were able to be generated from $\mathbf{P}$ and the forecast can be formed from averaging the multiple forecasts obtained from the multiple training sets. Ideally, if $\mathbf{P}$ were known and multiple training sets $\mathcal{D}_t^{(j)}$ $(j = 1, \ldots, J)$ may be drawn from $\mathbf{P}$, an ensemble aggregating predictor $\varphi_A(\mathbf{X}_t)$ can be constructed by the weighted averaging of $\varphi(\mathbf{X}_t, \mathcal{D}_t^{(j)})$ over j, i.e.,

$$\varphi_A(\mathbf{X}_t) \equiv \mathbb{E}_{\mathcal{D}_t}\varphi(\mathbf{X}_t, \mathcal{D}_t) \equiv \sum_{j=1}^{J} w_{j,t}\varphi\left(\mathbf{X}_t, \mathcal{D}_t^{(j)}\right), \tag{1}$$

where $\mathbb{E}_{\mathcal{D}_t}(\cdot)$ denotes the expectation over $\mathbf{P}$, $w_{j,t}$ is the weight function with $\sum_{j=1}^{J} w_{j,t} = 1$, and the subscript A in φ_A denotes "aggregation."

Lee and Yang (2006, Propositions 1 and 4) show that the ensemble aggregating predictor $\varphi_A(X_t)$ has no larger expected loss than the original predictor $\varphi(X_t, \mathcal{D}_t)$. For any convex loss function $c(\cdot)$ on the forecast error z_{t+1}, we will have

$$\mathbb{E}_{\mathcal{D}_t, Y_{t+1}, \mathbf{X}_t} c(z_{t+1}) \geqslant \mathbb{E}_{Y_{t+1}, \mathbf{X}_t} c\left(\mathbb{E}_{\mathcal{D}_t}(z_{t+1})\right),$$

where $\mathbb{E}_{\mathcal{D}_t}(z_{t+1})$ is the aggregating forecast error and $\mathbb{E}_{\mathcal{D}_t, Y_{t+1}, \mathbf{X}_t}(\cdot) \equiv \mathbb{E}_{\mathbf{X}_t}[\mathbb{E}_{Y_{t+1}|\mathbf{X}_t}\{\mathbb{E}_{\mathcal{D}_t}(\cdot)|X_t\}]$ denotes the expectation $\mathbb{E}_{\mathcal{D}_t}(\cdot)$ taken over $\mathbf{P}$ (i.e., averaging over the multiple training sets generated from $\mathbf{P}$), then taking an expectation of Y_{t+1} conditioning on X_t, and then taking an expectation of X_t. Similarly, we define the notation $\mathbb{E}_{Y_{t+1}, \mathbf{X}_t}(\cdot) \equiv \mathbb{E}_{\mathbf{X}_t}[\mathbb{E}_{Y_{t+1}|\mathbf{X}_t}(\cdot)|X_t]$. Therefore, the aggregating predictor will always have no larger expected cost than the original predictor for a convex loss function $\varphi(X_t, D_t)$. Examples of the convex loss function includes the squared error loss and a tick (or check) loss of Koenker and Basset (1978),

$$\rho_\alpha(z) \equiv \left[\alpha - \mathbf{1}(z < 0)\right]z. \tag{2}$$

How much improvement the aggregating predictor can improve depends on the distance between $\mathbb{E}_{\mathcal{D}_t, Y_{t+1}, \mathbf{X}_t} c(z_{t+1})$ and $\mathbb{E}_{Y_{t+1}, \mathbf{X}_t} c(\mathbb{E}_{\mathcal{D}_t}(z_{t+1}))$. We can define this distance by $\Delta \equiv \mathbb{E}_{\mathcal{D}_t, Y_{t+1}, \mathbf{X}_t} c(z_{t+1}) - \mathbb{E}_{Y_{t+1}, \mathbf{X}_t} c(\mathbb{E}_{\mathcal{D}_t}(z_{t+1}))$. Therefore, the effectiveness of the aggregating predictor depends on the *convexity* of the cost function. The more convex the cost function, the more effective the aggregating predictor can be. We will see the effect of the convexity on the performance of bagging later in this chapter (Section 6). If the loss function is squared error loss, then it can be shown that $\Delta = \mathbb{V}_{\mathcal{D}_t}[\varphi(\mathbf{X}_t, \mathcal{D}_t)]$ is the variance of the predictor, which measures the "instability" of the predictor; see Lee and Yang (2006, Proposition 1) and Breiman (1996). If the loss is the tick function, the effectiveness of bagging is also different for different quantile predictions: bagging works better for tail-quantile prediction than for mid-quantile prediction.

In practice, however, $\mathbf{P}$ is not known. In this case, we may estimate $\mathbf{P}$ by its empirical distribution, $\hat{\mathbf{P}}(\mathcal{D}_t)$, for a given $\mathcal{D}_t$. Then, from the empirical distribution $\hat{\mathbf{P}}(\mathcal{D}_t)$, multiple training sets may be drawn by the bootstrap method. Bagging predictors, $\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*)$, can then be computed by taking weighted average of the predictors trained over a set of bootstrap training sets. More specifically, the bagging predictor $\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*)$ can be obtained by the following steps:

1. Given a training set of data at time t, $\mathcal{D}_t \equiv \{(Y_s, \mathbf{X}_{s-1})\}_{s=t-R+1}^t$, construct the jth bootstrap sample $\mathcal{D}_t^{*(j)} \equiv \{(Y_s^{*(j)}, \mathbf{X}_{s-1}^{*(j)})\}_{s=t-R+1}^t$, $j = 1, \ldots, J$, according to the empirical distribution of $\hat{\mathbf{P}}(\mathcal{D}_t)$ of $\mathcal{D}_t$.
2. Train the model (estimate parameters) from the jth bootstrapped sample $\mathcal{D}_t^{*(j)}$.
3. Compute the bootstrap predictor $\varphi^{*(j)}(\mathbf{X}_t, \mathcal{D}_t^{*(j)})$ from the jth bootstrapped sample $\mathcal{D}_t^{*(j)}$.
4. Finally, for mean and quantile forecasts, the bagging predictor $\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*)$ can be constructed by averaging over J bootstrap predictors:

$$\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*) \equiv \sum_{j=1}^{J} \hat{w}_{j,t} \varphi^{*(j)}(\mathbf{X}_t, \mathcal{D}_t^{*(j)});$$

for binary forecasts, the bagging binary predictor $\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*)$ can be constructed by majority voting over J bootstrap predictors:

$$\varphi^B(\mathbf{X}_t, \mathcal{D}_t^*) \equiv \mathbf{1}\left(\sum_{j=1}^{J} \hat{w}_{j,t} \varphi^{*(j)}(\mathbf{X}_t, \mathcal{D}_t^{*(j)}) > 1/2\right),$$

with $\sum_{j=1}^{J} \hat{w}_{j,t} = 1$ in both cases.

One concern with applying bagging to time series is whether a bootstrap can provide a sound simulation sample for dependent data, for which the bootstrap is required to be consistent. It has been shown that some bootstrap procedure (such as the moving block bootstrap) can provide consistent densities for moment estimators and quantile estimators; see, e.g., Fitzenberger (1997).

3. Bagging with different averaging schemes

There are several ways to generate the averaging weight $\hat{w}_{j,t}$ for bagging predictors introduced in the previous section. The most commonly used one is equal-weighting across all bootstrap samples, i.e., $\hat{w}_{j,t} = 1/J$, $j = 1, \ldots, J$. However, one problem with equal weighted bagging is that some bootstrap samples could (and typically do) make extreme forecasts. Possible sources of these extreme forecasts include random procedures of generating bootstrap samples (especially from small samples), difficulties arising from multiple local optima for nonlinear models, and estimation difficulties for non-differentiable loss functions. In these cases, we may get some erratic values for the predictive parameter $\hat{\boldsymbol{\beta}}_t^{*(j)}(\mathcal{D}_t^{*(j)})$ and hence "crazy" bootstrap predictors $\varphi^{*(j)}(\mathbf{X}_t, \mathcal{D}_t^{*(j)})$. The extreme forecasts may happen more frequently for conditional quantile predictions than for conditional mean predictions. The effect may be large, so that such crazy bootstrap sample predictors may deteriorate performance of bagging predictors. By finding a way to alleviate or eliminate the effect of such crazy bootstrap predictors, we may improve bagging predictors.

We consider several ways to solve these extreme forecast problems. One is to estimate the combination weight based on in-sample performance of each predictor, for example, using Bayesian model averaging (BMA) weighting. By setting

$$\hat{w}_{j,t} \equiv \Pr\left[\hat{\boldsymbol{\beta}}_\alpha\left(\mathcal{D}_t^{*(j)}\right)|\mathcal{D}_t\right], \quad j = 1, \ldots, J,$$

a bootstrap predictor with better in-sample performance will be assigned a larger weight. Extreme-valued predictors are generated when parameters in the forecasting model are poorly estimated for bootstrap samples, in which case it is expected that the in-sample performance of the bootstrap estimators will not be good either. Therefore, by assigning the weights according to the in-sample performance, BMA-bagging predictors can alleviate the extreme-valued predictor problem to a certain extent. However, BMA-bagging predictors still put some positive weight on the extreme value predictors and thus does not completely eliminate the effects of crazy forecasts.

Another way to deal with these extreme value predictors is to sort all the bootstrap predictors and trim a certain number of bootstrap predictors from both tails before the averaging procedure. This procedure will be called the trimmed bagging. The user can decide the number of bootstrap predictors to trim depending on the seriousness of the extreme value predictors problem. However, it is hard to decide *a priori*, and thus in our Monte Carlo and empirical analysis, we choose to trim a fixed number (e.g., 5 and 10) of bootstrap predictors on each tail of the sorted bootstrap predictors without checking whether they are extreme or not.

Alternatively, we can simply use the median of the bootstrap predictors (instead of the mean or trimmed mean of the bootstrap predictors), which is the extreme case of trimmed bagging that uses only the middle one or two bootstrap predictors. In median bagging, we can avoid the arbitrary choice of how many bootstrap predictors are to be discarded in the trimmed bagging predictor.

We use a set of Monte Carlo simulation to gain further insights on how these different bootstrap aggregating weighting schemes work. For quantile predictions, we obtain the out-of-sample mean loss values for the unbagged predictors with $J = 1$ (S_1) and for bagging predictors with $J = 50$ (S_a, $a \geqslant 2$). We consider nine quantile levels with left tail probability $\alpha = 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9, 0.95$, and 0.99. It will be said that bagging "works" if $S_1 > S_a$. To rule out the chance of pure luck by a certain criterion, we compute the following three summary performance statistics from 100 Monte Carlo replications ($r = 1, \ldots, 100$):

$$T_{1,a} \equiv \frac{1}{100}\sum_{r=1}^{100} S_a^r,$$

$$T_{2,a} \equiv \left(\frac{1}{100}\sum_{r=1}^{100}\left(S_a^r - T_{1,a}\right)^2\right)^{1/2},$$

$$T_{3,a} \equiv \frac{1}{100}\sum_{r=1}^{100}\mathbf{1}\left(S_1^r > S_a^r\right),$$

where $a = 1$ for the non-bagged predictor ($J = 1$) and $a \geqslant 2$ for various bagging predictors with different weighting (equally weighted, BMA, median, and trimmed mean). T_1 measures the Monte Carlo mean of the out-of-sample mean loss, T_2 measures the Monte Carlo standard deviation of the out-of-sample mean loss, T_3 measures the Monte Carlo frequency that bagging works. We present T_1, T_2, and T_3 in Tables 1A–1F. To make the comparison of the bagging predictors and unbagged predictors easier, we also report two *relative* performance statistics (Figure 1, panels (a)–(f)): $T_{1,a}/T_{1,1}$ and $T_{2,a}/T_{2,1}$. For both,

Table 1A. AR(0)-ARCH(1)-Gaussian. Bagging quantile predictions for AR-ARCH models

		$R = 200$							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.01$	T_1	2.92	2.62	2.61	2.61	2.62	2.55	2.60	2.57
	T_2	1.04	0.72	0.70	0.70	0.72	0.85	0.72	0.71
	T_3		0.77	0.78	0.78	0.78	0.85	0.78	0.81

(*continued on next page*)

Table 1A. (continued)

		R = 200							
		$J=1$	$J=50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha=0.05$	T_1	9.97	9.60	9.60	9.60	9.60	9.57	9.58	9.57
	T_2	1.83	1.61	1.62	1.61	1.61	1.70	1.62	1.64
	T_3		0.83	0.82	0.83	0.83	0.92	0.85	0.87
$\alpha=0.10$	T_1	16.73	16.30	16.30	16.30	16.30	16.23	16.28	16.26
	T_2	2.73	2.50	2.51	2.50	2.50	2.50	2.50	2.49
	T_3		0.81	0.82	0.83	0.81	0.86	0.83	0.85
$\alpha=0.30$	T_1	32.86	32.27	32.31	32.27	32.27	32.25	32.27	32.27
	T_2	4.51	4.27	4.31	4.28	4.27	4.26	4.27	4.27
	T_3		0.88	0.89	0.88	0.88	0.89	0.90	0.91
$\alpha=0.50$	T_1	37.53	36.76	36.77	36.76	36.76	36.76	36.76	36.76
	T_2	4.90	4.74	4.75	4.75	4.74	4.72	4.73	4.72
	T_3		0.90	0.92	0.91	0.90	0.91	0.91	0.90
$\alpha=0.70$	T_1	32.70	32.01	32.05	32.02	32.01	32.02	32.02	32.02
	T_2	4.42	4.20	4.24	4.21	4.20	4.19	4.20	4.20
	T_3		0.92	0.93	0.92	0.92	0.94	0.91	0.93
$\alpha=0.90$	T_1	16.86	16.36	16.36	16.36	16.36	16.35	16.35	16.35
	T_2	2.58	2.36	2.37	2.36	2.36	2.37	2.36	2.36
	T_3		0.88	0.89	0.89	0.88	0.90	0.92	0.93
$\alpha=0.95$	T_1	10.12	9.72	9.72	9.72	9.72	9.66	9.70	9.68
	T_2	1.74	1.53	1.53	1.53	1.53	1.51	1.52	1.51
	T_3		0.87	0.85	0.87	0.86	0.90	0.88	0.88
$\alpha=0.99$	T_1	2.96	2.63	2.62	2.63	2.63	2.61	2.64	2.63
	T_2	1.05	0.69	0.68	0.69	0.69	0.78	0.72	0.73
	T_3		0.74	0.77	0.75	0.74	0.83	0.81	0.80
		R = 500							
$\alpha=0.01$	T_1	2.64	2.57	2.57	2.57	2.57	2.57	2.57	2.57
	T_2	0.57	0.46	0.47	0.47	0.46	0.50	0.47	0.48
	T_3		0.58	0.58	0.58	0.58	0.61	0.61	0.63
$\alpha=0.05$	T_1	9.97	9.84	9.83	9.84	9.84	9.83	9.84	9.84
	T_2	1.53	1.43	1.42	1.43	1.43	1.43	1.43	1.43
	T_3		0.68	0.71	0.69	0.68	0.76	0.72	0.73
$\alpha=0.10$	T_1	16.74	16.69	16.68	16.69	16.69	16.68	16.69	16.69
	T_2	2.28	2.32	2.30	2.31	2.32	2.32	2.32	2.32
	T_3		0.71	0.74	0.74	0.71	0.75	0.72	0.73
$\alpha=0.30$	T_1	32.52	32.30	32.29	32.30	32.30	32.30	32.30	32.30
	T_2	3.86	3.88	3.88	3.88	3.88	3.89	3.89	3.89
	T_3		0.70	0.70	0.70	0.70	0.70	0.69	0.69
$\alpha=0.50$	T_1	37.12	36.70	36.69	36.70	36.70	36.70	36.69	36.69
	T_2	4.49	4.41	4.40	4.40	4.41	4.40	4.40	4.40
	T_3		0.86	0.86	0.86	0.86	0.88	0.86	0.87
$\alpha=0.70$	T_1	32.48	32.30	32.28	32.30	32.30	32.31	32.30	32.30
	T_2	4.07	4.22	4.17	4.21	4.22	4.24	4.23	4.23
	T_3		0.77	0.76	0.77	0.77	0.75	0.75	0.75

Table 1A. (continued)

		R = 500							
		J = 1	J = 50						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.90$	T_1	16.62	16.51	16.52	16.51	16.51	16.52	16.52	16.52
	T_2	2.36	2.38	2.37	2.38	2.38	2.40	2.39	2.39
	T_3		0.74	0.71	0.74	0.74	0.79	0.73	0.72
$\alpha = 0.95$	T_1	9.84	9.71	9.72	9.71	9.71	9.70	9.70	9.70
	T_2	1.67	1.58	1.59	1.58	1.58	1.59	1.58	1.59
	T_3		0.71	0.73	0.72	0.71	0.76	0.75	0.79
$\alpha = 0.99$	T_1	2.65	2.59	2.59	2.59	2.59	2.57	2.58	2.58
	T_2	0.76	0.64	0.65	0.64	0.64	0.68	0.65	0.66
	T_3		0.58	0.58	0.58	0.58	0.66	0.60	0.63

Note: The ARCH(1) parameter is $\theta = 0.5$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

Table 1B. AR(1)-ARCH(0)-Gaussian. Bagging quantile predictions for AR-ARCH models

		R = 200							
		J = 1	J = 50						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.01$	T_1	3.07	2.81	2.79	2.80	2.80	2.77	2.79	2.78
	T_2	0.85	0.51	0.43	0.48	0.51	0.53	0.51	0.50
	T_3		0.64	0.67	0.66	0.64	0.64	0.64	0.65
$\alpha = 0.05$	T_1	10.64	10.55	10.54	10.55	10.55	10.51	10.53	10.52
	T_2	1.23	1.03	1.02	1.03	1.03	1.05	1.03	1.04
	T_3		0.55	0.54	0.55	0.55	0.60	0.59	0.58
$\alpha = 0.10$	T_1	17.90	17.85	17.83	17.84	17.85	17.83	17.83	17.83
	T_2	1.76	1.73	1.71	1.73	1.73	1.74	1.72	1.72
	T_3		0.58	0.58	0.58	0.58	0.55	0.58	0.59
$\alpha = 0.30$	T_1	35.29	35.27	35.22	35.25	35.26	35.25	35.25	35.25
	T_2	2.92	2.98	2.96	2.97	2.98	2.99	2.98	2.98
	T_3		0.58	0.58	0.58	0.58	0.55	0.59	0.57
$\alpha = 0.50$	T_1	40.23	40.16	40.12	40.14	40.16	40.12	40.15	40.14
	T_2	3.11	3.17	3.15	3.16	3.17	3.17	3.16	3.16
	T_3		0.58	0.61	0.60	0.59	0.61	0.59	0.59
$\alpha = 0.70$	T_1	34.93	34.89	34.85	34.87	34.89	34.86	34.87	34.86
	T_2	2.65	2.68	2.66	2.67	2.68	2.66	2.67	2.66
	T_3		0.58	0.62	0.58	0.58	0.61	0.61	0.61
$\alpha = 0.90$	T_1	17.80	17.69	17.67	17.68	17.69	17.66	17.68	17.67
	T_2	1.60	1.43	1.42	1.43	1.43	1.45	1.43	1.43
	T_3		0.54	0.54	0.54	0.54	0.57	0.57	0.57
$\alpha = 0.95$	T_1	10.55	10.46	10.45	10.45	10.46	10.39	10.44	10.42
	T_2	1.13	0.94	0.93	0.94	0.94	0.93	0.93	0.93
	T_3		0.53	0.53	0.53	0.53	0.61	0.55	0.57

