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Multicointegration

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Abstract

This paper introduces a deeper level of cointegration, which might be expected to occur in economics. It can arise from special optimal control situations and can improve short- and long-run forecasts. It seems to be particularly appropriate for considerations of inventory.

1. Introduction

If Q_t is a stationary series with finite variance, then its accumulated sum

$$y_t = \sum_{j=0}^{t} Q_{t-j}$$

is called integrated of order one, denoted $y_t \sim I(1)$. It is assumed that Q_t has a spectrum $f(\omega)$ with the property that $0 < f(\omega) < \infty$, and Q_t is called integrated of order zero, denoted $Q_t \sim I(0)$. The change of an $I(0)$ series will be denoted $I(-1)$, so that if $Q_t \sim I(0)$, then $\Delta Q_t \sim I(-1)$. An $I(-1)$ series will have spectrum having a zero at zero frequency and, of course, its accumulation will be $I(0)$. A series may be nonstationary and still be $I(0)$, but discussion of such possibilities is not necessary in this paper. A stationary series will have no trend, will frequently cross its mean value, will have short memory, and will be relatively unsmooth. An $I(1)$ series will (generally) have an increasing variance, will contain dominant long-swing components (from its infinite spectrum at zero frequencies) and so be smooth, will have long memory (the optimum forecast of y_{n+h} will involve y_n nontrivally for all h), and will not regularly cross any particular level. Thus $I(0)$ and $I(1)$ series have quite different appearances and, generally, the regression of one on the other will result in an (asymptotically) zero regression coefficient. It is an interesting empirical fact that many macroeconomic series appear to be $I(1)$, although possibly with a trend.

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If x_t, y_t are both $I(1)$ then it is typically true that any linear combination $x_t + by_t$ will also be $I(1)$. However, for some pairs of $I(1)$ series there does exist a linear combination

$$z_t = x_t - Ay_t \tag{1.1}$$

that is $I(0)$. When this occurs, x_t, y_t are said to be cointegrated. This will only occur when the two series have a decomposition of the form

$$x_t = AW_t + x_{1t},$$
$$y_t = W_t + y_{1t},$$

where x_{1t}, y_{1t} are both $I(0)$ and W_t is $I(1)$. Thus the $I(1)$ property of x_t, y_t comes from the single $I(1)$ common factor W_t. Further, if x_t, y_t are cointegrated they may be considered to be generated by an error-correcting model of the form

$$\Delta x_t = \rho_1 z_{t-1} + \text{lagged}(\Delta x_t, \Delta y_t) + \varepsilon_{xt},$$
$$\Delta y_t = \rho_2 z_{t-1} + \text{lagged}(\Delta x_t, \Delta y_t) + \varepsilon_{yt},$$

where at least one of ρ_1, ρ_2 nonzero, z_t is from (1.1), and ε_{xt}, ε_{yt} are jointly white noise.

These properties of cointegrated series, their generalizations to $I(d)$ processes, and testing questions are discussed in Granger 1983, 1986 and Engle and Granger 1987.

It is generally true that for any vector X_t of $NI(1)$ series, there will be at most r vectors α such that $\alpha' X_t$ is $I(0)$, with $r \leq N - 1$. However, it is also true that any pair of $I(1)$ series may be cointegrated, and this does allow the possibility of a deeper form of cointegration occurring, which can be illustrated in the following bivariate case. Suppose that x_t, y_t are both $I(1)$, have no trend, and are cointegrated, so that $z_t = x_t - Ay_t$ is $I(0)$. It follows that

$$S_t = \sum_{j=0}^{t} z_{t-j}$$

will be $I(1)$ and x_t, y_t will be said to be *multicointegrated* if S_t and x_t are also cointegrated. It follows that S_t and y_t will also be cointegrated. As S_t is a function of x_t, y_t and their lags, multicointegration allows two cointegrations at different levels, between just two series. A possible example might be x_t = income, y_t = total consumption, $z_t = x_t - y_t$ being savings, S_t being wealth, and wealth and consumption being cointegrated. The example investigated in this paper has x_t = sales, y_t = production, for some industry, $z_t = y_t - x_t$ = change in inventory (apart from a constant, being the initial inventory), and inventory and production being cointegrated.

