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Short Paper
Stock Adjustment for Multicointegrated Series

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Abstract: A typical stock adjustment model is a partial adjustment process to maintain simultaneously the two kinds of equilibrium relationships: a flow-flow relationship and a stock-flow relationship. We show that the stock adjustment model is an error correction model of 'multicointegrated' time series, and also an optimal decision rule generated from an intertemporal optimization problem. Economic examples in inventory model, housing construction, and consumption function are discussed.

JEL Classification System-Numbers: C5, E2

1 Introduction

After Metzler (1941) introduces the idea of the inventory accelerator mechanism, the best known extension is in the work of Holt, Modigliani, Muth and Simon (1960). The following equation can be regarded as a simple form of the model:

$$\Delta Q_t = \alpha + \beta(Q_{t-1} - ky_{t-1}) + \gamma(x_{t-1} - y_{t-1}) , \quad (1)$$

where $\Delta = 1 - B$, B is the backshift operator, Q_t is the level of stock (inventory) at the end of the period t, x_t and y_t are flow variables (such as production or delivery, and sales or shipment). The parameters β and γ measure the speeds of adjustment and k is the stock-flow ratio at a steady state (such as inventory-sales ratio). This is the stock adjustment framework studied by Lovell (1961), which is to maintain two kinds of long-run equilibrium relationships at the same time if both β and γ are not zero.

This property may be discussed using a recently developed time series method. If x_t and y_t are both I(1) and if $z_t \equiv x_t - y_t$ is I(0), x_t and y_t are cointegrated. If a pair of I(1) series are cointegrated it may allow the possibility of a deeper form of cointegration. Since z_t is I(0), $Q_t \equiv \sum_{j=0}^{t} z_{t-j}$ will be I(1). Then x_t and y_t will be

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said to be *multicointegrated* if Q_t and x_t are cointegrated (Granger, 1986). Q_t and y_t will also be cointegrated. Thus it follows that $u_t \equiv Q_t - ky_t$ is I(0) for a constant k. Granger and Lee (1990) show that multicointegrated series may be generated by the following error correction models:

$$\Delta x_t = \alpha_1 + \beta_1 u_{t-1} + \gamma_1 z_{t-1} + \textit{lagged}\,(\Delta x_t, \Delta y_t) + \textit{residual} \tag{2a}$$

$$\Delta y_t = \alpha_2 + \beta_2 u_{t-1} + \gamma_2 z_{t-1} + \textit{lagged}\,(\Delta x_t, \Delta y_t) + \textit{residual}\ . \tag{2b}$$

It may be noted that the stock adjustment model (1) can be derived from (2) with $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$, and $\gamma = \gamma_1 - \gamma_2 + 1$. The further properties of multicointegrated series are discussed in Granger and Lee (1990).[2] The stock adjustment may be considered as an optimal control rule. In section 2 we thus relate it to a particular cost minimization. The purpose of this note is to elaborate some of the results in Granger and Lee (1990). This note uses an intertemporal cost function, while Granger and Lee demonstrate it in a simpler setting. Three examples are discussed in section 3.

2 Optimization

We consider the following situation: x_t is a series that one is attempting to control, x_t^* is the target series for x_t, and $z_t = x_t - x_t^*$ is the control error, being the extent to which the target is missed. If x_t is I(1), x_t^* should be cointegrated with x_t with cointegrating vector $(1\ -1)'$. The control error z_t should have a bounded variance, i.e. I(0) (Kloek, 1984).

The accumulated control error is $Q_t = \sum_{j=0}^{t} z_{t-j}$ and it is assumed this series also has a target series Q_t^*. If z_t is I(0), Q_t is I(1). Q_t should also be cointegrated with Q_t^* with cointegrating vector $(1\ -1)'$, and the the control error for Q_t defined as $u_t = Q_t - Q_t^*$ is I(0). We call z_t a proportional control error and u_t an integral control error. Costs to the controller will arise from three sources, the size of z_t, the size of u_t and the amount of adjustment in x_t.