(*continued on next page*)

Table 1B. (continued)

		R = 200							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.99$	T_1	3.01	2.76	2.75	2.75	2.76	2.73	2.74	2.74
	T_2	0.67	0.39	0.39	0.39	0.39	0.44	0.39	0.40
	T_3		0.65	0.66	0.65	0.65	0.67	0.66	0.66
		R = 500							
$\alpha = 0.01$	T_1	2.78	2.78	2.79	2.79	2.78	2.78	2.78	2.78
	T_2	0.53	0.37	0.37	0.37	0.37	0.40	0.38	0.39
	T_3		0.37	0.37	0.37	0.37	0.37	0.39	0.37
$\alpha = 0.05$	T_1	10.59	10.67	10.66	10.67	10.67	10.66	10.67	10.66
	T_2	1.33	1.27	1.24	1.26	1.27	1.26	1.27	1.27
	T_3		0.43	0.44	0.43	0.43	0.39	0.4	0.38
$\alpha = 0.10$	T_1	17.86	18.05	18.04	18.05	18.05	18.04	18.05	18.05
	T_2	1.80	1.85	1.84	1.84	1.85	1.84	1.85	1.85
	T_3		0.32	0.32	0.32	0.32	0.32	0.32	0.32
$\alpha = 0.30$	T_1	34.83	35.07	35.05	35.06	35.07	35.04	35.06	35.05
	T_2	2.55	2.56	2.56	2.56	2.56	2.54	2.55	2.54
	T_3		0.33	0.32	0.33	0.33	0.34	0.32	0.33
$\alpha = 0.50$	T_1	39.81	40.01	39.99	40.01	40.01	40.01	40.00	40.00
	T_2	2.79	2.84	2.83	2.84	2.84	2.84	2.83	2.83
	T_3		0.43	0.43	0.43	0.43	0.44	0.44	0.43
$\alpha = 0.70$	T_1	34.69	34.86	34.84	34.85	34.85	34.83	34.84	34.84
	T_2	2.53	2.55	2.55	2.55	2.55	2.53	2.55	2.54
	T_3		0.45	0.44	0.45	0.45	0.44	0.45	0.46
$\alpha = 0.90$	T_1	17.61	17.67	17.67	17.67	17.67	17.66	17.67	17.67
	T_2	1.66	1.62	1.62	1.62	1.62	1.62	1.62	1.62
	T_3		0.51	0.5	0.5	0.51	0.5	0.49	0.49
$\alpha = 0.95$	T_1	10.33	10.42	10.42	10.42	10.42	10.41	10.42	10.42
	T_2	1.22	1.13	1.13	1.13	1.13	1.13	1.13	1.12
	T_3		0.4	0.4	0.4	0.4	0.4	0.41	0.42
$\alpha = 0.99$	T_1	2.81	2.76	2.76	2.76	2.76	2.77	2.77	2.76
	T_2	0.57	0.48	0.48	0.47	0.48	0.51	0.49	0.49
	T_3		0.48	0.48	0.48	0.48	0.48	0.48	0.48

Note: The AR(1) parameter is $\rho = 0.6$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

a value smaller than 1 indicates bagging predictors work better than the unbagged predictor.

We generate the data from

$$
\begin{aligned}
Y_t &= \rho Y_{t-1} + \varepsilon_t, \\
\varepsilon_t &= z_t\left[(1-\theta) + \theta\varepsilon_{t-1}^2\right]^{1/2}, \\
z_t &\sim \text{i.i.d. MW}_i,
\end{aligned} \tag{3}
$$

Table 1C. AR(1)-ARCH(0)-Skewed unimodal. Bagging quantile predictions for AR-ARCH models

		R = 200							
		J = 1	J = 50						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.01$	T_1	4.15	3.55	3.53	3.55	3.55	3.56	3.57	3.57
	T_2	1.36	0.75	0.73	0.75	0.75	0.92	0.80	0.82
	T_3		0.79	0.81	0.79	0.79	0.80	0.76	0.81
$\alpha = 0.05$	T_1	13.13	12.73	12.71	12.72	12.73	12.67	12.71	12.70
	T_2	2.00	1.67	1.65	1.66	1.67	1.70	1.67	1.68
	T_3		0.74	0.77	0.74	0.74	0.78	0.76	0.77
$\alpha = 0.10$	T_1	21.03	20.71	20.67	20.70	20.71	20.67	20.69	20.67
	T_2	2.89	2.54	2.53	2.53	2.54	2.58	2.54	2.55
	T_3		0.65	0.67	0.65	0.65	0.72	0.69	0.70
$\alpha = 0.30$	T_1	36.34	36.37	36.32	36.35	36.37	36.35	36.36	36.36
	T_2	3.80	3.81	3.78	3.80	3.81	3.83	3.81	3.81
	T_3		0.51	0.55	0.53	0.51	0.53	0.53	0.53
$\alpha = 0.50$	T_1	38.53	38.57	38.52	38.55	38.57	38.50	38.54	38.53
	T_2	3.52	3.56	3.54	3.55	3.56	3.54	3.55	3.55
	T_3		0.49	0.51	0.50	0.49	0.58	0.50	0.51
$\alpha = 0.70$	T_1	31.59	31.63	31.58	31.61	31.63	31.60	31.61	31.60
	T_2	2.78	2.83	2.81	2.82	2.83	2.83	2.82	2.82
	T_3		0.50	0.55	0.50	0.50	0.50	0.50	0.49
$\alpha = 0.90$	T_1	15.32	15.39	15.38	15.38	15.39	15.35	15.37	15.36
	T_2	1.64	1.43	1.44	1.43	1.43	1.44	1.44	1.44
	T_3		0.47	0.47	0.47	0.47	0.46	0.47	0.48
$\alpha = 0.95$	T_1	9.01	9.02	9.02	9.02	9.02	8.98	9.01	9.00
	T_2	1.09	0.94	0.95	0.94	0.94	0.96	0.95	0.95
	T_3		0.50	0.51	0.51	0.51	0.51	0.51	0.50
$\alpha = 0.99$	T_1	2.60	2.35	2.35	2.35	2.35	2.32	2.34	2.33
	T_2	0.73	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	T_3		0.58	0.58	0.58	0.58	0.64	0.60	0.63
		R = 500							
$\alpha = 0.01$	T_1	3.50	3.40	3.40	3.40	3.40	3.41	3.40	3.40
	T_2	0.80	0.60	0.59	0.60	0.60	0.66	0.62	0.64
	T_3		0.53	0.54	0.53	0.53	0.52	0.54	0.53
$\alpha = 0.05$	T_1	12.67	12.61	12.58	12.60	12.61	12.59	12.61	12.60
	T_2	1.80	1.71	1.67	1.70	1.71	1.69	1.72	1.71
	T_3		0.57	0.59	0.58	0.57	0.53	0.58	0.58
$\alpha = 0.10$	T_1	20.61	20.50	20.49	20.50	20.50	20.48	20.50	20.50
	T_2	2.58	2.36	2.37	2.36	2.36	2.40	2.37	2.39
	T_3		0.57	0.56	0.57	0.57	0.61	0.59	0.60
$\alpha = 0.30$	T_1	36.10	36.17	36.16	36.16	36.17	36.16	36.16	36.15
	T_2	3.57	3.48	3.47	3.47	3.48	3.47	3.48	3.48
	T_3		0.47	0.48	0.48	0.47	0.48	0.49	0.48
$\alpha = 0.50$	T_1	38.36	38.40	38.40	38.40	38.40	38.38	38.40	38.39
	T_2	3.12	3.11	3.11	3.11	3.11	3.12	3.11	3.12
	T_3		0.49	0.50	0.50	0.49	0.52	0.49	0.50

(*continued on next page*)

Table 1C. (continued)

		R = 500							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.70$	T_1	31.66	31.72	31.72	31.72	31.72	31.70	31.71	31.71
	T_2	2.40	2.44	2.44	2.44	2.44	2.43	2.44	2.44
	T_3		0.46	0.46	0.46	0.46	0.49	0.46	0.46
$\alpha = 0.90$	T_1	15.11	15.33	15.33	15.33	15.33	15.30	15.32	15.31
	T_2	1.29	1.28	1.28	1.28	1.28	1.27	1.28	1.28
	T_3		0.30	0.30	0.30	0.30	0.31	0.30	0.31
$\alpha = 0.95$	T_1	8.78	9.03	9.03	9.03	9.03	9.02	9.03	9.02
	T_2	0.87	0.89	0.89	0.89	0.89	0.90	0.89	0.90
	T_3		0.22	0.22	0.22	0.22	0.22	0.20	0.21
$\alpha = 0.99$	T_1	2.34	2.37	2.37	2.37	2.37	2.36	2.37	2.37
	T_2	0.58	0.32	0.32	0.32	0.32	0.33	0.32	0.33
	T_3		0.36	0.36	0.36	0.36	0.36	0.34	0.36

Note: The AR(1) parameter is $\rho = 0.6$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

Table 1D. AR(1)-ARCH(0)-Strongly skewed. Bagging quantile predictions for AR-ARCH models

		R = 200							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.01$	T_1	1.08	1.53	1.53	1.53	1.53	1.54	1.54	1.54
	T_2	0.16	0.19	0.19	0.19	0.19	0.19	0.19	0.19
	T_3		0.02	0.02	0.02	0.02	0.02	0.02	0.02
$\alpha = 0.05$	T_1	4.79	5.64	5.63	5.63	5.64	5.61	5.60	5.60
	T_2	0.45	0.68	0.68	0.68	0.68	0.69	0.68	0.68
	T_3		0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 0.10$	T_1	9.25	9.87	9.85	9.85	9.86	9.61	9.77	9.70
	T_2	0.92	1.12	1.11	1.12	1.12	1.10	1.11	1.10
	T_3		0.01	0.01	0.01	0.01	0.07	0.01	0.03
$\alpha = 0.30$	T_1	25.27	25.31	25.29	25.30	25.31	25.24	25.27	25.26
	T_2	2.68	2.68	2.68	2.68	2.68	2.67	2.67	2.67
	T_3		0.50	0.50	0.51	0.50	0.56	0.52	0.52
$\alpha = 0.50$	T_1	36.90	36.96	36.94	36.95	36.96	36.90	36.94	36.93
	T_2	3.97	3.93	3.92	3.93	3.93	3.94	3.94	3.94
	T_3		0.47	0.51	0.49	0.47	0.55	0.51	0.52
$\alpha = 0.70$	T_1	39.40	39.36	39.31	39.33	39.35	39.29	39.33	39.32
	T_2	4.40	4.21	4.18	4.20	4.21	4.23	4.22	4.22
	T_3		0.56	0.54	0.55	0.56	0.58	0.55	0.56
$\alpha = 0.90$	T_1	24.17	23.68	23.66	23.66	23.68	23.55	23.65	23.62
	T_2	2.70	2.58	2.57	2.57	2.58	2.63	2.58	2.59
	T_3		0.72	0.73	0.74	0.72	0.81	0.75	0.78

Table 1D. (continued)

		$R = 200$							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.95$	T_1	14.94	14.42	14.39	14.39	14.42	14.37	14.40	14.39
	T_2	1.91	1.67	1.63	1.64	1.66	1.68	1.68	1.68
	T_3		0.76	0.79	0.78	0.76	0.83	0.79	0.82
$\alpha = 0.99$	T_1	4.35	3.85	3.83	3.84	3.85	3.77	3.84	3.81
	T_2	1.23	0.76	0.68	0.72	0.76	0.92	0.83	0.84
	T_3		0.76	0.76	0.76	0.76	0.82	0.77	0.83
		$R = 500$							
$\alpha = 0.01$	T_1	1.04	1.58	1.58	1.58	1.58	1.59	1.59	1.59
	T_2	0.15	0.22	0.22	0.22	0.22	0.23	0.23	0.23
	T_3		0.02	0.02	0.02	0.02	0.01	0.02	0.01
$\alpha = 0.05$	T_1	4.76	5.66	5.65	5.65	5.66	5.69	5.66	5.68
	T_2	0.51	0.79	0.79	0.79	0.79	0.81	0.80	0.81
	T_3		0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 0.10$	T_1	9.18	9.63	9.62	9.63	9.63	9.46	9.56	9.52
	T_2	0.98	1.15	1.15	1.15	1.15	1.11	1.13	1.12
	T_3		0.00	0.00	0.00	0.00	0.05	0.01	0.02
$\alpha = 0.30$	T_1	24.97	25.01	25.01	25.01	25.01	25.01	25.01	25.01
	T_2	2.74	2.75	2.75	2.75	2.75	2.75	2.75	2.75
	T_3		0.46	0.46	0.46	0.46	0.46	0.45	0.45
$\alpha = 0.50$	T_1	36.33	36.39	36.38	36.39	36.39	36.38	36.38	36.38
	T_2	3.93	3.91	3.91	3.91	3.91	3.93	3.92	3.93
	T_3		0.45	0.43	0.45	0.45	0.46	0.46	0.46
$\alpha = 0.70$	T_1	38.80	38.88	38.87	38.88	38.88	38.87	38.87	38.87
	T_2	4.22	4.13	4.12	4.13	4.13	4.15	4.14	4.14
	T_3		0.53	0.53	0.53	0.53	0.51	0.51	0.51
$\alpha = 0.90$	T_1	23.23	23.18	23.16	23.17	23.18	23.12	23.17	23.15
	T_2	2.56	2.40	2.37	2.39	2.40	2.39	2.40	2.40
	T_3		0.59	0.57	0.58	0.59	0.58	0.58	0.59
$\alpha = 0.95$	T_1	14.14	14.03	14.02	14.03	14.03	14.00	14.02	14.02
	T_2	1.72	1.54	1.55	1.54	1.54	1.61	1.56	1.57
	T_3		0.60	0.62	0.61	0.60	0.59	0.61	0.59
$\alpha = 0.99$	T_1	3.75	3.66	3.67	3.66	3.66	3.63	3.66	3.65
	T_2	0.82	0.56	0.58	0.57	0.56	0.53	0.58	0.58
	T_3		0.52	0.52	0.52	0.52	0.60	0.59	0.63

Note: The AR(1) parameter is $\rho = 0.6$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

where the i.i.d. innovation z_t is generated from the first eight mixture normal distributions of Marron and Wand (1992, p. 717), each of which will be denoted as MW_i $(i = 1, \ldots, 8)$.[1] In Table 1A, and Figure 1, panel (a), we con-

[1] MW_1 is Gaussian, MW_2 is Skewed unimodal, MW_3 Strongly skewed, MW_4 Kurtoic unimodal, MW_5 Outlier, MW_6 Bimodal, MW_7 Separated bimodal, and MW_8 is Skewed bimodal; see Mar-

Table 1E. AR(1)-ARCH(0)-Kurtotic unimodal. Bagging quantile predictions for AR-ARCH models

		R = 200							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.01$	T_1	3.59	3.16	3.13	3.15	3.16	3.16	3.16	3.15
	T_2	1.02	0.73	0.67	0.72	0.73	0.83	0.75	0.77
	T_3		0.71	0.73	0.73	0.72	0.71	0.74	0.73
$\alpha = 0.05$	T_1	11.82	11.61	11.57	11.59	11.61	11.55	11.58	11.56
	T_2	1.68	1.63	1.57	1.60	1.63	1.59	1.61	1.60
	T_3		0.63	0.63	0.64	0.63	0.64	0.63	0.62
$\alpha = 0.10$	T_1	19.33	19.10	19.06	19.08	19.10	19.05	19.08	19.07
	T_2	2.44	2.34	2.31	2.32	2.34	2.37	2.34	2.35
	T_3		0.58	0.59	0.58	0.58	0.60	0.58	0.60
$\alpha = 0.30$	T_1	32.65	32.88	32.83	32.86	32.87	32.78	32.83	32.81
	T_2	3.85	3.79	3.78	3.78	3.79	3.81	3.79	3.80
	T_3		0.42	0.44	0.42	0.42	0.45	0.43	0.43
$\alpha = 0.50$	T_1	34.17	34.43	34.40	34.41	34.43	34.28	34.35	34.32
	T_2	3.74	3.74	3.74	3.74	3.74	3.71	3.73	3.73
	T_3		0.29	0.29	0.29	0.29	0.37	0.32	0.35
$\alpha = 0.70$	T_1	32.75	33.01	32.96	32.99	33.01	32.96	32.98	32.96
	T_2	3.67	3.53	3.53	3.53	3.54	3.59	3.55	3.56
	T_3		0.38	0.40	0.38	0.38	0.37	0.38	0.39
$\alpha = 0.90$	T_1	19.51	19.37	19.35	19.35	19.37	19.36	19.36	19.36
	T_2	2.30	2.18	2.19	2.18	2.18	2.18	2.18	2.17
	T_3		0.60	0.63	0.63	0.60	0.60	0.60	0.60
$\alpha = 0.95$	T_1	12.11	11.86	11.85	11.85	11.86	11.81	11.84	11.83
	T_2	1.80	1.53	1.53	1.53	1.53	1.56	1.53	1.54
	T_3		0.62	0.60	0.62	0.62	0.65	0.63	0.64
$\alpha = 0.99$	T_1	3.76	3.27	3.28	3.27	3.27	3.26	3.27	3.26
	T_2	1.12	0.66	0.67	0.66	0.66	0.76	0.67	0.68
	T_3		0.73	0.74	0.73	0.73	0.73	0.73	0.74
		R = 500							
$\alpha = 0.01$	T_1	3.20	3.12	3.12	3.12	3.12	3.10	3.11	3.10
	T_2	0.56	0.42	0.42	0.42	0.42	0.42	0.41	0.41
	T_3		0.57	0.57	0.57	0.57	0.57	0.57	0.57
$\alpha = 0.05$	T_1	11.61	11.66	11.65	11.65	11.66	11.66	11.66	11.66
	T_2	1.44	1.31	1.30	1.30	1.31	1.33	1.32	1.32
	T_3		0.48	0.48	0.48	0.48	0.45	0.47	0.47
$\alpha = 0.10$	T_1	18.93	19.08	19.07	19.08	19.08	19.07	19.08	19.07
	T_2	2.00	1.95	1.95	1.95	1.95	1.96	1.96	1.96
	T_3		0.37	0.38	0.38	0.37	0.37	0.38	0.38

ron and Wand (1992, p. 717). To save space, we only report results for MW_i ($i = 1, \ldots, 5$) in Tables 1A–1F and in each panel of Figure 1. The results for $i = 5, \ldots, 8$ are basically similar in pattern as to how bagging works and are available upon request.