Section II discusses some properties of multicointegrated processes, Section III relates them to some optimization and control situations; Section IV discusses an empirical example, and finally Section V is a conclusion. Only the bivariate case is considered; the extension to higher-order vectors is straightforward. The models considered here are special cases of the general dynamic cointegration process considered by Yoo (1987) and by Johansen (1988). In this paper a simple case that is most likely to be of relevance in economics is considered in some detail.

2. Properties of Multicointegrated Process

Suppose that x_t, y_t are $I(1)$ and cointegrated, with

$$z_t = x_t - Ay_t \sim I(0).$$

The standard common factor representation is

$$x_t = AW_t + x_{1t}, \qquad y_t = W_t + y_{1t},$$

where W_t is $I(1)$ and x_{1t}, y_{1t} are both $I(0)$. It follows that

$$S_t = \sum_{j=0}^{t} z_{t-j} = \sum_{j=0}^{t} (x_{1,t-j} - Ay_{1,t-j}),$$

and to be cointegrated with x_t it is necessary that this variable has ΔW_t as a component. This will occur if the full decompositions are

$$x_t = AW_t + \alpha_1 \Delta W_t + x_{2t}, \quad y_t = W_t + \alpha_2 \Delta W_t + y_{2t},$$

where x_{2t}, y_{2t} are both $I(-1)$, giving

$$S_t = CW_t + \delta x_{2t} - A\delta y_{2t},$$

where $C = \alpha_1 - A\alpha_2 \neq 0$, $\delta = \Delta^{-1}$, and $\delta x_{2t} - A\delta y_{2t}$ is $I(0)$, being the accumulation of $I(-1)$ variables. It follows that

$$p_t = x_t - DS_t \sim I(0),$$

where $D = A/C$. It should be noted that, using $\delta = \Delta^{-1}$,

$$p_t = x_t - D\delta z_t = (1 - D\delta \quad AD\delta)X_t,$$

where $X_t = (x_t \quad y_t)'$.

The Cramer representation of the vector $I(0)$ series is

$$\Delta X_t = C(B)\varepsilon_t. \tag{2.1}$$

It was shown in Granger (1983) and Engle and Granger (1988) that for the components of X_t to be cointegrated it is necessary and sufficient that the determinant of $C(B)$ has a root $(1 - B)$. It is shown in Appendix A that the requirement for X_t to be multicointegrated is that the determinant of $C(B)$ has a root $(1 - B)^2$. If

$$\det C(B) = (1 - B)^2 d(B),$$

and if $A(B)$ is the adjunct matrix of $C(B)$, then (2.1) may then be written

$$A(B)\Delta X_t = (1 - B)^2 d(B)\varepsilon_t. \tag{2.2}$$

Using the notation

$$A(B) = A(1) + \Delta A^*(B), \qquad A^*(B) = A^*(1) + \Delta A^{**}(B),$$

then after some algebra outlined in Appendix B, (2.2) can be written

$$\bar{A}(B)\Delta X_t = -\gamma_1 p_{t-1} - \gamma_2 z_{t-1} + d(B)\varepsilon_t, \tag{2.3}$$

where

$$p_t = (1 - D\delta \quad AD\delta)X_t, \qquad \delta = \Delta^{-1},$$

$$z_t = \alpha' X_t, \quad \alpha' = (1 \quad -A),$$

$$\gamma\alpha' = A(1), \quad \gamma = \begin{pmatrix} A_{11}(1) \\ A_{21}(1) \end{pmatrix}$$

$$\gamma_1 = -D^{-1}\gamma, \quad \gamma_2 = \gamma - A^{-1}\begin{pmatrix} A^*_{12}(1) \\ A^*_{22}(1) \end{pmatrix}$$

$$\bar{A}(B) = A(1) + A^*(1) + A^{**}(B).$$

Equation (2.3) is the error correction model for a pair of multicointegrated series, in which changes of X_t are related to the pair of lagged cointegration errors $z_t = x_t - Ay_t$ and $p_t = x_t - DS_t$. For multicointegrator, ΔX_t is generated by (2.3), with the necessary condition that at least one component of each of γ_1 and γ_2 is nonzero. Equation (2.3) is the generalized error correction model for multicointegrated series. It should be noted that the extra term in the error correction representation does lead to potentially improved forecasts of component of ΔX_t.