Assuming an additively separable and quadratic cost functions, the quantity to be minimized is given below. To be consistent with multicointegration one has to add the condition that x_t^* and Q_t^* are cointegrated. It is assumed that $x_t^* = y_t$ and $Q_t^* = ky_t$. Given y_t an agent chooses x_t and Q_t to minimize the expected discounted present value of costs

$$J_t = E_t \textstyle\sum_{j=0}^{\infty} \delta^j [z_{t+j}^2 + \lambda_1 u_{t+j}^2 + \lambda_2 (\Delta x_{t+j})^2] \tag{3}$$

subject to

$$Q_{t+j} - Q_{t+j-1} = x_{t+j} - y_{t+j} \tag{4}$$

[2] Engle and Yoo (1991) relate multicointegration to I(2) cointegration. For more discussion, see Gregoir and Laroque (1993), Stock and Watson (1993), and Engsted et al. (1995).

where $z_t = x_t - y_t$ and $u_t = Q_t - ky_t$. Three elements in the cost function, z_t, u_t and Δx_t, are all I(0). E_t denotes the conditional expectation given an information set up to time t, δ a discount factor that is less than one, and both λ_1, λ_2 are nonnegative.

The stochastic Euler equation is derived by substituting the equation of motion (4) into (3) and minimizing with respect to Q_t. The result is given by:

$$E_t a(B) Q_{t+2} = E_t c(B) y_{t+2} \qquad \text{for all } t \geq 0 \tag{5}$$

where

$$a(B) = 1 - \left(\frac{1}{\delta\lambda_2} + \frac{2}{\delta} + 2\right)B + \left(\frac{1+\lambda_1}{\delta^2\lambda_2} + \frac{1}{\delta^2} + \frac{1}{\delta\lambda_2} + \frac{4}{\delta} + 1\right)B^2$$
$$- \left(\frac{1}{\delta^2\lambda_2} + \frac{2}{\delta^2} + \frac{2}{\delta}\right)B^3 + \frac{1}{\delta^2}B^4$$

$$c(B) = -1 - \left(\frac{2}{\delta} + 1\right)B + \left(\frac{k\lambda_1}{\delta^2\lambda_2} - \frac{1}{\delta^2} - \frac{2}{\delta}\right)B^2 + \frac{1}{\delta^2}B^3 \ .$$

Note that $a(1) = \lambda_1/\delta^2\lambda_2$, $c(1) = k\lambda_1/\delta^2\lambda_2$, and thus $c(1) = ka(1)$. The transversality condition for this problem is given by $\lim_{j\to\infty} E_t Q_{t+j} \partial J_t / \partial Q_{t+j} = 0$.

The first step in obtaining the reduced form solution involves finding the roots of $a(B)$. Let $a(B) = (1 - \mu_1 B)(1 - \mu_2 B)(1 - \mu_3 B)(1 - \mu_4 B)$. It can be shown that if μ is a root $(\mu\delta)^{-1}$ is a root. Let μ_1 and μ_2 be the two smallest roots in absolute value. These roots may be real or complex conjugates. In order to satisfy the transversality condition, the stable roots must be solved backward and the unstable roots, forward. The existence of a stationary solution requires that two of the roots have modulus less than one and that two have modulus greater than one. If this condition holds, $|\mu_1| < 1$, $|\mu_2| < 1$, $\mu_3 = (\mu_1\delta)^{-1}$, and $\mu_4 = (\mu_2\delta)^{-1}$. Then (5) can be written

$$(1 - \mu_1 B)(1 - \mu_2 B)Q_t = E_t \frac{\mu_1\mu_2\delta}{\mu_1 - \mu_2} \textstyle\sum_{i=0}^{\infty} [(\mu_1\delta)^i - (\mu_2\delta)^i] B^{-i-1} c(B) y_t \tag{6}$$

We must now be specific about the data generation process for y_t series to make (6) empirically implementable. Let us assume, just for simplicity, that y_t is a random walk. Then (6) becomes substantially simpler:

$$(1 - \mu_1 B)(1 - \mu_2 B)Q_t = \frac{\mu_1\mu_2\delta^2}{(1 - \mu_1\delta)(1 - \mu_2\delta)} c(1) y_t + I(0) \ . \tag{7}$$

Since $a(1) = (1 - \mu_1)(1 - \mu_2)(1 - \mu_1\delta)(1 - \mu_2\delta)/\mu_1\mu_2\delta^2$ and $c(1) = ka(1)$, (7) becomes

$$\Delta Q_t = -(1 - \mu_1)(1 - \mu_2)u_{t-1} + \mu_1\mu_2 z_{t-1} + (1 - \mu_1)(1 - \mu_2)k\Delta y_t + I(0) \ . \tag{8}$$

This is the stock adjustment model shown in equation (1), and is a proportional-integral-derivative (PID) feedback controller familiar to control engineers

(Kwakernaak and Sivan, 1972; Phillips, 1954). Even if μ_1 and μ_2 are complex, $\mu_1\mu_2$ and $\mu_1 + \mu_2$ are real. Thus the coefficients in (8) are real.