Table 1E. (continued)

		$R = 500$							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.30$	T_1	32.47	32.71	32.70	32.70	32.71	32.69	32.69	32.69
	T_2	3.55	3.49	3.48	3.48	3.49	3.51	3.49	3.50
	T_3		0.35	0.35	0.35	0.35	0.35	0.35	0.34
$\alpha = 0.50$	T_1	34.31	34.40	34.40	34.40	34.40	34.38	34.39	34.38
	T_2	3.79	3.78	3.78	3.78	3.78	3.78	3.78	3.78
	T_3		0.41	0.40	0.41	0.41	0.44	0.43	0.43
$\alpha = 0.70$	T_1	32.71	32.93	32.92	32.92	32.93	32.91	32.92	32.92
	T_2	4.29	4.02	4.02	4.02	4.02	4.06	4.04	4.05
	T_3		0.37	0.38	0.37	0.37	0.37	0.38	0.38
$\alpha = 0.90$	T_1	19.38	19.34	19.33	19.34	19.34	19.33	19.34	19.34
	T_2	2.54	2.36	2.35	2.36	2.36	2.37	2.37	2.37
	T_3		0.55	0.52	0.54	0.55	0.52	0.50	0.50
$\alpha = 0.95$	T_1	11.84	11.72	11.71	11.71	11.72	11.70	11.71	11.71
	T_2	1.83	1.58	1.58	1.58	1.58	1.59	1.59	1.59
	T_3		0.58	0.58	0.58	0.58	0.57	0.57	0.58
$\alpha = 0.99$	T_1	3.27	3.14	3.14	3.14	3.14	3.13	3.14	3.13
	T_2	0.81	0.57	0.57	0.57	0.57	0.58	0.57	0.57
	T_3		0.49	0.49	0.49	0.49	0.54	0.51	0.52

Note: The AR(1) parameter is $\rho = 0.6$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

sider data-generating processes for ARCH-MW_1 with $\theta = 0.5$ (and $\rho = 0$), while in Tables 1B–1F and Figure 1, panels (b)–(f), we consider data-generating processes for AR-MW_i ($i = 1, \ldots, 5$) with $\rho = 0.6$ (and $\theta = 0$). Therefore, our data generating processes fall into two categories: the (mean-unpredictable) martingale-difference ARCH(1) processes without AR structure and the mean-predictable AR(1) processes without ARCH structure.

For each series, 100 extra observations are generated and then discarded to alleviate the effect of the starting values in random number generation. We consider one fixed out-of-sample size $P = 100$, and a range of estimation sample sizes, $R = 200$ and 500. Our bagging predictors are generated by averaging over $J = 50$ bootstrap predictors.

We consider a group of simple univariate polynomial quantile regression function of Chernozhukov and Umantsev (2001) as our predictive method:

$$Q_\alpha(Y_{t+h}|\mathbf{X}_t) = \tilde{\mathbf{X}}_t' \boldsymbol{\beta}_{\alpha,h}, \tag{4}$$

with h representing the forecast horizons, $\mathbf{X}_t = (Y_t \ \ldots \ Y_{t-h+1})$, $\tilde{\mathbf{X}}_t = (1 \ Y_t \ Y_t^2 \ \ldots \ Y_{t-h+1} \ Y_{t-h+1}^2)'$, and $\boldsymbol{\beta}_{\alpha,h} = [\boldsymbol{\beta}_{\alpha,h,0} \ \boldsymbol{\beta}_{\alpha,h,1} \ \boldsymbol{\beta}_{\alpha,h,2} \ \cdots \ \boldsymbol{\beta}_{\alpha,h,2h-1} \ \boldsymbol{\beta}_{\alpha,h,2h}]'$. For now, we set $h = 1$ to generate one-step-ahead forecasts, and we will talk about multi-step forecasts later in this chapter.

Table 1F. AR(1)-ARCH(0)-Outlier. Bagging quantile predictions for AR-ARCH models

		R = 200							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.05$	T_1	13.41	13.43	13.42	13.42	13.42	13.22	13.36	13.32
	T_2	5.34	4.74	4.79	4.75	4.74	4.78	4.79	4.81
	T_3		0.49	0.48	0.49	0.49	0.49	0.49	0.49
$\alpha = 0.10$	T_1	15.83	16.18	16.18	16.18	16.17	16.12	16.12	16.11
	T_2	5.45	5.30	5.31	5.30	5.30	5.33	5.31	5.32
	T_3		0.27	0.26	0.27	0.27	0.29	0.29	0.31
$\alpha = 0.30$	T_1	21.62	21.88	21.85	21.86	21.88	21.69	21.81	21.76
	T_2	4.78	4.75	4.74	4.75	4.75	4.70	4.74	4.72
	T_3		0.28	0.31	0.31	0.31	0.43	0.31	0.40
$\alpha = 0.50$	T_1	23.19	23.44	23.39	23.42	23.43	23.25	23.35	23.31
	T_2	4.48	4.49	4.47	4.48	4.49	4.40	4.45	4.43
	T_3		0.27	0.28	0.28	0.27	0.45	0.36	0.39
$\alpha = 0.70$	T_1	21.38	21.66	21.61	21.64	21.65	21.48	21.57	21.53
	T_2	4.85	4.80	4.80	4.80	4.80	4.79	4.79	4.78
	T_3		0.24	0.25	0.25	0.24	0.39	0.30	0.36
$\alpha = 0.90$	T_1	15.54	15.85	15.83	15.83	15.85	15.76	15.79	15.77
	T_2	5.47	5.25	5.25	5.24	5.25	5.33	5.26	5.28
	T_3		0.31	0.33	0.31	0.32	0.37	0.35	0.35
$\alpha = 0.95$	T_1	13.01	13.02	12.99	13.00	13.02	12.79	12.92	12.87
	T_2	5.12	4.45	4.45	4.46	4.46	4.54	4.50	4.51
	T_3		0.47	0.48	0.47	0.49	0.54	0.49	0.51
$\alpha = 0.99$	T_1	7.84	5.36	5.36	5.36	5.36	5.42	5.40	5.45
	T_2	4.82	1.87	1.83	1.88	1.87	2.18	1.98	2.14
	T_3		0.89	0.89	0.89	0.89	0.94	0.90	0.93
		R = 500							
$\alpha = 0.01$	T_1	5.91	5.39	5.40	5.39	5.39	5.38	5.38	5.39
	T_2	2.67	1.98	2.01	1.99	1.98	2.10	2.00	2.05
	T_3		0.69	0.69	0.69	0.69	0.74	0.71	0.73
$\alpha = 0.05$	T_1	12.86	13.13	13.15	13.12	13.13	13.07	13.09	13.08
	T_2	5.42	4.81	4.87	4.80	4.82	4.90	4.85	4.87
	T_3		0.36	0.36	0.36	0.37	0.37	0.36	0.38
$\alpha = 0.10$	T_1	15.49	16.00	16.00	16.00	16.00	16.00	16.00	16.00
	T_2	5.32	5.21	5.23	5.22	5.21	5.21	5.22	5.21
	T_3		0.17	0.18	0.16	0.17	0.17	0.16	0.15
$\alpha = 0.30$	T_1	21.34	21.53	21.52	21.52	21.53	21.46	21.49	21.47
	T_2	4.66	4.66	4.66	4.66	4.66	4.65	4.65	4.65
	T_3		0.29	0.29	0.29	0.29	0.33	0.29	0.32
$\alpha = 0.50$	T_1	23.15	23.22	23.21	23.21	23.22	23.17	23.19	23.19
	T_2	4.45	4.50	4.50	4.50	4.50	4.46	4.48	4.48
	T_3		0.40	0.40	0.40	0.40	0.44	0.41	0.40
$\alpha = 0.70$	T_1	21.48	21.58	21.57	21.58	21.58	21.53	21.55	21.54
	T_2	4.80	4.78	4.78	4.78	4.78	4.77	4.77	4.77
	T_3		0.31	0.33	0.32	0.32	0.44	0.35	0.38

Table 1F. (continued)

		$R = 500$							
		$J = 1$	$J = 50$						
			mean	BMA_1	BMA_5	BMA_R	med	$trim_5$	$trim_{10}$
$\alpha = 0.90$	T_1	15.54	15.98	15.98	15.98	15.98	15.99	15.98	15.98
	T_2	5.34	5.13	5.14	5.13	5.13	5.14	5.14	5.14
	T_3		0.19	0.21	0.21	0.19	0.21	0.23	0.21
$\alpha = 0.95$	T_1	13.02	13.22	13.22	13.21	13.21	13.16	13.18	13.17
	T_2	5.21	4.65	4.67	4.66	4.66	4.72	4.68	4.70
	T_3		0.38	0.39	0.38	0.38	0.41	0.40	0.42
$\alpha = 0.99$	T_1	6.24	5.61	5.64	5.62	5.62	5.65	5.63	5.65
	T_2	2.71	2.01	2.05	2.01	2.01	2.12	2.06	2.09
	T_3		0.71	0.71	0.71	0.71	0.69	0.70	0.73

Note: The AR(1) parameter is $\rho = 0.6$ in Equation (3). See the definition of T_1, T_2, and T_3 in the text, which are computed from 100 Monte Carlo replications.

We estimate $\boldsymbol{\beta}_{\alpha,h}$ recursively using the "rolling" samples of size $R - 2h + 1$. Suppose there are $T (\equiv R + P)$ observations in total. We use the most recent $R - 2h + 1$ observations available at time t, $R \leqslant t < T - h$, as a training sample, $\mathcal{D}_t \equiv \{(Y_s, \mathbf{X}_{s-h})\}_{s=t-R+2h}^{t}$. We then generate $P (= T - R)$ h-step-ahead forecasts for the remaining forecast validation sample. For each time t in the P prediction periods, we use a rolling training sample $\mathcal{D}_t$ of size $R - 2h + 1$ to estimate model parameters:

$$\hat{\boldsymbol{\beta}}_{\alpha,h}(\mathcal{D}_t) \equiv \arg\min_{\boldsymbol{\beta}_{\alpha,h}} \sum_{s=t-R+h+1}^{t} \rho_\alpha(u_s), \quad t = R, \ldots, T, \tag{5}$$

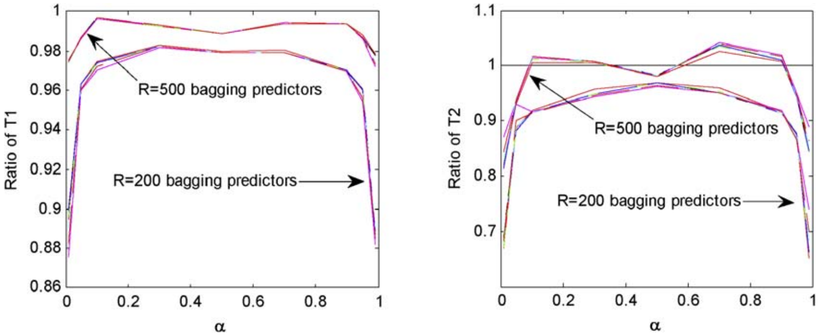
where $u_s \equiv Y_s - Q_\alpha(Y_s|\mathbf{X}_{s-h}) = Y_s - \tilde{\mathbf{X}}'_{s-h}\boldsymbol{\beta}_{\alpha,h}$. $\hat{\boldsymbol{\beta}}_{\alpha,h}(\mathcal{D}_t)$ is estimated using the interior-point algorithm suggested by Portnoy and Koenker (1997).

To generate bootstrap samples, we use the block bootstrap for both the Monte Carlo experiments and empirical applications. We choose the block size that minimizes the in-sample average cost recursively and therefore we use a different block size at each forecasting time t and for each loss function with different α's.

The Monte Carlo results are reported in Tables 1A–1F and Figure 1, panels (a)–(f), where mean, BMA_k, med, and $trim_k$ denote the equal-weighted bagging predictors, BMA-weighted bagging predictors using the k-most recent in-sample observations, median-weighted bagging predictors, and k-trimmed on each tail weighted bagging predictors, respectively.

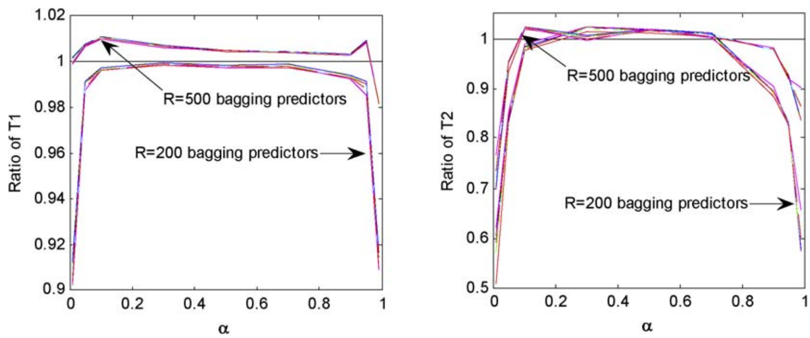
According to our Monte Carlo results on quantile predictions shown in Tables 1A–1F, we summarize our observations as follows. First, in most of cases, BMA-weighted, median, and trimmed bagging predictors have better predictive

(a) AR(0)-ARCH(1)-Gaussian



Note: The ARCH(1) parameter in (1) is $\theta = 0.5$. The two figures report the tick loss ratio and standard error ratio of bagging predictors over unbagged predictors for 100 Monte Carlo replications.

(b) AR(1)-ARCH(0)-Gaussian



Note: The AR(1) parameter in (1) is $\rho = 0.6$.

(c) AR(1)-ARCH(0)-Skewed unimodal

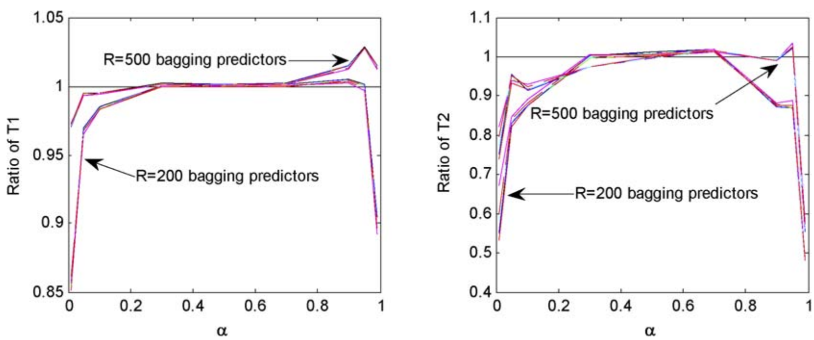
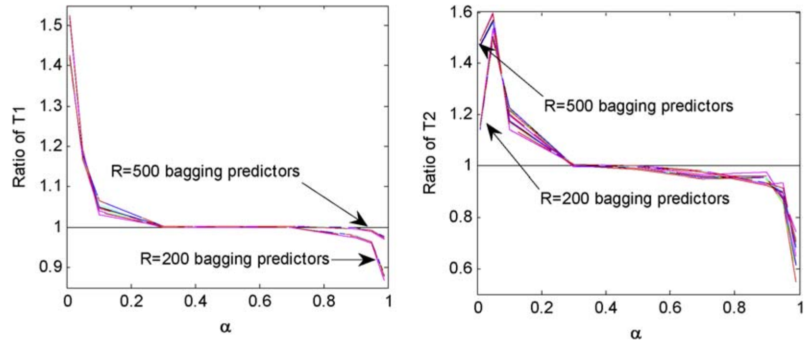
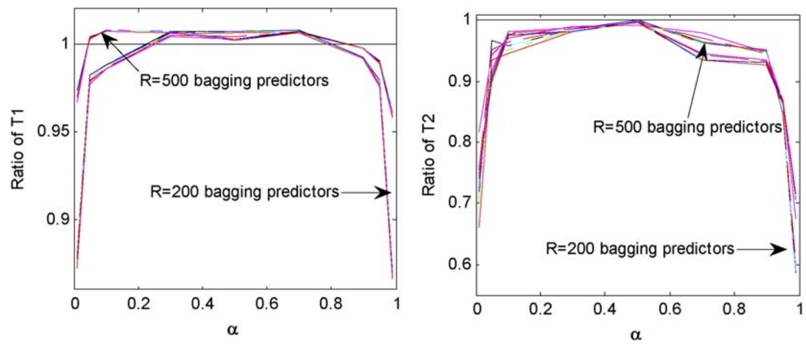


Fig. 1. Bagging quantile prediction for AR-ARCH models.

(d) AR(1)-ARCH(0)-Strongly skewed



(e) AR(1)-ARCH(0)-Kurtotic unimodal



(f) AR(1)-ARCH(0)-Outlier

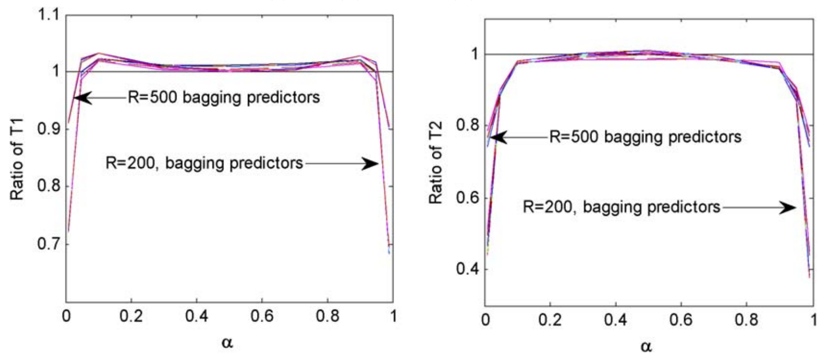


Fig. 1. (Continued.)

performance (smaller T_1 and T_2 and larger T_3) compared to the mean bagging predictor, even when we have a relatively large sample size. Second, on average, the improvement brought by median bagging is larger than the trimmed bagging and BMA-weighted bagging predictors, so median bagging tends to give the smallest T_1 and largest T_3 among all predictors. Third, the outstanding performance of the median bagging predictors is most obvious when α values are close to 0 or 1, where the extreme value problem are most serious because there are fewer observations in the tails and the parameter regression estimates are sensitive to the estimation sample. When the sample size R is 200 and α values are close to 0 or 1, the median bagging predictors can further reduce the average loss (T_1) by about 1% on average and increase the percentage of times bagging works (T_3) by about 4% on average compared with mean bagging predictors. However, the advantage of median bagging predictors are not as evident when α values are close to 0.5.