An example of a generating process that produces a pair of multicointegrated series is

$$\Delta X_t = \begin{bmatrix} A + \Delta(1 - A) & -A^2(1 - \Delta) \\ 1 - \Delta & -A + \Delta(1 + A) \end{bmatrix} \varepsilon_t.$$

In this case $\alpha' = (1 - A)$, $D = A$, and the error correction models are

$$\Delta x_t = -p_{t-1} + \varepsilon_{1t}, \qquad \Delta y_t = -\lambda p_{t-1} + \lambda z_{t-1} + \varepsilon_{2t},$$

where $\lambda = A^{-1}$.

So far, the series have been assumed to be without trends in mean. To generalize this case, the common factor W_t can be assumed to be the sum of a trend $m(t)$, plus an $I(1)$ component without drift. Thus, multicointegration does allow trends, but of a very limited form.

3. Generation from Optimum Control

As an example of how multicointegration can arise from an optimum control situation, involving both proportional and integral control, consider the following situation: y_t is a series that one is attempting to control (e.g. inflation); y^*_{t-1} is the target series for y_t, determined at time $t-1$ by the controller; $e_t = y_t - y^*_{t-1}$ is the control error, being the extent to which the target is missed, perhaps due to imperfect control; c_t is the control series, whose value is set at time t by the controller.

Assume that y_t and c_t are related by the 'plant equation'

$$y_t = c_{t-1} + x_{t-1} + \varepsilon_t, \tag{3.1}$$

where x_t is some unspecified set of predetermined variables (possibly including expectations made at time $t-1$ of some contemporaneous variables).

The accumulated control error is

$$S_t = \sum_{j=0}^{t} e_{t-j},$$

and it is assumed this series also has a target series S^*_{t-1}. Costs to the controller will arise from three sources: the size of e_t (i.e. $y_t - y^*_{t-1}$), the size of the control error for S_t (i.e. $S_t - S^*_{t-1}$), and the amount of change in c_t, the cost of changing the control series. Assuming quadratic costs, the quantity to be minimized is thus

$$J = E[(y_{t+1} - y^*_t)^2 + \lambda_1(S_{t+1} - S^*_t)^2 + \lambda_2(c_t - c_{t-1})^2], \tag{3.2}$$

the expectation being made at time t conditional on quantities known at time t. It is naturally assumed that both λ_1, λ_2 are $\geqslant 0$.

Using $S_{t+1} = y_{t+1} - y^*_t + S_t$ and substituting from (3.1) with t replaced by $t+1$ gives

$$J = (1+\lambda_1)\sigma^2_\varepsilon + (c_t + x_t - y^*_t)^2 + \lambda_1(c_t + x_t - y^*_t + S_t - S^*_t)^2 + \lambda_2(c_t - c_{t-1})^2.$$

Differentiating with respect to c_t and equating to zero gives

$$c_t = \theta[(1+\lambda_1)y^*_t - (1+\lambda_1)x_t - \lambda_1 S_t + \lambda_2 c_{t-1} + \lambda_1 S^*_t],$$

where $\theta = (1 + \lambda_1 + \lambda_2)^{-1}$, substituting for c_{t-1} from (3.1) gives

$$c_t = \theta[(1+\lambda_1)y^*_t - (1+\lambda_1)x_t - \lambda_1(S_t - S^*_t) + \lambda_2 y_t - \lambda_2 x_{t-1} - \lambda_2\varepsilon_t].$$

Finally, replacing t by $t-1$ substituting into the plant equation (3.1) gives

$$\Delta y_t = -\theta(1+\lambda_1)(y_{t-1} - y^*_{t-1}) - \theta\lambda_1(S_{t-1} - S^*_{t-1}) + \theta\lambda_2\Delta x_{t-1} + d(B)\varepsilon_t, \tag{3.3}$$

where $d(B) = 1 - \theta\lambda_2 B$, and it should be noted that $0 \leq \theta\lambda_2 < 1$.