The relative size of the integral correction factor $(1-\mu_1)(1-\mu_2)$ and the proportional correction factor $\mu_1\mu_2$ may be expressed in terms of the ratio of the penalty coefficients λ_1/λ_2 in the cost function (3). Since $a(1) = \lambda_1/\delta^2\lambda_2$ and $a(1) = (1-\mu_1)(1-\mu_2)(1-\mu_1\delta)(1-\mu_2\delta)/\mu_1\mu_2\delta^2$,

$$\lambda_1/\lambda_2 = (1-\mu_1)(1-\mu_2)(1-\mu_1\delta)(1-\mu_2\delta)/\mu_1\mu_2 \; .$$

If the ratio λ_1/λ_2 is large the relative importance of an integral error correction becomes large.

3 Examples of the Stock Adjustment

3.1 *Inventory Adjustment in the US Retail Trade*

The first empirical example makes use of monthly series of inventory (Q_t = Citibase code IVRRQ) and sales (y_t = RTQ) from 1967:01 to 1994:12 (336 observations) for the U.S. retail trade, obtained from Citibase. The series are in 1987 constant dollars and seasonally adjusted at the source. The unit root hypothesis cannot be rejected for both Q_t and y_t series. Testing multicointegration involves an OLS regression $Q_t = \hat{a} + \hat{b}y_t + u_t$, and the unit root test for the residual u_t. From the test statistics reported in Tables 1 and 2 the inventory and

Table 1. Tests for unit roots in z_t and u_t

	ADF_1	PP_1	ADF_2	PP_2
Example 1 (inventory):				
z_t	−12.89(0)**	−12.65**	−14.03(0)**	−14.25**
u_t	−5.31(0)**	−5.02**	−5.31(0)**	−5.01**
Example 2 (housing):				
z_t	−4.14(1)**	−5.81**	−4.13(1)**	−5.80**
u_t	−6.81(1)**	−11.96**	−6.80(1)**	−11.94**
Example 3 (consumption):				
z_t	−0.56(3)	−0.65	−2.91(3)	−4.57**
u_t	−1.81(1)	−2.01	−1.81(1)	−2.01

ADF_i and PP_i denote the augmented Dickey-Fuller statistic and the Phillips-Perron statistic, respectively. The statistics with $i = 1$ are computed without a constant and a trend term, and those with $i = 2$ are with a constant but without a trend term. ** and * denote the significance at 1% and 5% levels, respectively. The critical values are taken from Fuller (1976) for z_t and from Engle and Yoo (1987) for u_t. The number in ( ) after the ADF statistics are number of lag-augmentation chosen using the SIC. The Phillips-Perron statistics are calculated with six non-zero autocovariances. The results are similar for the other values of the lag truncation in autocovariances.

Table 2. The Johansen tests for cointegration between Q_t and y_t

	Trace statistic	Maximum eigenvalue statistic
Example 1 (inventory):	46.55**	45.60**
Example 2 (housing):	46.11**	34.34**
Example 3 (consumption):	4.48	4.24

The Johansen (1991) statistics are reported. ** denotes the significance at 1% level.

sales series are cointegrated at 1% level. Suppose that agents minimize the cost expressed in (3), with z_t being the cost rising from the gap between delivery and sales, u_t being due to inventory holding or stock-out, and with Δx_t for delivery cost. Then the estimated optimal stock adjustment is

$$\underset{(5.78)}{\Delta Q_t = 399.41} \; - \; \underset{(-5.52)}{0.07u_{t-1}} \; + \; \underset{(3.89)}{0.23z_{t-1}} \; - \; \underset{(-0.98)}{0.66\Delta y_{t-1}} \; + \; residual \; .$$

White's t-values are in brackets. The adjusted coefficient of determination ($\bar{R}^2$) of the regression is 0.17. No significant residual autocorrelations are found. For example, $LM_1 = 0.85$ and $LM_6 = 0.25$, where LM_i denotes the p-value of a Lagrange multiplier test for residual autocorrelation of order i. If the coefficient estimate for u_{t-1} is small, it is very significant. Both the integral and proportional error corrections are very significant.