From Figure 1, panels (a)–(e), we can see that different bagging predictors work in similar trends. First, bagging predictors work better when the sample size is smaller, so the $R = 200$ lines lie below the $R = 500$ lines in the figures, and both $R = 200$ and $R = 500$ lie below the unit line most of the time. Second, bagging predictors works better when α values are close to 0 or 1, so the bagging lines look like the letter "*n*," especially when $R = 200$. Third, bagging predictors work better when α-quantiles lie on the sparse part of the error distribution. Our explanation is that for the sparse part of the error distribution, there are fewer observations, therefore quantile predictions are sensitive to the estimation sample and bagging predictors work better for unstable predictions. For example, when the error term is left skewed, as in Figure 1, panel (c), bagging predictors give larger loss reductions for the prediction of small α-quantiles than for large α-quantiles; when the error term are right skewed, as in Figure 1, panel (d), bagging predictors give large loss reduction for the prediction of larger α-quantiles but do not work for small α-quantiles; and among panels (a)–(f) of Figure 1, panel (f) has the sparsest distribution on both tails among all DGPs, and bagging predictors deliver the best performance (smallest T_1 and T_2 and largest T_3).

Our conclusions on the performance of BMA bagging predictors and median bagging predictors are further supported by empirical experiments. We make pseudo real-time forecasts of the daily returns of six major US stock indices and two major foreign exchange rates. We split the series into two parts: one for in-sample estimation with sizes $R = 100$ and 300 and another for out-of-sample forecast validation with sample size $P = 250$ (fixed for both R's). We choose the most recent $P = 250$ days in the sample as the out-of-sample validation sample. We use a rolling-sample scheme, that is, the first forecast is based on observations $T - P - R + 1$ through $T - P$, the second forecast is based on observations $T-P-R+2$ through $T-P+1$, and so on. The eight series are the Dow Jones Industrial Averages (Dow Jones), New York Stock Exchange Composite (NYSE), Standard and Poor's 500 (S&P 500), National Association of Securities Dealers Automated Quotations Composite (NASDAQ), Russell 2000 index (Rusell 2000), Pacific Exchange Technology (PET), US Dollar per Euro (USD/EUR), and US Dollar per Japanese Yen (USD/JPY). The total sample period and the out-of-sample forecasting period are summarized as follows:

	Total sample period	Out-of-sample period ($P = 250$)
Dow Jones	10/27/1998–12/31/2000	01/05/2000–12/31/2000
NYSE	10/27/1998–12/31/2000	01/05/2000–12/31/2000
S&P 500	10/27/1998–12/31/2000	01/05/2000–12/31/2000
NASDAQ	10/27/1998–12/31/2000	01/05/2000–12/31/2000
Rusell 2000	10/27/1998–12/31/2000	01/05/2000–12/31/2000
PET	10/27/1998–12/31/2000	01/05/2000–12/31/2000
USD/EUR	10/10/2003–04/11/2005	08/05/2004–04/11/2005
USD/YEN	10/10/2003–04/11/2005	08/05/2004–04/11/2005

We consider nine quantile parameters, $\alpha = 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9,$ $0.95,$ and 0.99. The empirical experiments are reported in Tables 2A–2H and Figure 2, panels (a)–(h). Our findings are as follows. First, bagging predictors works better when the sample size is smaller. Second, bagging predictors work better for α values close to 0 or 1 than for α values close to 0.5. Third, for the six stock return series, bagging predictors work better for α values close to 0 than α values close to 1 because the distribution of stock returns all have long left tails. However, for the two foreign exchange series, bagging works rather symmetrically for α values close to 0 and 1 because they have symmetric distributions.

4. Bagging multi-step quantile forecasts

We will show how bagging works for multi-step predictions in this section. It is important to make multi-step forecasts in the real world. For example, a group of users for time series predictions are policy makers, and since it potentially takes a long time for monetary and fiscal policies to generate expected effects in the economy, policy makers have to produce predictions more than one-period-ahead.

We check four multi-step horizons, $h = 1, 2, 3, 4$. If we have a simple linear model, then multi-step forecasts can be achieved by simple iteration of the one-step-ahead predictors. However, we may not apply this naïve iteration method to generate multi-step forecasts for nonlinear models. We use polynomial quantile regression models to take account of the nonlinear structures in the data. As mentioned by Tsay (1993), Lin and Tsay (1996), and Chevillon and Hendry (2004), the "direct" multi-step method will suffer less from model misspecification than the "iterated" multi-step methods; therefore, the direct multi-step method is also called "adaptive estimation," and it should be able to generate better or at least as good predictions as iterated methods in the case of model uncertainty or misspecification.

There are few papers discussing how to make multi-step conditional quantile forecasts. We can either iterate one-step-ahead forecast or model the relation-

Table 2A. USD/EUR Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	5.59	3.51	3.51	3.51	3.51	3.39	4.07	3.94	3.94	3.94	3.94	4.01
$\alpha = 0.05$	14.62	12.21	12.23	12.21	12.21	12.01	14.18	13.95	13.94	13.93	13.95	13.77
$\alpha = 0.10$	23.35	22.01	22.03	22.01	22.02	21.98	22.95	22.70	22.68	22.69	22.70	22.74
$\alpha = 0.30$	37.76	36.58	36.55	36.57	36.58	36.50	36.80	36.10	36.09	36.10	36.10	36.04
$\alpha = 0.50$	40.90	40.22	40.20	40.21	40.22	40.06	40.38	39.74	39.73	39.74	39.74	39.70
$\alpha = 0.70$	38.43	36.92	36.90	36.93	36.93	36.80	38.29	37.59	37.58	37.59	37.59	37.50
$\alpha = 0.90$	23.80	22.53	22.53	22.54	22.53	22.67	23.24	22.86	22.86	22.87	22.86	22.74
$\alpha = 0.95$	15.15	13.96	13.99	13.97	13.96	14.09	14.37	14.35	14.35	14.35	14.35	14.32
$\alpha = 0.99$	7.18	3.71	3.71	3.71	3.71	3.43	4.32	4.18	4.18	4.18	4.18	4.14

Note: Each cell gives the tick loss of quantile prediction over the period 08/05/2004–04/11/2005.

Table 2B. USD/JPY Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	7.20	3.81	3.81	3.81	3.82	3.21	4.36	3.98	3.98	3.98	3.98	4.09
$\alpha = 0.05$	15.74	13.68	13.67	13.67	13.68	13.64	15.23	14.41	14.4	14.4	14.41	14.47
$\alpha = 0.10$	23.63	22.91	22.89	22.91	22.91	23.17	22.59	22.3	22.3	22.3	22.3	22.17
$\alpha = 0.30$	39.43	38.58	38.58	38.58	38.58	38.54	38.12	37.49	37.49	37.49	37.49	37.53
$\alpha = 0.50$	40.60	39.86	39.86	39.86	39.86	39.65	40.79	40.83	40.84	40.83	40.83	40.82
$\alpha = 0.70$	37.17	36.66	36.65	36.66	36.66	36.23	37.35	38.08	38.08	38.08	38.08	38.16
$\alpha = 0.90$	24.66	22.09	22.05	22.09	22.09	21.96	23.06	22.01	22	22.01	22.01	22.07
$\alpha = 0.95$	16.10	14.47	14.40	14.46	14.47	14.41	13.91	13.88	13.87	13.88	13.88	13.8
$\alpha = 0.99$	6.88	4.18	4.17	4.18	4.18	4.87	3.75	3.64	3.64	3.64	3.64	3.62

Note: Each cell gives the tick loss of quantile prediction over the period 08/05/2004–04/11/2005.

Table 2C. Dow Jones Industrial Averages Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	14.17	10.43	10.53	10.47	10.43	9.15	11.30	10.17	10.22	10.18	10.17	10.12
$\alpha = 0.05$	38.67	33.43	33.98	33.51	33.46	32.07	39.10	37.60	37.89	37.65	37.60	37.76
$\alpha = 0.10$	65.82	58.75	59.25	58.78	58.78	57.27	62.58	59.27	59.34	59.28	59.28	59.41
$\alpha = 0.30$	114.68	110.87	111.26	110.84	110.88	110.72	110.42	109.17	109.18	109.18	109.17	109.14
$\alpha = 0.50$	129.14	124.29	124.52	124.24	124.33	124.17	125.10	123.48	123.54	123.49	123.48	123.49
$\alpha = 0.70$	109.85	108.48	108.59	108.39	108.47	107.43	106.85	106.09	106.25	106.14	106.09	106.36
$\alpha = 0.90$	61.60	57.59	58.56	57.82	57.59	56.72	57.81	54.69	55.12	54.83	54.69	54.26
$\alpha = 0.95$	37.73	34.09	34.07	34.19	34.11	33.03	32.44	31.45	31.67	31.52	31.45	31.90
$\alpha = 0.99$	13.34	14.45	14.18	14.42	14.40	14.11	9.30	8.66	8.40	8.61	8.66	8.31

Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

Table 2D. New York Stock Exchange Composite Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	11.39	8.43	8.50	8.45	8.43	7.73	9.31	7.85	7.93	7.87	7.85	8.05
$\alpha = 0.05$	36.33	28.23	28.56	28.32	28.24	26.65	32.25	29.88	30.08	29.93	29.88	29.20
$\alpha = 0.10$	55.37	47.77	48.21	47.86	47.78	46.96	49.57	47.83	47.95	47.86	47.84	47.27
$\alpha = 0.30$	94.59	90.80	91.00	90.81	90.82	90.72	91.18	89.62	89.63	89.62	89.63	89.69
$\alpha = 0.50$	105.49	102.21	102.37	102.27	102.23	101.42	103.02	102.45	102.56	102.49	102.46	102.67
$\alpha = 0.70$	94.18	92.39	92.34	92.37	92.39	91.97	92.73	90.92	91.03	90.96	90.92	91.12
$\alpha = 0.90$	56.58	49.43	49.77	49.61	49.46	49.03	52.01	49.05	49.43	49.12	49.05	48.77
$\alpha = 0.95$	34.57	30.16	30.54	30.31	30.18	31.27	31.68	29.49	29.71	29.57	29.49	29.73
$\alpha = 0.99$	15.57	10.01	10.11	9.86	9.99	9.34	9.34	8.29	8.34	8.31	8.29	8.72

Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

Table 2E. Standard and Poor's 500 Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	16.54	10.13	10.21	10.16	10.13	7.99	13.18	9.82	9.94	9.86	9.82	9.49
$\alpha = 0.05$	43.20	37.60	38.29	37.67	37.62	35.67	36.58	35.53	35.56	35.53	35.53	35.69
$\alpha = 0.10$	67.41	62.86	63.10	62.83	62.87	62.66	62.62	58.85	58.84	58.82	58.85	58.58
$\alpha = 0.30$	113.91	113.15	113.16	113.09	113.15	112.82	112.75	111.89	111.89	111.88	111.89	112.04
$\alpha = 0.50$	135.85	129.14	129.27	129.16	129.17	128.88	131.30	130.63	130.63	130.62	130.63	130.40
$\alpha = 0.70$	120.03	117.25	117.23	117.16	117.26	117.11	118.37	116.01	116.04	116.05	116.01	116.35
$\alpha = 0.90$	71.74	66.46	66.82	66.39	66.48	67.83	67.71	65.53	65.30	65.49	65.53	65.82
$\alpha = 0.95$	46.60	39.49	39.59	39.51	39.50	40.25	39.98	39.30	39.58	39.38	39.31	38.67
$\alpha = 0.99$	18.44	13.18	13.49	13.19	13.14	15.19	13.66	11.40	11.45	11.45	11.40	11.92

Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

Table 2F. NASDAQ Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	40.75	28.79	28.33	28.07	28.75	27.01	30.00	24.83	25.30	24.49	24.84	25.20
$\alpha = 0.05$	100.35	88.50	88.61	87.38	88.51	89.28	90.40	83.17	83.21	82.92	83.18	82.97
$\alpha = 0.10$	144.47	132.42	132.46	131.75	132.54	129.04	143.22	138.43	138.91	138.48	138.44	136.49
$\alpha = 0.30$	274.40	254.31	253.92	255.04	254.56	252.35	277.28	269.61	269.51	269.44	269.64	270.23
$\alpha = 0.50$	310.14	294.82	295.50	295.34	295.09	295.45	309.47	303.89	304.19	304.16	303.91	304.09
$\alpha = 0.70$	270.05	261.54	261.76	262.32	261.62	261.50	267.21	262.99	262.75	263.18	263.00	263.09
$\alpha = 0.90$	153.03	142.64	140.41	142.08	142.73	142.44	143.95	134.93	135.16	135.11	134.96	133.81
$\alpha = 0.95$	96.11	83.19	81.37	81.95	83.22	80.51	96.98	84.92	84.67	84.67	84.95	84.00
$\alpha = 0.99$	40.68	31.65	30.55	29.94	31.44	34.98	33.91	29.85	28.49	28.56	29.84	28.95

Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

Table 2G. Russell 2000 Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	39.12	16.12	16.18	16.05	16.14	1605	22.33	16.17	16.28	16.19	16.18	15.84
$\alpha = 0.05$	55.79	50.81	50.92	50.51	50.82	48.96	54.96	53.36	53.59	53.24	53.36	52.77
$\alpha = 0.10$	94.53	85.34	85.74	85.01	85.37	85.57	92.39	86.78	87.12	86.80	86.79	87.01
$\alpha = 0.30$	170.80	161.92	161.54	161.87	162.03	162.53	165.96	165.76	165.72	165.75	165.76	165.63
$\alpha = 0.50$	192.64	184.05	184.83	184.58	184.11	182.66	187.40	183.92	184.03	184.00	183.92	183.65
$\alpha = 0.70$	165.53	163.11	163.30	163.36	163.11	162.96	164.40	161.70	161.79	161.76	161.70	161.37
$\alpha = 0.90$	101.89	86.55	86.96	86.67	86.61	85.78	97.04	91.45	91.44	91.43	91.46	91.76
$\alpha = 0.95$	65.35	51.60	51.47	51.50	51.63	50.06	60.60	56.25	56.46	56.30	56.25	55.55
$\alpha = 0.99$	29.24	22.36	21.93	21.82	22.31	24.83	21.75	14.49	14.86	14.56	14.51	15.26

Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

Table 2H. Pacific Exchange Technology Daily Returns. Empirical applications of bagging quantile predictions

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	30.17	21.15	20.12	20.08	21.13	21.27	26.16	19.83	19.99	19.81	19.83	20.49
$\alpha = 0.05$	83.32	70.34	70.04	69.47	70.35	66.23	74.15	71.39	71.67	71.14	71.40	71.44
$\alpha = 0.10$	130.88	114.82	115.41	114.58	114.91	115.68	126.60	121.82	122.29	121.77	121.82	120.31
$\alpha = 0.30$	257.32	249.79	247.66	249.31	249.99	249.57	256.27	253.02	252.03	252.96	253.04	253.02
$\alpha = 0.50$	289.98	283.59	282.72	283.59	283.66	283.96	288.93	282.36	281.78	282.61	282.38	282.32
$\alpha = 0.70$	245.96	239.35	238.88	239.53	239.40	239.37	249.67	245.34	245.17	245.49	245.35	245.76
$\alpha = 0.90$	145.41	134.41	132.97	134.11	134.39	132.29	134.88	126.13	126.57	126.45	126.17	124.15
$\alpha = 0.95$	83.24	79.86	79.50	80.31	79.83	78.72	87.76	79.15	79.28	79.30	79.17	78.42
$\alpha = 0.99$	29.51	23.79	23.37	23.35	23.75	25.42	25.86	22.16	22.04	21.97	22.17	23.42

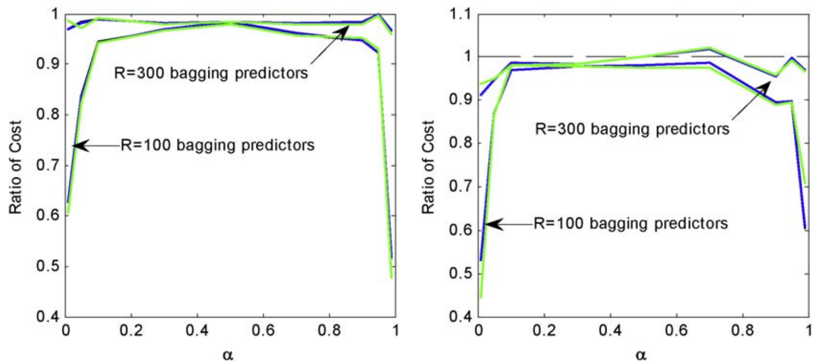
Note: Each cell gives the tick loss of quantile prediction over the period 01/05/2000–12/31/2000.

ship between $Q_\alpha(Y_{t+h}|\mathbf{X}_t)$ and $\mathbf{X}_t$ directly. To apply the iterated method for multi-step quantile, we need to set up a quantile regression model based on the lags of quantile itself, such as the CaViaR model of Engle and Manganelli (2004):

$$Q_\alpha(Y_{t+1}|\mathbf{X}_t) = b_0 + b_1 Q_\alpha(Y_t|\mathbf{X}_{t-1}) + e_{t+1}. \tag{6}$$

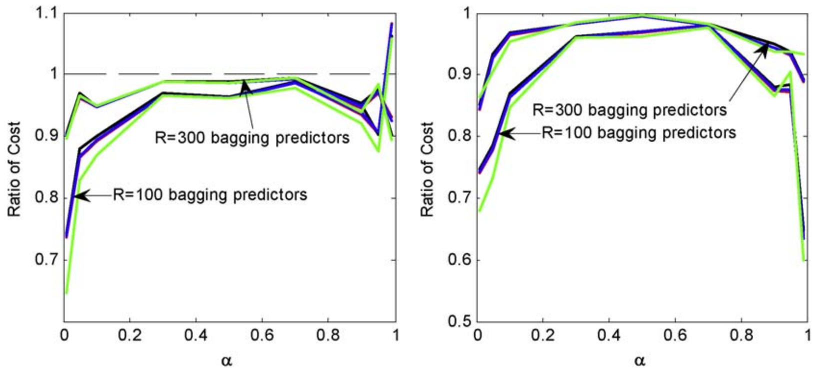
Even with the CaViaR model, we can only use "naïve" iteration to get the multi-step quantile forecast. The naïve iterated multi-step quantile method may generate poor forecasts. To be comparable with the results from other part of this chapter, we model the relationship between $Q_\alpha(Y_{t+h}|\mathbf{X}_t)$ and $\mathbf{X}_t$ directly

(a) USD/EUR Daily Returns (b) USD/JPY Daily Return



Note: Panels (a) and (b) report the tick loss ratio of bagging predictors over unbagged predictors for 08/05/2004–04/11/2005.

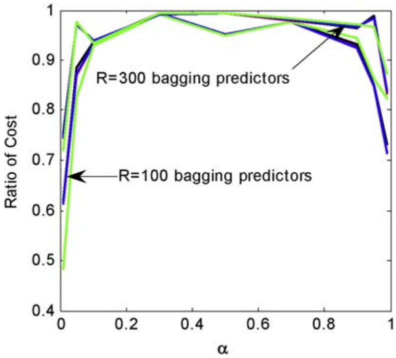
(c) Dow Jones Industrial Averages Daily Returns (d) New York Stock Exchange Composite Daily Returns



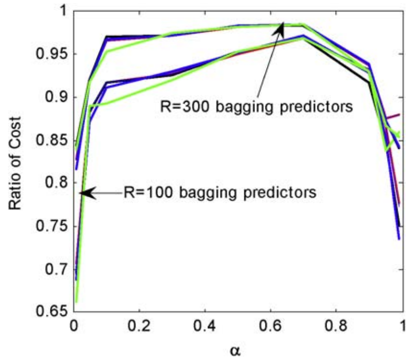
Note: Panels (c) and (d) report the tick loss ratio of bagging predictors over unbagged predictors for 01/05/2000–12/31/2000.