If it is assumed that the two target series y_t^* and S_t^* are both $I(1)$, (3.3) is consistent with y_t, S_t being cointegrated with y_t^*, S_t^*, respectively. For (3.3) to be consistent with multicointegration one has to add the condition that y_t^* and S_t^* are cointegrated. It is thus seen that multicointegration can arise from a special control situation. See also Granger, 1988.

For the inventory example, y_t^* would be expected sales, y_t actual production, c_t planned production, $x_t = 0$, e_t change in inventory, S_t level of inventory, S_t^* planned level to inventory, which is linearly related to planned production or expected sales.

4. Empirical Example: Inventories

The question considered in this section is the form of the relationships between sales, production, and inventory. For a company there is an obvious identity

$$\text{production} - \text{sales} = \text{change in inventory}, \tag{4.1}$$

and if sales is $I(1)$ and the change in inventory is $I(0)$ then production and sales will be cointegrated with a known cointegrating vector $(1 - 1)$. For multicointegration, inventory and sales (and hence production) will also need to be cointegrated. For this particular situation, standard tests for cointegration can be used between inventories and sales as the cointegrating vector at the first level is known. If, using the notation of Section II, both A and D are estimated, new test critical levels for D, the second level, may need to be found. Fortunately, this question can be left for later study. A further advantage of this example is that the first level of cointegration, identity (4.1), will aggregate perfectly from an individual company to an industry and to gross macro variables. The second level of cointegration will not necessarily aggregate unless the D values are (virtually) identical across companies. Aggregation questions are considered by Gonzalo (1989).

The general process of testing for multicointegration and its modeling is based on the methods discussed in Engle and Granger 1987. For a pair of series x_t, y_t the steps are

1. test that both x_t, y_t are $I(1)$,
2. run a least-squares regression

$$x_t = a + by_t + \text{residual } (z_{1t})$$

   to estimate a, b,
3. test that the residual (z_{1t}) is $I(0)$,
4. run the OLS regression

$$x_t = c + dS_{1t} + \text{residual } (w_{1t}),$$

where $S_{1t} = \sum_{j=0}^{t} z_{1,t-1}$, and

5. test if that residual (w_{1t}) is $I(0)$.

The test used in (1) is the augmented Dickey–Fuller (ADF) test in which an OLS regression

$$\Delta x_t = \beta x_{t-1} + \text{lags of } \Delta x_t \tag{4.2}$$

is run and the t-statistic of β used as the test statistic. The null is $\beta = 0$, which corresponds to $x_t \sim I(1)$. The t-statistic does not have the t-distribution, so that critical values provided by Dickey and Fuller (1979) have to be used. In step (3) the same test statistic is used with x_t in (4.2) replaced by z_t, but as b is estimated somewhat different critical values have to be used, as provided by Engle and Granger (1987). Presumably as b and d are both estimated, the critical values of the ADF test will require further modification. However, for our empirical example, step (2) is unnecessary, and thus regular ADF test can be used in steps (1) and (3) and the Engle–Granger modified test used in step (5). It may be noted that b and d are estimated with extra efficiency, as proved by Stock (1987), when the series involved are cointegrated. The error correction model takes the form

$$\Delta x_t = \rho_1 z_{1,t-1} + \rho_2 w_{1,t-1} + \text{lagged}(\Delta x_t, \Delta y_t) + \text{white noise residual},$$

which is estimated by OLS and the significance of ρ_1, ρ_2 can be tested using standard t-tests. It has become standard practice to repeat all the steps, reversing x_t, y_t in (2), to give z_{2t} for use in (3) and similarly in steps (4) and (5), giving as new residual w_{1t}. The error correction model for Δy_t, then uses z_{2t}, w_{2t} in its construction.

To produce an example, series are taken from the U.S. Department of Commerce, Bureau of Economic Analysis, as available on the Citibank data tape. Monthly figures for the period 1967:1 to 1987:4 for final sales in manufacturing and trade in constant (1982) dollars provide the sales figures. The same tables provide figures for inventories in constant dollars and the 'production' series is then generated by the identity (4.1). The sample size is 244 observations.

The notation used in p_t is production, s_t sales, z_t change in inventory and I_t inventory $= \sum_{j=0}^{t} z_{t-j}$. Note that I_t is the inventory level apart from the initial level I_0, which appears as a constant throughout the sample period.