3.2 *Stock Adjustment in US Housing Construction*

The second empirical example is the relationship between housing starts (or completions) and housing stocks under construction. The data obtained from Citibase are: new privately owned housing units started (x_t = HSFR) and new privately owned housing units completed (y_t = HCP). The sample is monthly from 1968:01 to 1994:12 with 324 observations and seasonally adjusted. As approximately 2% of new housing starts is found never completed (due to bankrupt, demolition, recording errors, conversion, and etc.), we let $z_t = 0.98x_t - y_t$ and the new housing units under construction at time t be $Q_t = \sum_{j=0}^{t} z_{t-j}$. The results in Tables 1 and 2 indicate that Q_t and y_t are cointegrated at 1% level. Assuming that builders minimize the cost function in (3), with z_t being the cost due to the uncompleted starts, u_t due to holding inadequate housing stocks under construction, and with Δx_t being the cost to adjust housing starts, the estimated optimal stock adjustment is

$$\underset{(0.07)}{\Delta Q_t = 0.46} - \underset{(-4.20)}{0.05u_{t-1}} + \underset{(9.51)}{0.63z_{t-1}} + \underset{(0.42)}{0.05\Delta y_{t-1}} + \underset{(4.13)}{0.25z_{t-2}} + \underset{(0.47)}{0.04\Delta y_{t-2}} + residual \; .$$

For the regression, we obtain $\bar{R}^2 = 0.71$, $LM_1 = 0.72$, and $LM_6 = 0.44$. Both the integral and proportional error corrections are very significant.

3.3 *Wealth Effects in US Consumption Function*

It may be expected that consumption and cumulated savings (which may be called wealth) are cointegrated. To check this possibility we use data from Citibase for disposable personal income (GMYD), total personal outlays (GMOUT), and real (in 1987 dollars) per capita disposable personal income (GMYDPQ). The series are seasonally adjusted and monthly from 1959:01 to 1994:12 with 432 observations. From these series real per capita disposable income (x_t), real per capita personal outlays (y_t), real per capita personal savings ($z_t \equiv x_t - y_t$), and cumulated savings (Q_t) are computed. The test statistics in Tables 1 and 2 show that Q_t and y_t are not cointegrated. With z_t being the cost rising from inadequate flow of savings, u_t being the extent to which expenditure and wealth depart each other, and with Δx_t being the cost to make more money (e.g., giving up leisure), a consumer who minimizes the cost in (3) has the following optimal program to adjust her wealth

$$\Delta Q_t = \underset{(3.21)}{54.56} - \underset{(-1.53)}{0.0005}u_{t-1} + \underset{(5.19)}{0.62}z_{t-1} - \underset{(-0.36)}{0.05}\Delta y_{t-1} + \underset{(1.21)}{0.14}z_{t-2} - \underset{(-0.27)}{0.03}\Delta y_{t-2}$$

$$+ \underset{(2.99)}{0.16}z_{t-3} + \underset{(1.81)}{0.27}\Delta y_{t-3} + \textit{residual} ,$$

where $\bar{R}^2 = 0.79$, $LM_1 = 0.65$, and $LM_6 = 0.21$. As Q_t and y_t are not cointegrated the integral error correction to maintain the wealth-consumption (stock-flow) relationship is weak. The stock (wealth) adjustment is mainly from the proportional error correction z_t to maintain the cointegrated relationship between income and consumption (flow-flow).

4 Concluding Remarks

We have related the stock adjustment model to error correction models of multicointegrated series and to a particular linear-quadratic cost function. The similar problem for a simpler case of cointegration may be found in Nickell (1985). If a set of time series are multicointegrated, the stock adjustment process reflects both the integral and proportional error correction mechanisms. If an objective of econometric modeling is to construct a parsimonious and data-coherent model consistent with a relevant economic theory (Hendry and Richard, 1982), then a correctly specified stock adjustment model may be such a model since it is an optimal control rule and it is data-based.

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