Fig. 2. Empirical applications of bagging quantile prediction.

(e) Standard and Poor's 500 Daily Returns

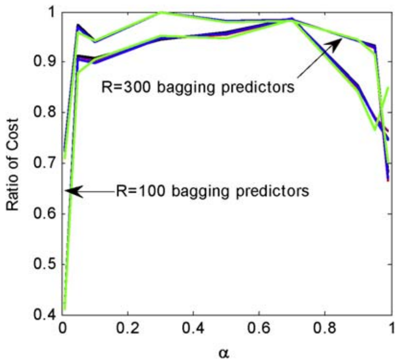


(f) NASDAQ Daily Returns

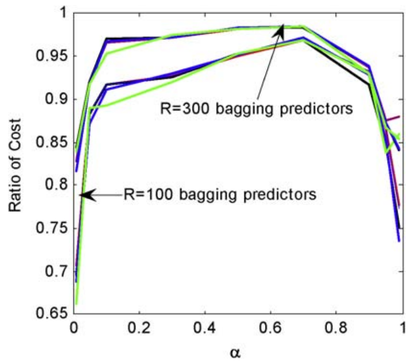


Note: Panels (e) and (f) report the tick loss ratio of bagging predictors over unbagged predictors for 01/05/2000–12/31/2000.

(g) Russell 2000 Daily Returns



(h) Pacific Exchange Technology Daily Returns



Note: Panels (g) and (h) report the tick loss ratio of bagging predictors over unbagged predictors for 01/05/2000–12/31/2000.

Fig. 2. (Continued.)

using the polynomial quantile regression model as shown in (4) in Section 2. We use the same DGP as in (3) for our Monte Carlo experiments. We will only make "direct" multi-step quantile forecasts.

According to our Monte Carlo results for quantile forecasts reported in Table 3 and Figure 3, we find the following: the loss level (T_1) for both unbagged and bagging predictors increases as the forecast horizon increase; the frequencies that bagging predictors outperform unbagged predictors (T_3) also increase with the forecast horizon; the relative average loss of bagging predictors compared to unbagged predictors ($T_{1,a}/T_{1,1}$) and relative standard error of loss for the bagging predictors compared to unbagged predictors ($T_{2,a}/T_{2,1}$) decrease with the forecast horizon.

Table 3. Bagging multi-step quantile forecasts for AR(0)-ARCH(1)-Gaussian model

$R=20,\ P=100$		$h=1$			$h=2$			$h=3$			$h=4$		
		$J=1$	$J=50$		$J=1$	$J=50$		$J=1$	$J=50$		$J=1$	$J=50$	
			mean	med		mean	med		mean	med		mean	med
$\alpha=0.01$	T_1	14.73	9.54	8.72	21.38	12.29	11.49	30.90	14.56	13.37	42.75	15.26	13.81
	T_2	6.90	2.89	2.84	13.25	4.15	3.61	23.99	6.42	5.00	53.20	6.12	4.37
	T_3		0.93	0.96		0.92	0.96		0.97	0.96		0.99	1.00
$\alpha=0.05$	T_1	19.97	14.14	13.39	26.44	17.10	16.32	35.68	18.74	17.82	46.80	19.62	18.47
	T_2	7.11	3.36	3.17	13.16	4.87	3.90	23.54	6.46	4.97	51.51	6.06	4.85
	T_3		0.97	0.98		0.98	0.97		0.98	0.98		0.99	0.99
$\alpha=0.10$	T_1	25.46	19.52	18.66	30.83	22.52	22.05	37.88	23.89	23.06	50.88	25.05	24.05
	T_2	6.75	3.39	3.32	11.72	5.20	4.24	17.48	5.94	4.92	48.86	6.55	5.19
	T_3		0.98	1.00		0.99	1.00		0.99	0.99		1.00	1.00
$\alpha=0.30$	T_1	40.75	34.52	33.55	43.05	36.70	36.36	45.89	37.50	37.00	51.27	38.32	37.78
	T_2	7.03	4.72	4.72	7.90	5.44	5.24	11.32	5.81	5.48	14.26	6.22	5.76
	T_3		1.00	1.00		0.99	0.99		0.98	1.00		1.00	1.00
$\alpha=0.50$	T_1	45.31	38.88	38.08	47.61	40.81	40.67	49.12	41.54	41.11	53.58	42.16	41.70
	T_2	7.12	5.25	5.22	8.27	5.91	5.86	9.02	5.80	5.67	13.66	6.43	6.07
	T_3		1.00	1.00		1.00	1.00		0.99	1.00		1.00	1.00
$\alpha=0.70$	T_1	40.10	34.36	33.38	42.91	36.14	35.89	44.81	37.24	36.83	49.11	37.74	37.21
	T_2	6.32	5.11	4.83	8.27	5.54	5.31	9.67	5.72	5.24	13.28	6.05	5.61
	T_3		0.99	1.00		0.99	0.99		0.99	0.99		0.98	0.99
$\alpha=0.90$	T_1	24.64	19.55	18.70	30.02	21.46	21.12	34.45	23.22	22.45	43.77	23.87	23.00
	T_2	5.18	3.79	3.37	10.98	4.80	4.59	17.75	5.67	4.91	24.49	5.53	5.01
	T_3		0.97	0.99		1.00	0.99		0.99	0.98		1.00	1.00
$\alpha=0.95$	T_1	19.04	14.07	13.17	26.34	16.44	15.54	31.06	18.29	17.27	39.54	18.62	17.72
	T_2	5.48	3.97	3.14	13.45	5.46	4.06	21.97	5.74	4.71	25.76	5.54	4.79
	T_3		0.96	0.97		0.96	0.98		0.95	0.95		1.00	0.98

Table 3. (continued)

$R = 20,\ P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$	
			mean	med		mean	med		mean	med		mean	med
$\alpha = 0.99$	T_1	14.00	9.51	8.49	21.31	11.40	10.63	26.30	13.49	12.35	35.31	14.48	13.07
	T_2	5.42	3.24	2.73	13.56	3.97	3.53	22.49	5.23	4.32	26.23	5.63	4.36
	T_3		0.93	0.94		0.98	0.98		0.92	0.94		0.98	0.99
$R = 50,\ P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
$\alpha = 0.01$	T_1	5.74	4.30	4.41	8.16	5.34	5.59	12.22	5.50	5.77	12.16	6.15	6.48
	T_2	2.45	1.39	1.49	6.66	2.32	2.38	41.34	2.53	2.72	17.53	3.10	3.51
	T_3		0.83	0.91		0.89	0.86		0.93	0.87		0.87	0.84
$\alpha = 0.05$	T_1	12.31	10.89	10.89	13.98	12.15	12.36	14.68	12.37	12.67	16.44	12.91	13.12
	T_2	2.38	1.81	1.82	3.97	2.86	2.80	4.57	3.13	3.38	8.73	3.68	3.74
	T_3		0.92	0.94		0.89	0.83		0.94	0.89		0.90	0.88
$\alpha = 0.10$	T_1	19.27	17.73	17.65	20.51	18.85	18.92	21.00	19.28	19.36	22.20	19.65	19.66
	T_2	3.02	2.57	2.61	4.52	3.25	3.27	5.14	3.81	3.86	6.97	4.29	3.91
	T_3		0.94	0.98		0.89	0.90		0.87	0.85		0.92	0.91
$\alpha = 0.30$	T_1	35.74	33.59	33.56	36.02	34.13	34.29	36.10	34.42	34.54	36.36	34.57	34.58
	T_2	4.83	4.26	4.32	5.42	4.41	4.56	5.59	4.68	4.69	5.47	4.82	4.66
	T_3		0.98	0.99		0.94	0.94		0.91	0.88		0.94	0.94
$\alpha = 0.50$	T_1	40.62	37.98	37.88	40.25	38.29	38.38	40.27	38.52	38.53	40.03	38.45	38.54
	T_2	5.43	4.42	4.45	6.03	4.56	4.64	5.44	4.72	4.70	5.31	4.67	4.70
	T_3		0.99	1.00		0.90	0.93		0.95	0.96		0.91	0.91
$\alpha = 0.70$	T_1	35.56	33.14	33.12	35.75	33.71	33.77	35.59	33.89	33.84	35.61	33.98	34.00
	T_2	5.12	3.91	4.01	6.13	4.28	4.32	5.22	4.23	4.17	4.86	4.44	4.35
	T_3		0.98	0.99		0.94	0.92		0.92	0.93		0.90	0.90

(*continued on next page*)

Table 3. (continued)

$R = 50,\ P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$	
			mean	med		mean	med		mean	med		mean	med
$\alpha = 0.90$	T_1	19.19	17.37	17.29	20.87	18.51	18.58	20.91	18.76	18.84	21.39	19.10	19.17
	T_2	3.96	2.63	2.52	6.46	3.50	3.54	5.32	3.29	3.39	7.79	4.56	4.24
	T_3		0.95	0.96		0.95	0.90		0.90	0.89		0.91	0.92
$\alpha = 0.95$	T_1	12.15	10.62	10.61	14.29	11.89	12.10	14.89	12.21	12.53	15.27	12.50	12.67
	T_2	3.57	1.92	2.03	6.16	2.88	3.05	5.69	2.96	3.07	5.63	3.69	3.67
	T_3		0.88	0.91		0.90	0.89		0.89	0.85		0.94	0.91
$\alpha = 0.99$	T_1	5.49	4.05	4.10	7.66	5.00	5.17	9.32	5.34	5.58	9.19	5.72	6.04
	T_2	3.96	1.49	1.56	5.98	2.16	2.34	9.26	2.50	2.54	6.27	3.14	2.99
	T_3		0.85	0.88		0.89	0.84		0.87	0.87		0.87	0.83
$R = 100,\ P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
$\alpha = 0.01$	T_1	3.69	3.19	3.22	5.32	3.87	4.20	5.41	3.92	4.23	6.01	4.12	4.40
	T_2	1.58	1.11	1.21	5.43	1.70	2.20	3.37	1.79	2.20	4.68	2.17	2.40
	T_3		0.78	0.83		0.83	0.75		0.82	0.78		0.89	0.86
$\alpha = 0.05$	T_1	10.97	10.27	10.31	12.23	11.51	11.65	12.60	11.59	11.75	12.84	11.68	11.79
	T_2	2.23	1.89	1.99	3.22	2.84	2.97	3.63	2.99	3.10	4.33	3.25	3.27
	T_3		0.83	0.91		0.89	0.83		0.90	0.85		0.92	0.88
$\alpha = 0.10$	T_1	17.83	17.10	17.12	18.98	18.26	18.32	19.09	18.30	18.38	19.44	18.36	18.40
	T_2	3.27	2.72	2.79	4.15	3.52	3.58	3.88	3.52	3.60	4.69	3.76	3.81
	T_3		0.85	0.87		0.87	0.85		0.89	0.86		0.92	0.92
$\alpha = 0.30$	T_1	34.28	33.18	33.17	34.29	33.62	33.65	34.29	33.60	33.61	34.43	33.64	33.63
	T_2	5.12	4.73	4.70	5.41	4.90	4.90	5.26	4.92	4.88	5.47	5.03	5.03
	T_3		0.95	0.95		0.84	0.86		0.84	0.86		0.84	0.86

Table 3. (continued)

$R = 100, P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$	
			mean	med		mean	med		mean	med		mean	med
$\alpha = 0.50$	T_1	38.87	37.58	37.59	38.59	37.84	37.86	38.58	37.91	37.94	38.57	37.90	37.93
	T_2	6.13	5.22	5.26	5.68	5.35	5.35	5.63	5.37	5.37	5.60	5.38	5.39
	T_3		0.98	0.99		0.87	0.89		0.91	0.90		0.85	0.83
$\alpha = 0.70$	T_1	33.86	32.78	32.77	34.04	33.32	33.33	33.97	33.30	33.32	33.94	33.37	33.40
	T_2	5.53	4.82	4.86	5.30	4.96	4.93	5.48	5.04	5.00	5.20	4.99	5.03
	T_3		0.90	0.94		0.80	0.85		0.88	0.84		0.84	0.84
$\alpha = 0.90$	T_1	17.58	16.86	16.83	18.66	17.92	18.00	18.86	18.04	18.07	18.89	18.18	18.22
	T_2	3.13	2.73	2.79	4.10	3.47	3.66	4.41	3.67	3.78	4.31	3.85	3.93
	T_3		0.87	0.89		0.85	0.85		0.84	0.86		0.86	0.87
$\alpha = 0.95$	T_1	10.79	10.14	10.16	12.13	11.32	11.42	12.30	11.39	11.52	12.55	11.58	11.71
	T_2	2.43	1.91	2.01	3.58	2.81	2.88	3.56	2.75	2.99	3.90	3.21	3.41
	T_3		0.82	0.90		0.79	0.79		0.88	0.82		0.86	0.87
$\alpha = 0.99$	T_1	4.07	3.21	3.28	5.20	3.87	4.13	5.20	3.92	4.21	6.34	4.09	4.43
	T_2	2.75	1.20	1.41	3.42	1.76	2.04	3.02	1.71	2.04	6.29	2.07	2.65
	T_3		0.85	0.89		0.80	0.78		0.84	0.77		0.91	0.90
$R = 200, P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
$\alpha = 0.01$	T_1	2.92	2.72	2.73	3.44	3.24	3.30	3.85	3.37	3.41	3.97	3.41	3.48
	T_2	1.04	0.89	1.03	1.51	1.37	1.44	1.95	1.53	1.58	2.48	1.82	1.81
	T_3		0.72	0.73		0.67	0.66		0.82	0.83		0.78	0.78
$\alpha = 0.05$	T_1	9.97	9.68	9.72	10.78	10.56	10.60	11.05	10.67	10.68	11.19	10.70	10.75
	T_2	1.83	1.63	1.69	2.31	2.19	2.28	2.76	2.40	2.41	3.07	2.42	2.47
	T_3		0.80	0.83		0.76	0.73		0.75	0.74		0.82	0.80

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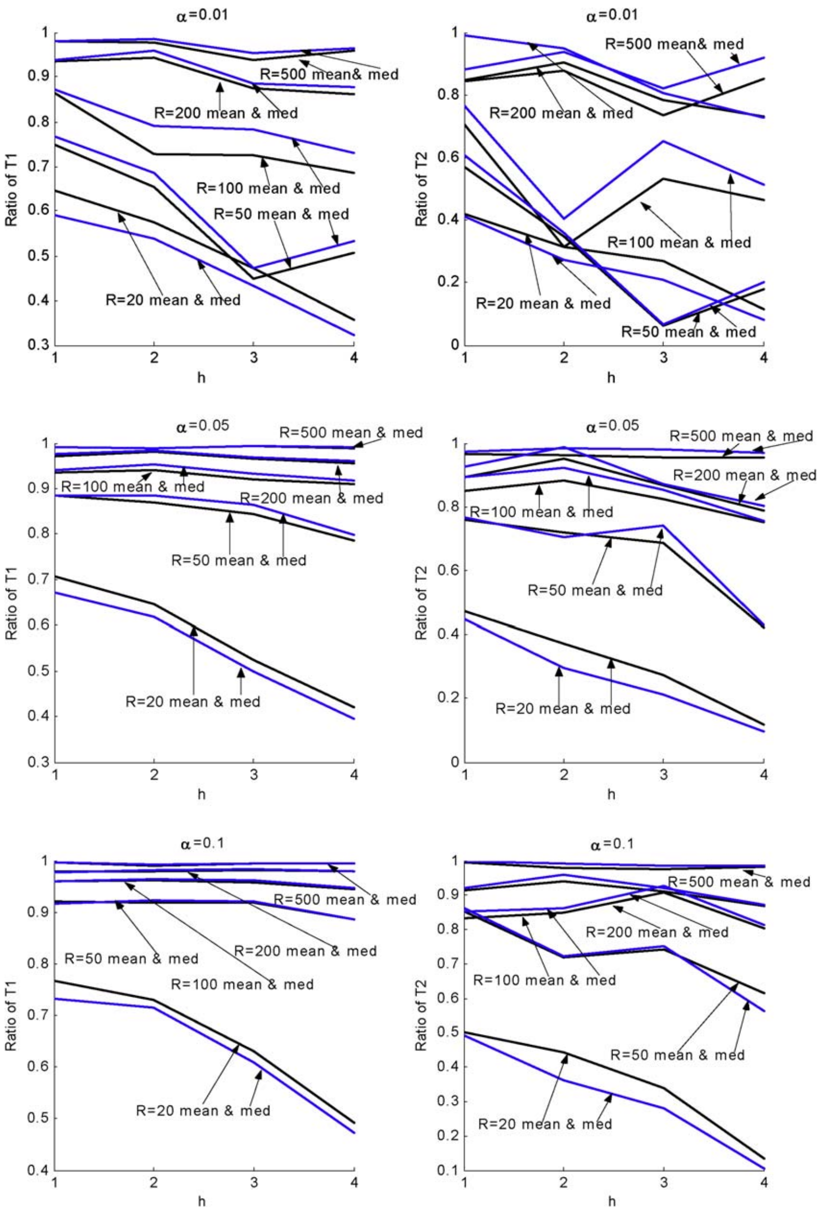
Table 3. (continued)

$R = 200$, $P = 100$		$h = 1$			$h = 2$			$h = 3$			$h = 4$		
		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$		$J = 1$	$J = 50$	
			mean	med		mean	med		mean	med		mean	med
$\alpha = 0.10$	T_1	16.73	16.38	16.35	17.66	17.31	17.33	17.67	17.34	17.39	17.74	17.36	17.39
	T_2	2.73	2.50	2.51	3.21	3.02	3.08	3.38	3.08	3.11	3.60	3.13	3.14
	T_3		0.75	0.82		0.75	0.69		0.67	0.68		0.73	0.77
$\alpha = 0.30$	T_1	32.86	32.37	32.37	33.02	32.72	32.73	33.09	32.73	32.74	32.98	32.75	32.77
	T_2	4.51	4.30	4.29	4.63	4.46	4.46	4.70	4.50	4.49	4.73	4.54	4.55
	T_3		0.80	0.83		0.66	0.67		0.74	0.75		0.68	0.65
$\alpha = 0.50$	T_1	37.53	37.02	37.04	37.44	37.16	37.18	37.53	37.21	37.24	37.50	37.22	37.24
	T_2	4.90	4.75	4.74	5.08	4.95	4.95	5.16	5.02	5.03	5.03	4.96	4.97
	T_3		0.86	0.88		0.66	0.67		0.81	0.83		0.74	0.79
$\alpha = 0.70$	T_1	32.70	32.23	32.23	32.96	32.71	32.73	32.95	32.70	32.72	32.93	32.69	32.73
	T_2	4.42	4.24	4.26	4.69	4.58	4.57	4.73	4.60	4.63	4.68	4.58	4.58
	T_3		0.83	0.87		0.80	0.75		0.74	0.76		0.74	0.71
$\alpha = 0.90$	T_1	16.86	16.48	16.53	17.77	17.49	17.48	17.75	17.50	17.51	17.95	17.60	17.61
	T_2	2.58	2.42	2.49	3.31	3.23	3.23	3.51	3.31	3.35	3.50	3.33	3.35
	T_3		0.76	0.80		0.76	0.78		0.74	0.79		0.79	0.82
$\alpha = 0.95$	T_1	10.12	9.87	9.88	11.15	10.96	11.02	11.30	10.99	11.03	11.39	11.07	11.06
	T_2	1.74	1.62	1.61	2.56	2.44	2.54	2.97	2.53	2.59	3.05	2.65	2.67
	T_3		0.78	0.80		0.70	0.66		0.72	0.74		0.71	0.75
$\alpha = 0.99$	T_1	2.96	2.75	2.77	3.73	3.39	3.48	4.13	3.48	3.61	3.97	3.43	3.54
	T_2	1.05	0.77	0.80	1.54	1.34	1.34	2.53	1.45	1.50	2.00	1.43	1.52
	T_3		0.70	0.73		0.71	0.65		0.82	0.77		0.87	0.85

Notes: The ARCH parameter is $\theta = 0.5$ as defined in Equation (3). The three rows of each multi-step forecast method report the average, the standard error and the frequency of better performance of bagging predictors in terms of tick loss computed from 100 Monte Carlo replications. See the definition of T_1, T_2, and T_3 in the text.