Using the manufacturing and trade data described above, the ADF test for p_t, s_t both indicated that they are $I(1)$, with test statistics having values -0.55 and -0.41, respectively. Twelve lags were used in the test, and the 95 per cent critical value is approximately 2.88. For z_t, the change in inventory, the ADF test statistic takes the value -4.28, which allows rejection of the null of $I(1)$ at least at a 99 per cent level. These initial tests thus indicate that sales and production are cointegrated, z_t is $I(0)$, and I_t will be $I(1)$. To back up the ADF test, it might be noted that the first six autocorrelations of the z_t series are 0.432, 0.317, 0.452, 0.304, 0.230, and 0.232.

The regression relating production and the level of inventory gave

$$p_t = 13.77 + 0.62I_t + w_{1t}, \qquad \bar{R}^2 = 0.93, \qquad DW = 0.10$$

w_{1t} has an ADF test statistic of -3.44, suggesting that the null hypothesis that w_{1t} is $I(1)$ can be rejected at the 5 per cent level. The first six autocorrelations of w_{1t} are 0.943, 0.894, 0.841, 0.776, 0.712, and 0.646. The corresponding error correction model is

$$\begin{aligned}\Delta p_t = {} & \underset{(5.07)}{1.90} - \underset{(3.96)}{0.81 z_{t-1}} + \underset{(3.00)}{0.08 w_{1t-1}} + \underset{(1.15)}{0.18 \Delta p_{t-1}} \\ & - \underset{(2.09)}{0.36 \Delta s_{t-1}} + \text{residual}, \qquad \bar{R}^2 = 0.06, \qquad DW = 1.97\end{aligned}$$

(moduli of t-values are shown below).

Reversing p_t, s_t in this sequence gives a residual to step (4) w_{1t} that is also $I(0)$ at the 5 per cent level, with an ADF statistic of -3.32 and giving an error correction model

$$\begin{aligned}\Delta s_t = {} & \underset{(2.79)}{0.99} - \underset{(0.16)}{0.03 z_{t-1}} + \underset{(0.63)}{0.02 w_{1,t-1}} - \underset{(2.47)}{0.42 \Delta s_{t-1}} \\ & + \underset{(1.51)}{0.24 \Delta p_{t-1}} + \text{residual}, \qquad \bar{R}^2 = 0.03, \qquad DW = 2.01.\end{aligned}$$

It is seen that the error correction models indicate that the corrections occur and are significant only in the production equation.

The results are generally supportive of multicointegration being present between gross production and sales.

The same analysis has been conducted for production and sales of each of 27 U.S. industries plus industrial groupings with generally similar conclusions. These results will be presented elsewhere (Ganger and Lee 1989).

5. Conclusion

This paper has introduced a deeper level of cointegration, which might be expected to occur in economics, at least in theory. It can arise from special optimal control situations and, if present, can further improve short- and long-run forecasts. It does seem to be particularly appropriate for considerations of inventory, as illustrated by the empirical example. The extent to which it is found in other economic series can only be established by further empirical work.

Appendix A. Proof of $(1-B)^2$ root in $\det C(B)$

The Cramer representation has

$$\Delta X_t = C(B)\varepsilon_t, \tag{A.1}$$

and using the notation

$$C(B) = C(1) + \Delta C^*(B), \qquad C^*(B) = C^*(1) + \Delta C^{**}(B), \tag{A.2}$$

and

$$C(B) = \begin{bmatrix} C_{11}(B) & C_{12}(B) \\ C_{21}(B) & C_{22}(B) \end{bmatrix}, \tag{A.3}$$

it is found that

$$\det C(B) = E_0 + E_1\Delta + O(\Delta^2),$$

where

$$E_0 = C_{11}(1)C_{22}(1) - C_{12}(1)C_{21}(1)$$
$$E_1 = [C_{11}(1)C^*_{22}(1) + C_{22}C^*_{11}(1) - C_{21}(1)C^*_{12}(1) - C_{12}(1)C^*_{21}(1)].$$