Note: The two figures in each row report the tick loss ratio and standard error ratio of bagging predictors over unbagged predictors over 100 Monte Carlo replications (see the detailed explanation in the main text).

Fig. 3. Bagging multi-step quantile forecasts for AR(0)-ARCH(1)-Gaussian model.

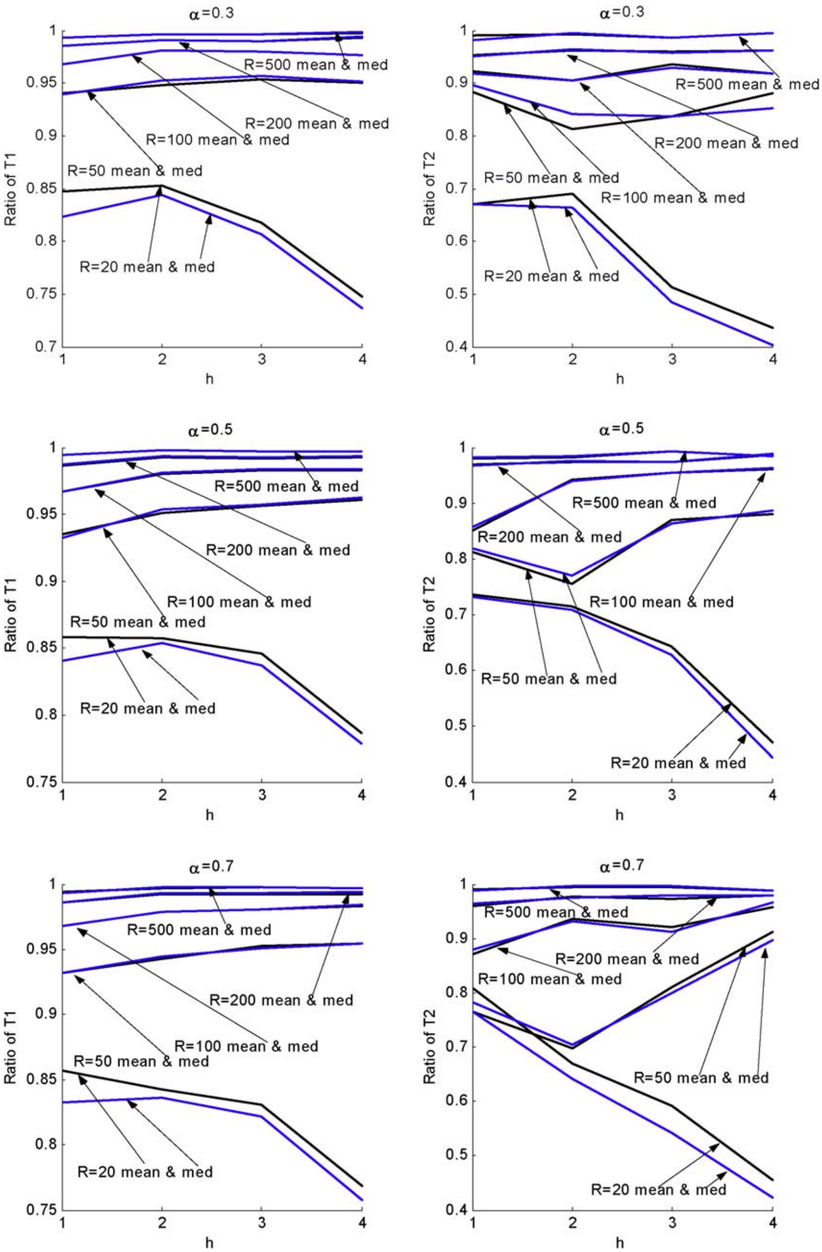


Fig. 3. (Continued.)

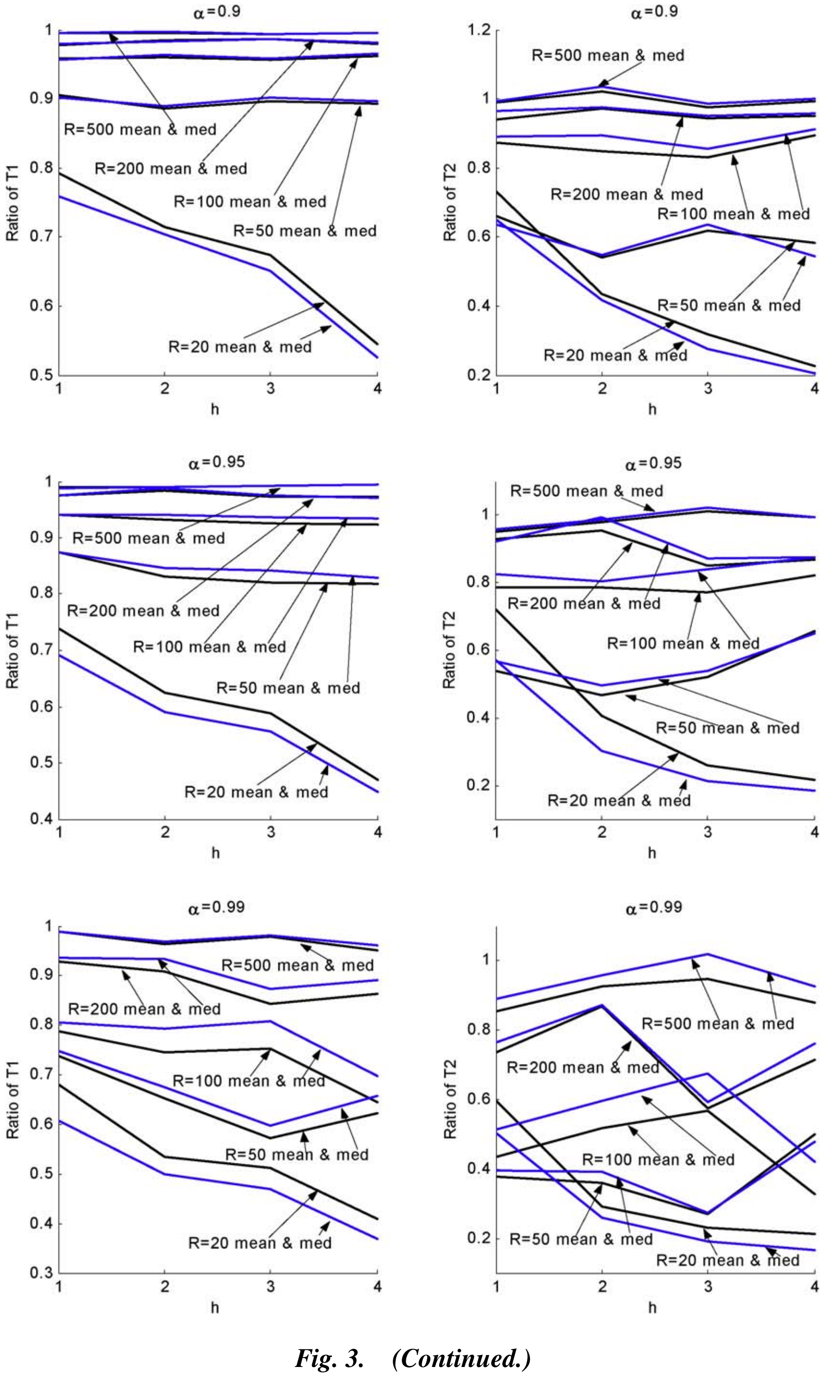


Fig. 3. (Continued.)

5. Bagging quantile forecasts with different tick losses

Komunjer (2005) introduced a tick-exponential family defined by:

$$\begin{aligned}&\varphi^{\alpha}_{t+h}\big(Y_{t+h}, Q_\alpha(Y_{t+h}|\mathbf{X}_t)\big)\\ &\quad = \exp\big(-(1-\alpha)\big[a_t\big(Q_\alpha(Y_{t+h}|\mathbf{X}_t)\big) - b_t(Y_{t+h})\big]\mathbf{1}\big\{Y_{t+h} \leqslant Q_\alpha(Y_{t+h}|\mathbf{X}_t)\big\}\\ &\qquad + \alpha\big[a_t\big(Q_\alpha(Y_{t+h}|\mathbf{X}_t)\big) - c_t(Y_{t+h})\big]\mathbf{1}\big\{Y_{t+h} > Q_\alpha(Y_{t+h}|\mathbf{X}_t)\big\}\big), \qquad (7)\end{aligned}$$

where (i) a_t is continuously differentiable function and b_t and c_t are $\mathcal{F}_t$-measurable functions; (ii) φ^{α}_{t+h} is a probability density; (iii) $Q_\alpha(Y_{t+h}|\mathbf{X}_t)$ is the α-quantile of $Y_{t+h}|\mathbf{X}_t$.

A class of quasi-maximum likelihood estimators (QMLEs), $\hat{\boldsymbol{\beta}}_{\alpha,h}(\mathcal{D}_t)$, can be obtained by solving

$$\boldsymbol{\beta}_{\alpha,h}(\mathcal{D}_t) = \arg\max_{\boldsymbol{\beta}_{\alpha,h}} R^{-1} \sum_{s=t-R+h+1}^{t} \ln \varphi^{\alpha}_{s}\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big). \qquad (8)$$

If $a_t(Q_\alpha(Y_{t+h}|\mathbf{X}_t)) = Q_\alpha(Y_{t+h}|\mathbf{X}_t)$ and $b_t(Y_{t+h}) = c_t(Y_{t+h}) = Y_{t+h}$, the maximizing problem in (8) is equivalent to the minimization problem of Koenker and Bassett (1978) as shown in (2). We also try another group of exponential tick family loss functions introduced by Komunjer (2005) by setting

$$a_t(\eta) = b_t(\eta) = c_t(\eta) = \frac{1}{\alpha(1-\alpha)}\operatorname{sgn}(\eta)\ln\big(1+|\eta|^p\big), \qquad (9)$$

where $\operatorname{sgn}(\eta) \equiv \mathbf{1}\{\eta \geqslant 0\} - \mathbf{1}\{\eta < 0\}$.

Our empirical results of quantile prediction for S&P 500 daily return during 01/13/2004–01/07/2005 ($P = 250$) using the rolling estimation samples with $R = 100$ and 300 (10/31/2002–01/07/2005) is shown in Table 4, where "tick"

Table 4. Bagging quantile predictions for S&P 500 daily returns using tick-exponential losses

		$R = 100$					
		$J = 1$	$J = 50$				
			mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	tick	5.59	5.45	5.45	5.45	5.45	5.81
	$p = 1$	5.95	5.30	5.30	5.30	5.30	5.21
	$p = 2$	4.90	4.22	4.22	4.22	4.22	4.22
	$p = 3$	4.51	4.26	4.26	4.26	4.26	4.29
$\alpha = 0.05$	tick	20.49	19.46	19.46	19.46	19.46	19.75
	$p = 1$	20.61	19.66	19.66	19.66	19.66	19.52
	$p = 2$	19.40	19.16	19.16	19.16	19.16	19.22
	$p = 3$	19.17	19.20	19.20	19.20	19.20	19.19
$\alpha = 0.10$	tick	33.95	33.08	33.08	33.08	33.08	33.25
	$p = 1$	33.98	33.19	33.19	33.19	33.19	33.26
	$p = 2$	33.60	33.12	33.11	33.12	33.12	33.03
	$p = 3$	33.01	33.20	33.20	33.20	33.20	33.04

Table 4. (continued)

		R = 100					
		J = 1	J = 50				
			mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.30$	tick	65.93	63.52	63.52	63.52	63.52	63.90
	$p = 1$	65.23	63.60	63.60	63.60	63.60	63.85
	$p = 2$	64.00	62.93	62.93	62.93	62.93	62.71
	$p = 3$	63.60	63.65	63.65	63.65	63.65	63.56
$\alpha = 0.50$	tick	69.99	68.66	68.66	68.66	68.66	68.65
	$p = 1$	70.03	68.66	68.66	68.66	68.66	68.67
	$p = 2$	68.81	68.25	68.25	68.25	68.25	68.26
	$p = 3$	74.22	73.62	73.62	73.62	73.62	73.87
$\alpha = 0.70$	tick	59.81	59.85	59.85	59.85	59.85	59.87
	$p = 1$	60.44	59.84	59.84	59.84	59.84	59.99
	$p = 2$	59.48	59.55	59.55	59.55	59.55	59.69
	$p = 3$	69.18	64.53	64.53	64.53	64.53	67.42
$\alpha = 0.90$	tick	32.37	31.95	31.95	31.95	31.95	32.13
	$p = 1$	31.55	31.91	31.91	31.91	31.91	32.07
	$p = 2$	31.72	31.87	31.87	31.87	31.87	32.03
	$p = 3$	32.68	31.83	31.83	31.83	31.83	32.19
$\alpha = 0.95$	tick	19.68	18.94	18.94	18.94	18.94	19.36
	$p = 1$	19.87	18.82	18.82	18.82	18.82	19.26
	$p = 2$	18.57	18.43	18.43	18.43	18.43	18.23
	$p = 3$	18.02	18.41	18.41	18.41	18.41	18.25
$\alpha = 0.99$	tick	5.44	5.33	5.33	5.33	5.33	5.52
	$p = 1$	5.19	5.38	5.39	5.38	5.38	5.49
	$p = 2$	4.66	4.69	4.69	4.69	4.69	4.55
	$p = 3$	4.52	4.26	4.25	4.26	4.25	4.16
		R = 200					
$\alpha = 0.01$	tick	4.66	4.74	4.74	4.74	4.74	4.72
	$p = 1$	4.84	4.81	4.81	4.81	4.81	4.74
	$p = 2$	4.28	4.54	4.43	4.42	4.43	4.22
	$p = 3$	4.95	4.64	4.64	4.64	4.64	4.31
$\alpha = 0.05$	tick	19.33	18.76	18.76	18.76	18.76	19.09
	$p = 1$	19.36	18.81	18.81	18.81	18.81	19.01
	$p = 2$	18.71	18.63	18.63	18.63	18.63	18.55
	$p = 3$	18.51	18.41	18.41	18.41	18.41	18.53
$\alpha = 0.10$	tick	33.68	33.03	33.03	33.03	33.03	33.05
	$p = 1$	33.55	32.99	32.99	32.99	32.99	32.98
	$p = 2$	33.13	32.56	32.56	32.56	32.56	32.55
	$p = 3$	32.50	32.53	32.53	32.53	32.53	32.58
$\alpha = 0.30$	tick	63.51	62.66	62.66	62.66	62.66	62.64
	$p = 1$	63.46	62.65	62.65	62.65	62.65	62.62
	$p = 2$	62.69	62.74	62.74	62.74	62.74	62.80
	$p = 3$	63.13	63.04	63.04	63.04	63.04	62.97
$\alpha = 0.50$	tick	68.74	68.12	68.12	68.12	68.12	68.02
	$p = 1$	68.65	68.17	68.17	68.17	68.17	68.16
	$p = 2$	68.55	68.05	68.05	68.05	68.05	68.18
	$p = 3$	72.07	71.93	71.93	71.93	71.93	72.03

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Table 4. (continued)

		R = 200					
		$J = 1$	$J = 50$				
			mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.70$	tick	60.08	59.49	59.49	59.49	59.49	59.41
	$p = 1$	60.10	59.48	59.48	59.48	59.48	59.44
	$p = 2$	59.59	59.77	59.77	59.77	59.77	59.96
	$p = 3$	65.21	60.55	60.55	60.55	60.55	60.79
$\alpha = 0.90$	tick	32.19	31.45	31.45	31.45	31.45	31.34
	$p = 1$	31.64	31.43	31.43	31.43	31.43	31.33
	$p = 2$	32.48	31.93	31.93	31.93	31.93	32.07
	$p = 3$	33.71	31.11	31.11	31.11	31.11	31.77
$\alpha = 0.95$	tick	18.99	18.84	18.84	18.84	18.84	18.80
	$p = 1$	18.94	18.92	18.92	18.92	18.92	18.84
	$p = 2$	19.01	18.55	18.55	18.55	18.55	18.40
	$p = 3$	18.58	18.99	18.99	18.99	18.98	18.44
$\alpha = 0.99$	tick	4.75	4.41	4.41	4.41	4.41	4.42
	$p = 1$	4.44	4.45	4.45	4.45	4.45	4.44
	$p = 2$	4.68	4.34	4.34	4.34	4.34	4.22
	$p = 3$	5.74	4.98	4.97	4.97	4.97	4.23
		R = 300					
$\alpha = 0.01$	tick	4.78	4.77	4.77	4.77	4.77	4.77
	$p = 1$	4.73	4.84	4.84	4.84	4.84	4.73
	$p = 2$	5.06	5.06	5.01	5.02	5.02	4.84
	$p = 3$	5.44	5.71	5.71	5.71	5.71	4.92
$\alpha = 0.05$	tick	18.51	18.40	18.40	18.40	18.40	18.45
	$p = 1$	18.48	18.27	18.27	18.27	18.27	18.20
	$p = 2$	19.41	18.26	18.26	18.26	18.26	18.29
	$p = 3$	18.39	18.28	18.28	18.28	18.28	18.23
$\alpha = 0.10$	tick	32.62	31.95	31.95	31.95	31.95	31.85
	$p = 1$	32.47	31.97	31.97	31.97	31.97	31.86
	$p = 2$	32.14	32.57	32.57	32.57	32.57	32.71
	$p = 3$	32.28	32.15	32.15	32.15	32.15	32.30
$\alpha = 0.30$	tick	63.18	62.84	62.84	62.84	62.84	62.72
	$p = 1$	63.02	62.87	62.87	62.87	62.87	62.72
	$p = 2$	62.84	62.72	62.72	62.72	62.72	62.90
	$p = 3$	62.88	62.80	62.80	62.80	62.80	62.74
$\alpha = 0.50$	tick	68.48	67.69	67.69	67.69	67.69	67.65
	$p = 1$	68.39	67.70	67.70	67.70	67.70	67.55
	$p = 2$	68.85	67.91	67.91	67.91	67.91	67.88
	$p = 3$	71.44	71.79	71.79	71.79	71.79	71.89
$\alpha = 0.70$	tick	60.54	60.10	60.10	60.10	60.10	60.20
	$p = 1$	60.52	60.08	60.08	60.08	60.08	60.08
	$p = 2$	60.39	59.60	59.60	59.60	59.60	59.60
	$p = 3$	68.61	59.50	59.50	59.50	59.50	61.56
$\alpha = 0.90$	tick	33.31	32.75	32.75	32.75	32.75	32.62
	$p = 1$	33.29	32.74	32.74	32.74	32.74	32.64
	$p = 2$	33.32	32.78	32.78	32.78	32.78	32.74
	$p = 3$	33.25	31.69	31.69	31.69	31.69	32.28

Table 4. (continued)

		$R = 300$					
		$J = 1$	$J = 50$				
			mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.95$	tick	20.03	19.40	19.40	19.40	19.40	19.33
	$p = 1$	19.89	19.38	19.38	19.38	19.38	19.36
	$p = 2$	19.74	19.45	19.45	19.45	19.45	19.45
	$p = 3$	20.96	19.28	19.28	19.28	19.28	19.44
$\alpha = 0.99$	tick	5.96	5.30	5.30	5.30	5.30	5.46
	$p = 1$	5.95	5.52	5.52	5.52	5.52	5.60
	$p = 2$	6.20	5.69	5.69	5.69	5.69	5.42
	$p = 3$	11.94	5.61	5.55	5.55	5.54	4.85

Notes: The four rows for each α report the tick loss of quantile predictors estimated by four different tick-exponential loss function as defined in text. The out-of-sample evaluation period is 01/13/2004–01/07/2005.

denotes $\boldsymbol{\beta}_{\alpha,h}(\mathcal{D}_t)$ is estimated using (2) and $p = 1, 2, 3$ denote that $\boldsymbol{\beta}_{\alpha,h}(\mathcal{D}_t)$ is estimated using (9). We can see from the table that no matter which tick losses we use, bagging predictors have lower quantile cost when α is small and have no obvious advantage over the unbagged predictors when α is large.

6. Bagging quantile forecasts with different estimation algorithms

If we want to forecast the conditional mean, usually it is not a big problem to estimate the parameters for linear models and most nonlinear models. However, for quantile forecasts, since the quantile loss function is not differentiable, it is very hard to estimate model parameters, especially when we use nonlinear quantile regression models. Algorithms that can be used for the quantile estimation have been reviewed by Buchinsky (1998), Koenker and Park (1996), Frenk *et al.* (1994), Chernozhukov and Hong (2003), and Komunjer (2005) for both linear quantile and nonlinear quantile models. We compare two different algorithms for quantile estimation in this chapter in terms of bagging. The two algorithms are the interior point algorithm introduced by Portnoy and Koenker (1997) and minimax algorithm introduced by Komunjer (2005).

Portnoy and Koenker (1997) propose statistical pre-processing for general quantile regression problems and combine it with "interior point" methods for solving linear programs. The following is a brief explanation on how to apply the interior algorithm for quantile estimation. If we put all the error terms u_s for $s = t - R + h + 1, t - R + h + 2, \ldots, t$ in (5) into positive numbers, the quantile estimation problem can be rewritten as

$$\hat{\boldsymbol{\beta}}_{\alpha,h}(\mathcal{D}_t) = \arg\min_{\boldsymbol{\beta}_{\alpha,h}} \big(\alpha u^+ + (1-\alpha)u^- | Y_s = \tilde{\mathbf{X}}'_{s-h}\boldsymbol{\beta}_{\alpha,h} + u_s\big), \tag{10}$$

where u^+ is an $(R-h)$-vector of positive errors or zeros and u^- is an $(R-h)$-vector of absolute values of negative errors or zeros. Portnoy and Koenker (1997) show that the optimization program (10) can be rewritten into the following dual formulations:

$$\omega = \arg\max_{\omega}\left(\sum_{s=t-R+h+1}^{t} Y_s\omega_s \,\middle|\, \sum_{s=t-R+h+1}^{t} \tilde{\mathbf{X}}_{s-h}\omega_s = 0,\ \omega_s \in [-1, 1] \right), \tag{11}$$

where $\omega_s = 1$ if $u_s > 0$, $\omega_s = -1$ if $u_s < 0$, $-1 < \omega_s < 1$ if $u_s = 0$; and ω_s is like a Lagrange multiplier on the constraints or marginal cost of relaxing the constraints. The optimization problem in (11) is the standard formulations of interior point methods for linear programs with bounded variables.

The interior point algorithm is easy to apply, runs fast, and is embodied in most popular computer software (for example, GAUSS and MATLAB). However, the interior point algorithm can only be used for linear quantile models with the tick loss function. If we have a nonlinear quantile regression model or use the tick-exponential family introduced by Komunjer (2005) for the quantile estimation, we have to choose another algorithm for parameter estimation.

Komunjer (2005) introduces a new quantile regression algorithm – the minimax algorithm, which is a more flexible method than the interior point algorithm and can be used for nonlinear quantile regression models and more general quantile loss functions. The idea is that the function $\varphi_{t+h}^{\alpha}(Y_{t+h}, Q_\alpha(Y_{t+h}|\mathbf{X}_t))$ in (7) is twice continuously differentiable by parts and the optimization problem in (8) can be represented as a maximum of two separated branches which are both convex and twice continuously differentiable. Defining

$$\psi_s^{\alpha}\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big) \equiv \exp\big\{\alpha\big[a_s\big(Q_\alpha(Y_s|\mathbf{X}_{s-h})\big) - c_t(Y_s)\big]\big\}$$

and

$$\phi_s^{\alpha}\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big) \equiv \exp\big\{-(1-\alpha)\big[a_s\big(Q_\alpha(Y_s|\mathbf{X}_{s-h})\big) - b_t(Y_s)\big]\big\},$$

the optimization problem in (8) becomes $\max \ln \psi_1^{\alpha}(Y_1, Q_\alpha(Y_1|\mathbf{X}_0))$ in the case of $t = 1$ and $h = 1$, i.e.,

$$\max\min\big\{\ln \psi_1^{\alpha}\big(Y_1, Q_\alpha(Y_1|\mathbf{X}_0)\big), \ln \phi_1^{\alpha}\big(Y_1, Q_\alpha(Y_1|\mathbf{X}_0)\big)\big\},$$

or equivalently,

$$-\min\big[\max\big\{-\ln \psi_1^{\alpha}\big(Y_1, Q_\alpha(Y_1|\mathbf{X}_0)\big), -\ln \phi_1^{\alpha}\big(Y_1, Q_\alpha(Y_1|\mathbf{X}_0)\big)\big\}\big].$$

Therefore, the maximization problem in (8) is transformed into a minimax problem.

Using this idea, Komunjer (2005, Theorem 6) shows that the QMLE estimator $\hat{\boldsymbol{\beta}}_{\alpha,h}(\mathcal{D}_t)$ from (8) can be written as a solution to a minimax problem:

$$\min_{\boldsymbol{\beta}_{\alpha,h}(\mathcal{D}_t)}\Big[\max_{t-R+h\leqslant k\leqslant t}\big\{-P_k\big(Y, Q_\alpha(Y|\mathbf{X})\big)\big\}\Big],$$

where

$$P_k\big(Y, Q_\alpha(Y|\mathbf{X})\big) \equiv \begin{cases} (R-h)^{-1} \displaystyle\sum_{s=t-R+h+1}^{t} \ln \psi_s^\alpha\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big), & \text{if } k = t-R+h, \\ (R-h)^{-1}\left[\displaystyle\sum_{s=t-R+h+1}^{k} \ln \phi_s^\alpha\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big) \right. & \\ \left. \quad + \displaystyle\sum_{s=k+1}^{t} \ln \psi_s^\alpha\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big)\right], & \text{if } t-R+h < k < t, \\ (R-h)^{-1} \displaystyle\sum_{s=t-R+h+1}^{t} \ln \phi_s^\alpha\big(Y_s, Q_\alpha(Y_s|\mathbf{X}_{s-h})\big), & \text{if } k = t. \end{cases}$$

Intuitively, Komunjer's minimax algorithm can be decomposed into two step. First, for a given set of parameters, we assign all the forecast errors proper costs to make sure all the forecast errors get positive punishment, i.e., maximize the punishment for a given set of parameters. The second step is to find the set of parameters that can minimize the forecast cost. However, the minimax algorithm runs slower than the interior point algorithm and is therefore more computationally costly.

To compare the two algorithms, we can check tick losses of quantile prediction for S&P 500 daily returns reported in Table 4 (minimax algorithm) and Table 5 (interior point algorithm). The interior point algorithm and minimax algorithm give somewhat different results. Therefore, in small samples, bagging may work differently depending on the estimation algorithm.

7. Bagging quantile forecasts with different quantile regression models

With the flexibility provided by the minimax algorithm, we check the performance of bagging predictors on highly nonlinear quantile regression models – artificial neural network models. Given model uncertainty, when the sample size is limited, it is usually hard to choose the number of hidden nodes and the number of inputs (lags) and estimate the large number of parameters in a neural network model. Therefore, a neural network model can generate poor predictions with a small sample. In such cases, bagging can do a wonderful job improving forecasting performance.

The nonlinear quantile regression function we use in this section is the univariate single-layer feed-forward artificial neural network function of White (1992). Following the definition in (4), the neural network models are set with $\mathbf{X}_t = (1 Y_t\ Y_{t-1}\ \ldots\ Y_{t-l+1})'$, $\tilde{X}_{t,j} = [1+\exp(-\mathbf{X}_t'\gamma_j)]^{-1}$ $(j = 2,\ldots,k)$, $\tilde{\mathbf{X}}_t = (\mathbf{X}_t'\ \tilde{X}_{t,2}\ \ldots\ \tilde{X}_{t,k})'$, $\boldsymbol{\beta}_{\alpha,h} = [\boldsymbol{\beta}_1'\ \beta_2\ \ldots\ \beta_k]'$ is an $(l+k)$ vector, $\boldsymbol{\beta}_1$ is an $(l+1)$ vector, and $Q_\alpha(Y_{t+h}|\mathbf{X}_t) = \tilde{\mathbf{X}}_t'\boldsymbol{\beta}_{\alpha,h}$. We consider the number of nodes

Table 5. Bagging quantile predictions for S&P 500 daily returns

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	6.72	5.04	5.01	5.03	5.04	4.59	4.71	4.80	4.80	4.80	4.80	4.62
$\alpha = 0.05$	20.63	19.37	19.28	19.33	19.36	19.13	18.66	18.41	18.42	18.41	18.41	18.24
$\alpha = 0.10$	34.69	31.95	31.95	31.95	31.95	31.47	32.64	32.00	32.00	32.00	32.00	32.00
$\alpha = 0.30$	66.22	62.68	62.84	62.75	62.70	62.64	63.28	61.72	61.70	61.72	61.72	61.77
$\alpha = 0.50$	70.19	68.54	68.6	68.59	68.55	68.61	68.40	67.96	67.95	67.97	67.96	67.96
$\alpha = 0.70$	59.95	59.54	59.51	59.53	59.54	59.16	60.45	60.19	60.19	60.19	60.19	60.22
$\alpha = 0.90$	31.96	31.38	31.39	31.39	31.38	31.33	33.15	32.40	32.41	32.40	32.40	32.37
$\alpha = 0.95$	19.98	18.38	18.41	18.4	18.38	18.86	19.91	19.69	19.68	19.68	19.69	19.57
$\alpha = 0.99$	5.36	5.55	5.56	5.56	5.55	6.01	5.97	5.40	5.40	5.40	5.40	5.41

Note: Each cell gives the tick loss of quantile prediction over the period 01/13/2004–01/07/2005.

$(k - 1)$ from 0 to 5 and number of lags (l) from 1 to 3. Both l and k are selected for each estimation process using the SIC. We choose one combination of p and l from 18 candidates for each prediction. When $k = 1$, we have a linear regression model; when $k \geqslant 2$, we have a nonlinear regression model.

The neural network model has been widely used in modeling unknown nonlinearities in economics and finance. However, with the choice of explanatory variables and number of nodes, the model uncertainty and parameter estimation problems can be very serious. Lee (2000) introduces a method called Bayesian Random Searching (BARS) to choose the optimal number of hidden nodes as well as the best subset of explanatory variables. Instead of choosing only one, he selects several best-performing models and averages over them. He also provides the asymptotic consistency proof of the posterior neural network regression based on the i.i.d. normal error term assumption. The BARS method is built upon the model space searching work by Raftery *et al.* (1997) and is similar to the approach of Chipman *et al.* (1998) in their implementation of Bayesian classification and regression tree (CART). The BARS method is simply BMA-weighted bagging when our basic model is the artificial neural network.

Because of the large number of parameters to be estimated and the highly nonlinear structure, we can expect that the neural network model will generate poor predictions if we have a small sample size, and we can expect that bagging can play a crucial rule in improving the performance of neural network models. The only problem with bagging neural network models is that we need to choose the number of lags and nodes and estimate all the parameters for each combination of lags and nodes, so it takes substantial computer time to generate predictions. Therefore, we only conduct one empirical experiment to give a rough idea on how bagging predictors work for neural network models. We make quantile predictions with $\alpha = 0.1, 0.3, 0.5, 0.7$, and 0.9 using S&P 500 monthly data which is summarized as follows:

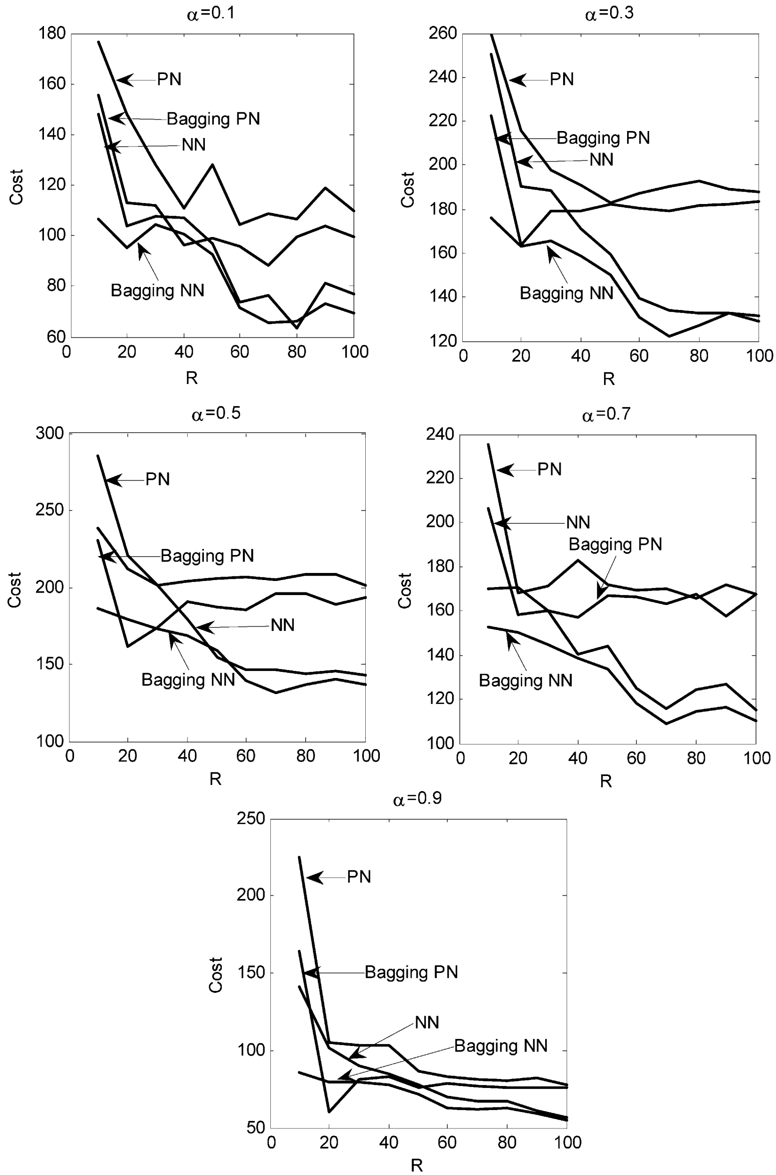
	In-sample period	Out-of-sample period	$T+1$	P
S&P 500	October 1982– October 1995	November 1995– February 2004	257	100

From Table 6 and Figure 4, we can see that even when the in-sample size R is small, unbagged neural network predictors already show some advantage over the simple polynomial (PN) predictors because of the flexibility of neural network (NN) models to capture nonlinearities in the data. Bagging works well for both PN and NN models, and the improvement by bagging is quite substantial when R is small. When the sample size R is large, the neural network model shows a much clearer advantage over polynomial predictors by always generating better predictions, and yet bagging neural network predictors still make further improvement over unbagged neural network predictors. Therefore, using bagging predictors, we can save a more complicated prediction model which has more flexibility to capture nonlinear structure but is more difficult to estimate.