It should be noted that

$$p_t = x_t - DS_t = (1 - D\Delta^{-1})x_t + DA\Delta^{-1}y_t.$$

Substitution from (A.2) and (A.3) gives

$$\begin{aligned} p_t = \{&-D\Delta^{-1}[C_{11}(1) - AC_{21}(1)] - [C_{11}(1) - DC^*_{11}(1) + ADC^*_{21}(1)] \\ &+ O(\Delta)\}\Delta^{-1}\varepsilon_{1t} \\ &+\{-D\Delta^{-1}[C_{21}(1) - AC_{22}(1)] + [C_{12}(1) - DC^*_{12}(1) + ADC^*_{22}(1)] \\ &+ O(\Delta)\}\Delta^{-1}\varepsilon_{2t}. \end{aligned}$$

For p_t to be $I(0)$, terms in Δ^{-1} and Δ^{-2} must be zero, giving the conditions

$$C_{11}(1) = AC_{21}(1), \qquad C_{12}(1) = AC_{22}(1), \tag{A.4}$$

which ensure that $E_0 = 0$, and

$$C_{11}(1) = D[C^*_{11}(1) - AC^*_{21}(1)], \qquad C_{12}(1) - D[C^*_{12}(1) - AC^*_{22}(1)], \tag{A.5}$$

which ensure that $E_0 = 0$, hence giving the result that $\det C(B) = O(\Delta)^2$ as required. Conditions (A.4) are sufficient to ensure that $z_t \sim I(0)$ and conditions (A.4) and (A.5) together are those required on $C(B)$ to guarantee multi-cointegration.

Appendix B. The error correction model

If x_t, y_t are multicointegrated, denoting $X_t = (x_t \quad y_t)'$ as a 2×1 vector the Cramer representation is

$$\Delta X_t = C(B)\varepsilon_t.$$

If $A(B)$ is the adjunct matrix of $C(B)$ this may then be written using the result of Appendix A as

$$A(B)\Delta X_t = d(B)\Delta^2 \varepsilon_t,$$

i.e.

$$A(B)X_t = d(B)\Delta\varepsilon_t. \tag{B.1}$$

Using the expansions

$$A(B) = A(1) + \Delta A^*(B), \qquad A^*(B) = A^*(1) + \Delta A^{**}(B),$$

we have

$$A(B) = A(1)B + \Delta\widetilde{A}(B), \tag{B.2}$$

$$\widetilde{A}(B) = \widetilde{A}(1)B + \Delta\bar{A}(B), \tag{B.3}$$

where

$$\widetilde{A}(B) = A(1) + A^*(B), \qquad \bar{A}(B) = A(1) + A^*(1) + A^{**}(B).$$

Let

$$z_t = x_t - Ay_t = (1 \quad -A)X_t, \qquad p_t = x_t - D\delta z_t = (1 - D\delta \quad AD\delta)X_t,$$

where $\delta = \Delta^{-1}$.

Using (B.2), (B.1) can be written

$$\widetilde{A}(B)\Delta X_t = -\gamma z_{t-1} + d(B)\Delta\varepsilon_t, \tag{B.4}$$

since $A(1) = \gamma\alpha'$ (see Engle and Granger, 1987). If $\alpha' = (1 - A)$ and

$$A(1) = \begin{bmatrix} A_{11}(1) & A_{12}(1) \\ A_{21}(1) & A_{22}(1) \end{bmatrix},$$

then

$$\gamma = \begin{pmatrix} A_{11}(1) \\ A_{21}(1) \end{pmatrix}.$$

Dividing (B.4) by Δ gives

$$\widetilde{A}(B)X_t = -\gamma\delta z_{t-1} + d(B)\varepsilon_t.$$

Substitution from (B.3) by Δ gives

$$\begin{aligned} \bar{A}(B)\Delta X_t &= -\widetilde{A}(1)X_{t-1} - \gamma\delta z_{t-1} + d(B)\varepsilon_t \\ &= -\widetilde{A}(1)X_{t-1} - \gamma D^{-1}(x_{t-1} - p_{t-1}) + d(B)\varepsilon_t \\ &= -[\widetilde{A}(1) + D^{-1}\gamma i']X_{t-1} + D^{-1}\gamma p_{t-1} + d(B)\varepsilon_t, \end{aligned}$$

where $i' = (1 \quad 0)$. Let

$$\widetilde{A}(1) + D^{-1}\gamma i' = \gamma_2\alpha', \qquad D^{-1}\gamma = -\gamma_1,$$

so that

$$\bar{A}(B)\Delta X_t = -\gamma_1 p_{t-1} - \gamma_2 z_{t-1} + d(B)\varepsilon_t.$$