Table 6. Bagging quantile predictions for S&P 500 monthly returns with different regression models

			$R = 10$	$R = 20$	$R = 30$	$R = 40$	$R = 50$	$R = 60$	$R = 70$	$R = 80$	$R = 90$	$R = 100$
$\alpha = 0.10$	PN	$J = 1$	176.40	148.05	127.88	110.75	128.19	104.49	108.95	106.48	118.67	109.68
		$J = 50$	155.54	113.05	111.78	96.33	98.96	95.82	88.37	99.42	103.92	99.36
	NN	$J = 1$	147.72	103.77	107.56	106.82	96.68	73.58	76.52	63.29	81.39	76.71
		$J = 50$	106.78	95.21	104.30	100.84	92.39	71.48	65.61	66.29	73.10	69.59
$\alpha = 0.30$	PN	$J = 1$	259.51	215.73	197.99	190.70	182.65	187.37	190.61	192.54	188.85	187.91
		$J = 50$	222.54	163.38	179.19	179.46	182.48	180.13	178.97	181.44	182.24	183.60
	NN	$J = 1$	250.14	190.15	188.59	170.97	159.59	139.24	134.02	132.63	133.01	131.44
		$J = 50$	176.34	163.32	165.44	158.73	150.32	130.68	122.14	127.02	132.52	129.04
$\alpha = 0.50$	PN	$J = 1$	285.33	221.18	201.47	203.87	205.49	206.28	204.95	208.30	208.57	201.01
		$J = 50$	230.49	161.15	173.50	190.35	187.59	185.38	196.26	196.06	188.81	193.78
	NN	$J = 1$	238.24	212.11	201.32	178.79	154.36	146.44	146.73	143.45	145.34	142.54
		$J = 50$	186.58	179.40	172.79	168.43	158.84	139.33	131.57	136.56	140.16	136.60
$\alpha = 0.70$	PN	$J = 1$	235.22	167.92	171.43	183.12	171.77	169.55	170.17	165.58	171.91	167.27
		$J = 50$	170.15	170.60	159.98	157.00	167.01	166.43	163.22	167.65	157.78	167.29
	NN	$J = 1$	206.36	158.48	160.24	140.66	144.30	125.15	115.57	124.37	126.95	115.15
		$J = 50$	152.51	150.00	144.50	138.59	133.36	118.25	109.09	114.22	116.41	110.17
$\alpha = 0.90$	PN	$J = 1$	224.54	105.44	103.49	102.94	86.34	82.77	80.88	80.34	82.36	77.69
		$J = 50$	164.33	60.45	81.57	83.29	75.61	78.54	76.53	75.60	75.68	76.43
	NN	$J = 1$	141.10	101.57	90.12	84.99	78.12	69.39	67.28	67.45	61.12	56.61
		$J = 50$	85.78	79.45	79.51	78.07	72.02	62.39	62.00	62.40	59.62	55.13

Note: The four rows for each α report the tick losses of quantile predictors of S&P 500 monthly returns over the period November 1995–February 2004 using polynomial (PN), mean bagging PN, neural network (NN), and mean bagging NN predictors.



Notes: The five panels report the tick losses of quantile predictors of S&P 500 monthly returns over the period November 1995–February 2004 using polynomial and neural network quantile regression models. PN represents the forecast loss from polynomial model and NN represents the forecast loss from neural network model.

Fig. 4. Bagging quantile predictions with different regression models.

8. Bagging binary and quantile forecasts in different frequencies

We are concerned about prediction at different frequencies because the predictability of time series may be different in different frequencies. As discussed by, e.g., Christofferson and Diebold (2006), the sign predictability of stock returns may depend on the frequency. The optimal binary prediction $G_{t,1}(\mathbf{X}_t)$ that minimizes $\mathbb{E}_{Y_{t+1}}(\rho_\alpha(G_{t+1}-G_{t,1}(\mathbf{X}_t))|\mathbf{X}_t)$ will be the α-quantile of G_{t+1} conditioning on $\mathbf{X}_t$, which can be achieved by an indicator function of the α-quantile of Y_{t+1} conditioning on $\mathbf{X}_t$ (Lee and Yang, 2006), i.e.,

$$G_{t,1}(\mathbf{X}_t) = Q_\alpha(G_{t+1}|\mathbf{X}_t) = \mathbf{1}\big(Q_\alpha(Y_{t+1}|\mathbf{X}_t) > 0\big),$$

where the second equation holds because the indicator function $\mathbf{1}(\cdot)$ is monotonic (Powell, 1986).

We conduct bagging predictions for S&P 500 binary and quantile prediction at both daily (Table 2A; Figure 2, panel (e); Table 5; Table 7) and monthly (Table 6, Figure 5) frequencies. We find that the bagging quantile prediction works in a similar pattern for both daily (Table 5) and monthly frequencies (Table 6). However, for binary predictions, bagging works much less effectively with high frequency (daily) series, perhaps because daily signs may be too noisy and difficult to forecast anyway. This result is consistent with Christofferson and Diebold (2006).

9. Pretesting and bagging

In this section we discuss a potential extension of this chapter, with pretesting as considered in Bühlmann and Yu (2002), Inoue and Kilian (2006), and Stock and Watson (2006). Bühlmann and Yu (2002) show that bagging works by smoothing the hard threshold function (e.g., an indicator function). To see this, suppose the bootstrap works for $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $Z_n \equiv n^{1/2}(\bar{Y}_n-\mu)/\sigma \to^d N(0,1)$ as $n \to \infty$. Let $y \equiv \mu + c\sigma n^{-1/2}$. Consider a binary model,

$$\begin{aligned}\hat{\theta}_n(y) &= \mathbf{1}(\bar{Y}_n > y)\\ &= \mathbf{1}\big(\bar{Y}_n > \mu + c\sigma n^{-1/2}\big)\\ &= \mathbf{1}\big(n^{1/2}(\bar{Y}_n - \mu)/\sigma > c\big)\\ &= \mathbf{1}(Z_n > c),\end{aligned}$$

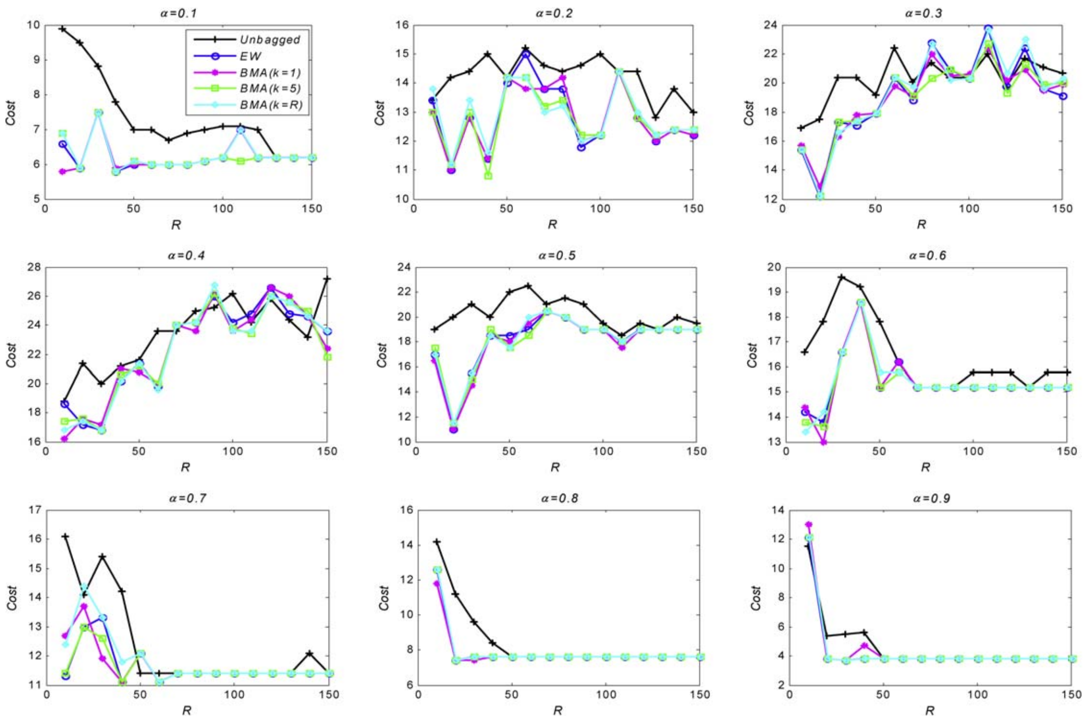
whose bagging predictor is

$$\begin{aligned}\hat{\theta}_{n,B}(y) &= \mathbb{E}^*\hat{\theta}_n^*(y)\\ &= \mathbb{E}^*\mathbf{1}\big(\bar{Y}_n^* > y\big)\\ &= \mathbb{E}^*\mathbf{1}\big(n^{1/2}\big(\bar{Y}_n^* - \bar{Y}_n\big)/\sigma > n^{1/2}(y - \bar{Y}_n)/\sigma\big)\\ &\approx 1 - \Phi\big(n^{1/2}(y - \bar{Y}_n)/\sigma\big)\\ &= 1 - \Phi\big(n^{1/2}\big(\mu + c\sigma n^{-1/2} - \bar{Y}_n\big)/\sigma\big)\\ &= 1 - \Phi(c - Z_n),\end{aligned}$$

Table 7. Bagging binary predictions for S&P 500 daily returns

	$R = 100$						$R = 300$					
	$J = 1$	$J = 50$					$J = 1$	$J = 50$				
		mean	BMA_1	BMA_5	BMA_R	med		mean	BMA_1	BMA_5	BMA_R	med
$\alpha = 0.01$	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.37
$\alpha = 0.05$	6.85	6.85	6.85	6.85	6.85	6.85	6.85	6.85	6.85	6.85	6.85	6.85
$\alpha = 0.10$	13.70	13.70	13.70	13.70	13.70	13.70	13.70	13.70	13.70	13.70	13.70	13.70
$\alpha = 0.30$	43.60	40.80	41.10	40.80	40.80	40.80	41.20	41.10	41.10	41.10	41.10	41.10
$\alpha = 0.50$	60.50	58.00	58.00	58.00	59.00	58.00	57.00	56.00	56.00	55.50	56.00	56.00
$\alpha = 0.70$	35.70	34.30	34.30	34.30	34.30	34.30	33.90	33.90	33.90	33.90	33.90	33.90
$\alpha = 0.90$	11.30	11.30	11.30	11.30	11.30	11.30	11.30	11.30	11.30	11.30	11.30	11.30
$\alpha = 0.95$	5.65	5.65	5.65	5.65	5.65	5.65	5.65	5.65	5.65	5.65	5.65	5.65
$\alpha = 0.99$	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13

Note: Each cell reports the asymmetric binary prediction loss with parameter α over the period 01/13/2004–01/07/2005.



Note: The nine graphs above report the asymmetric losses of binary predictors of S&P 500 monthly returns over the period November 1995–February 2004.

Fig. 5. Bagging binary predictions for S&P 500 Monthly Returns.

where $\approx$ denotes asymptotic equivalence as $n \to \infty$. When $y = \mu$, $\hat{\theta}_n(y)$ is most unstable. Compare the predictors at this value $y = \mu$ (or $c = 0$). When $y = \mu$ ($c = 0$), $\hat{\theta}_n(\mu)$ has mean 1/2 and variance $(1/2)(1 - 1/2) = 1/4$. In comparison, when $y = \mu$ ($c = 0$), the bagging predictor $\hat{\theta}_{n,B}(\mu) \approx 1 - \Phi(-Z_n) = \Phi(Z_n) = U$ has mean 1/2 and variance 1/12. Hence, bagging reduces the variance of the predictor from 1/4 to 1/12.

Bühlmann and Yu (2002) use the idea that bagging works via smoothing hard-thresholding into soft-thresholding for the location and regression models. Consider a location model with pretesting (PT),

$$\mathrm{PT} = \hat{\theta}_n(y) = \hat{\beta}_{0,n}\mathbf{1}(\hat{\beta}_{0,n} > y) = \hat{\beta}_{0,n}\mathbf{1}(Z_n > c),$$

and its bagging predictor (BA),

$$\mathrm{BA} = \hat{\theta}_{n,B}(y) = \mathbb{E}^*\hat{\theta}_n^*(y) = \mathbb{E}^*\hat{\beta}_{0,n}^*\mathbf{1}\big(\hat{\beta}_{0,n}^* > y\big) = \mathbb{E}^*\hat{\beta}_{0,n}^*\mathbf{1}\big(Z_n^* > c\big).$$

Here the location parameter is β_0 if $Z_n > c$ and zero otherwise. The PT model has hard thresholding around $Z_n = c$, while BA has smooth soft-thresholding.

Bühlmann and Yu (2002) also consider the variable-selection in a regression model by pretesting:

$$\mathrm{PT} = \hat{\theta}_n(y) = \sum_{j=0}^{M} \hat{\beta}_{j,n}\mathbf{1}(\hat{\beta}_{j,n} > y)x_n^{(j)} = \sum_{j=0}^{M} \hat{\beta}_{j,n}\mathbf{1}(Z_{n,j} > c)x_n^{(j)},$$

where the jth variable $x_n^{(j)}$ is included if its coefficient is bigger than a given threshold c. The variable-selection conducted via pretesting introduces hard-thresholding. Bagging can smooth hard-thresholding in this case as follows:

$$\mathrm{BA} = \hat{\theta}_{n,B}(y) = \mathbb{E}^*\hat{\theta}_n^*(y) = \mathbb{E}^* \sum_{j=0}^{M} \hat{\beta}_{j,n}^*\mathbf{1}\big(Z_j^* > c\big)x_n^{(j)}.$$

Inoue and Kilian (2006) exploit the idea that bagging can reduce the variance of the predictor from a regression model when the predictors/regressors are selected by pretesting to show how bagging works for forecasting inflation.

Breiman (1996) and Lee and Yang (2006) consider the case when $c = 0$. In other words, they did not consider pretesting, and bagging is applied to an unrestricted regression (UR) model with all the M predictors/regressors included (without selecting a subset of them by pretesting). In this case, bagging would still work, especially when the UR model is bad (particularly in small samples). Certainly $c = 0$ is not optimal, as bagging would work better with some larger values of c. If $c = 0$, bagging is not asymptotically admissible (Stock and Watson, 2006). An example is shown in Lee and Yang (2006) for bagging binary prediction with majority voting, where bagging works well in small samples but does not work asymptotically with $c = 0$. The choice of c is like the choice of the shrinkage parameter, as shown in Stock and Watson (2006) and also noted in Inoue and Kilian (2006). Stock and Watson (2006) show that $c = 1.96$ is too small for bagging to be comparable to factor methods. As they note, $c = 2.58$ makes bagging work better. In this chapter, we consider only $c = 0$ (no pretesting) for both binary and quantile prediction, as in Lee and Yang (2006). With pretesting ($c > 0$), we expect that bagging would work better, based on the results of Bühlmann and Yu (2002), Inoue and Kilian (2006), and Stock and Watson (2006). Investigation of bagging with pretesting for the binary and quantile prediction is left for future work. It can be conjectured that pretesting would be more beneficial in improving bagging, particularly for longer multi-step forecasting.

10. Summary and conclusion

Bagging is a smoothing method designed to improve predictive ability under the presence of parameter estimation uncertainty and model uncertainty. There are two ways of aggregating – averaging or voting. Bagging quantile predictors are constructed via weighted averaging over predictors trained on bootstrapped training samples. Bagging binary predictors are conducted via (majority) voting on predictors trained on the bootstrapped training samples.

To understand how bagging works, various explanations have been made. It may be hard to understand the meaning of multiple training set $\mathcal{D}_t^{(j)}$ in the time series circumstances, since time is not repeatable. However, considering an example of estimating and forecasting with panel data may be helpful. Suppose we want to forecast consumption of a household next period. When historical observations of the interested household are very limited, estimated parameters and predictions will have large variances, especially for nonlinear regression models. If we can find some other households that have similar consumption patterns (similar underlying probability distribution, $\mathbf{P}$), it would be better to use historical observations from all similar households rather than just from a particular household in the estimation process, though we only use data for this particular household to forecast. Therefore, the ensemble aggregating predictor is like finding similar households, and the bootstrap aggregating predictor is like finding similar bootstrapped (artificial) households.

What was done in Lee and Yang (2006) is to examine how bagging works: (i) with equal-weighted and BMA-weighted averaging; (ii) for one-step-ahead binary prediction (with voting) and for one-step-ahead quantile prediction (with averaging); (iii) with a particular choice of loss function (linlin, check); and (iv) with a particular choice of regression model (linear, polynomial).

What we do in this chapter ("Further Issues") is to consider: (i) different aggregating schemes (trimmed-mean bagging, median bagging); (ii) multi-step forecast horizons (to see how bagging performs in situations with greater uncertainty); (iii) a more general class of loss functions, i.e., the so-called tick-exponential family, to examine the effect of the convexity of the loss function (in addition to the check loss for quantile estimation); (iv) different algorithms (the minimax algorithm vs. the interior point algorithm for the estimation of the quantile model); (v) different regression models (polynomial quantile and neural network quantile models); and (vi) different data frequencies (monthly and daily S&P 500 returns).

We find the following. (i) Median bagging and trimmed-mean bagging can be more robust to extreme predictors from bootstrap samples and have better performance than equally weighted bagging predictors. (ii) Bagging works more with longer forecast horizons. (iii) Bagging works well under more general tick loss functions. (iv) Bagging may work differently with different quantile estimation algorithms. (v) Bagging works well with highly nonlinear quantile regression models (e.g., artificial neural network). (vi) Bagging quantile predictors is not affected by the frequency of the data, while bagging binary predictor is significantly affected when daily returns are considered instead of monthly returns.

From comparing different averaging schemes, we find that (i) the BMA-, median-, and trimmed-bagging predictors have better predictive performance than equal-weighted bagging predictors, even when we have a relatively large sample size. (ii) The median bagging is generally the best. (iii) The outstanding performance of median bagging predictors is most obvious when α values are close to 0 or 1, where the extreme value problem is most serious because there are fewer observations in the tails and parameters estimates are sensitive to the sample. However, the advantage of median bagging predictors is not as clear when α values are close to 0.5. (iv) Bagging works more when the sample size is smaller. (v) Bagging works more when α-quantiles lie on the sparse part of the error distribution. Our explanation is that for the sparse part of the error distribution, there are fewer observations, and therefore quantile predictions are sensitive to the estimation sample and bagging predictors work better for unstable predictions.

From bagging multi-step quantile forecasts, we find that the performance of bagging relative to unbagged predictor gets better as the forecast horizon increases. From examining how other algorithms may work for the bagging, we find that the interior point algorithm and minimax algorithm give somewhat different results. Therefore, in small samples, bagging may work differently depending on the estimation algorithm. From checking the performance of bagging predictors on highly nonlinear quantile regression models (artificial neural network models), we find that, given model uncertainty when the sample size is limited, it is usually hard to choose the number of hidden nodes and the number of inputs (lags), and to estimate the large number of parameters in a neural network model, in which cases, using bagging predictors, we can save the complicated model (more flexible to capture nonlinear structure but harder to estimate) for out-of-sample forecasting.

Finally, we conclude with some additional comments on how and/or why bagging may be useful in the presence of structural breaks and model uncertainty.

In the presence of structural breaks. In this chapter, we find that bagging may work more when the size of the training sample is small and the predictor is unstable. Bagging seems to smooth the parameter estimation uncertainty due to a small sample to improve the forecasting performance. The potential advantage of bagging lies in areas where small samples are common. Bagging may be useful when structural breaks are frequent so that simply using as many observations as possible is not a wise choice for out-of-sample prediction, and forecasts can "fail" in the presence of breaks. It is very likely that the optimal estimation window size for generating forecasts will be affected by the breaks, as recently shown by Clark and McCracken (2004) and Pesaran and Timmermann (2007). It would be interesting in future work to examine more extensively how bagging performs in the presence of structural breaks.

In the presence of model uncertainty. Bagging is a smoothing method to improve predictive ability under the presence of parameter estimation and model uncertainty. For example, as we find in Section 4, the gains to bagging increase with the forecast horizon, as there is more uncertainty at longer forecast horizons and more smoothing can operate. Bagging may also improve forecasting when there is uncertainty concerning the measurement of a variable, functional form, and best proxy to use for a variable of interest. These are forms of model uncertainty that bagging may smooth it out, and it would also be interesting to examines this in future research. A number of issues are left for future research, even after the "Further Issues" considered in the present chapter.

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