Since

$$\widetilde{A}(1) + D^{-1}\gamma i' = \begin{bmatrix} (1 + D^{-1})A_{11}(1) + A_{11}^*(1) & -AA_{11}(1) + A_{12}^*(1) \\ (1 + D^{-1})A_{21}(1) + A_{21}^*(1) & -AA_{21}(1) + A_{22}^*(1) \end{bmatrix},$$

then

$$\gamma_2 = (1 + D^{-1})\gamma + \begin{pmatrix} A_{11}^*(1) \\ A_{21}^*(1) \end{pmatrix} \tag{B.5}$$

or

$$\gamma_2 = \gamma - A^{-1}\begin{pmatrix} A_{12}^*(1) \\ A_{22}^*(1) \end{pmatrix}. \tag{B.6}$$

Finally, it suffices to show that the columns of $A^*(1)$ satisfy that (B.5) = (B.6). Noting that $A(B)$ is the adjunct matrix of $C(B)$, i.e.

$$A(B) = \begin{bmatrix} C_{22}(B) & -C_{12}(B) \\ -C_{21}(B) & C_{11}(B) \end{bmatrix},$$

the relation (B.5) = (B.6) can be written

$$(1 + D^{-1})C_{22}(1) + C_{22}^*(1) = C_{22}(1) = A^{-1}C_{12}^*(1),$$
$$(1 + D^{-1})C_{21}(1) + C_{21}^*(1) = C_{21}(1) + A^{-1}C_{11}^*(1),$$

i.e.

$$C_{22}(1) = DA^{-1}[C_{12}^*(1) - AC_{22}^*(1)],$$
$$C_{21}(1) = DA^{-1}[C_{11}^*(1) - AC_{21}^*(1)],$$

which hold if x_t and y_t are multicointegrated, since then (A.4) and (A.5) hold.

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References

Dickey, D. A. and W. A. Fuller (1979), 'Distribution of estimates for autoregressive time series with unit root', *Journal of American Statistical Association*, 427–31.

Engle, R. F. and C. W. J. Granger (1987), 'Cointegration and error correction: representation, estimation and testing', *Econometrica*, **55**, 251–71.

—— and B.-S. Yoo (1987), 'Forecasting and testing in cointegrated systems', *Journal of Econometrics*, **35**, 143–59.

Gonzalo, J. (1988), 'Cointegration and aggregation', unpublished Working Paper, University of California, San Diego.

Granger, C. W. J. (1983), 'Cointegrated variables and error correcting models', University of California, San Diego.

—— (1986), 'Developments in the study of cointegrated economic Variables', *Oxford Bulletin of Economics and Statistics*, **48**, 213–28.

—— (1988), 'Causality, cointegration and control', *Journal of Economic Dynamics and Control*, **12**, 551–9.

——and T.-H. Lee (1988), 'Investigation of production, sales and inventory relationships using multicointegration and nonsymmetric error correction models', *Journal of Applied Econometrics*, **4**, 5145–59.

Johansen, S. (1988), 'The mathematical structure of the error correction models', *Contemporary Mathematics*, American Math. Assoc.

Stock, J. (1987), 'Asymptotic properties of least squares estimators of cointegrating Vectors', *Econometrica*, **55**, 1035–56.

Yoo, B.-S. (1987), 'Cointegrated time series structure, forecasting and testing', Ph.D. dissertation, University of California, San Diego.

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Application of some advances in
vector autoregressive models) enabl
rational expectations present value
incomplete data on information of
some relatively encouraging new re
the term structure and some puzzlin
prices.

Present value models are among
of economics. A present value m
that Y_t is a linear function of th
future y_t:

$$Y_t = \theta(1 -$$

where c, the constant, θ, the c
discount factor, are parameters t
to be estimated. Here and in
expectation, conditional on the
includes y_t and Y_t themselves an
$\mathbf{H}_t$ available to the econometri
expectations theory for interest
the one-period rate), the present

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this paper. We are responsible for any
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