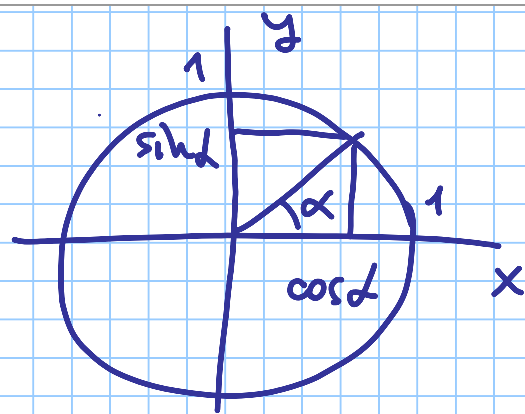
1. Trigonometry review



$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = 1 \quad \text{(definition of radians)}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 \Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \qquad \sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$$

Derivative of cosine

$$\frac{\cos(x+\Delta x) - \cos x}{\Delta x} = \frac{\cos x\cos(\Delta x) - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$\left(\frac{\cos\Delta x - 1}{\Delta x} \to 0 \qquad \frac{\sin\Delta x}{\Delta x} \to 1\right)$$

$$\to -\sin x \qquad \frac{d(\cos x)}{dx} = -\sin x$$

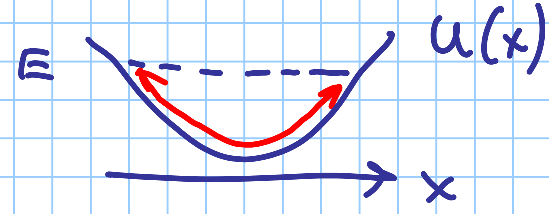
$$\frac{d(\sin x)}{dx} = +\cos x \quad \text{(similar derivation)}$$

$$\frac{d}{dt}(\cos(\omega t)) = -\sin(\omega t)\cdot\frac{d}{dt}(\omega t) = -\omega\sin(\omega t)$$

(chain rule)

$$\frac{d}{dt}(\sin\omega t) = +\omega\cos(\omega t)$$

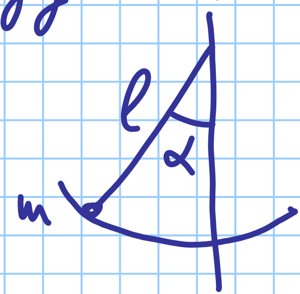
2. Oscillatory motion



Motion in a potential well (no friction)

$$\frac{mv^2}{2} + U(x) = E \quad \text{(energy conservation)}$$

Consider a pendulum

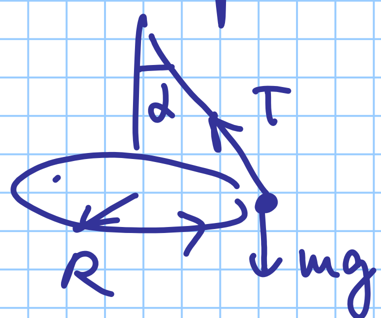


$$U(\alpha) = mgl(1-\cos\alpha)$$

$$= 2mgl\sin^2\frac{\alpha}{2} \approx mgl\cdot\frac{\alpha^2}{2} \quad (\alpha - \text{small})$$

Consider spherical pendulum – rotational motion

$$a = \frac{v^2}{R}$$



$$ma = T\sin\alpha$$

$$T\cos\alpha = mg$$

$$m\cdot v^2/R = mg\,\frac{\sin\alpha}{\cos\alpha} = mg\tan\alpha$$

$$R = l\sin\alpha$$

$$v = \omega R$$

$$\omega = \frac{2\pi}{\text{period}} \quad \text{– angular velocity}$$

$$\omega^2 = \frac{g\tan\alpha}{l\sin\alpha} = \frac{g}{l\cos\alpha} \approx \frac{g}{l} \Rightarrow \omega = \sqrt{\frac{g}{l}}$$

$$\left.\begin{aligned} x &= R\cos\omega t \\ y &= R\sin\omega t \end{aligned}\right\}$$

This is a sum of two oscillations – one along x one along y!!

Check: $x = R\cos\omega t = \alpha\cdot l \Rightarrow \alpha = \alpha_0\cos\omega t$

$$v = \frac{dx}{dt} = -R\omega\sin(\omega t)$$

$$R \approx l\cdot\alpha_0$$

$$K = \frac{mv^2}{2} = \frac{m R^2 \omega^2}{2} \sin^2(\omega t) = \frac{mg\ell}{2} \alpha_0^2 \sin^2(\omega t)$$

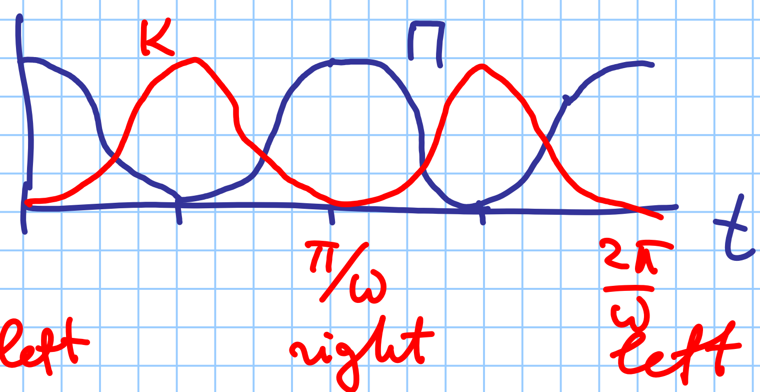
$$\Pi = mg\ell \frac{\alpha^2}{2} = mg\ell \frac{1}{2} \alpha_0^2 \cos^2 \omega t$$

$$K + \Pi = \frac{1}{2} mg\ell \cdot \alpha_0^2 (\sin^2 \omega t + \cos^2 \omega t) = \frac{1}{2} mg\ell \, (!)$$

conserved energy!

* Many situations where energy quadratic near a minimum $\Rightarrow$ small oscillations harmonic

* $K \leftrightarrow \Pi$ all the time (at twice the frequency)



Frequency (Hz) $\frac{1}{\text{period}} = \frac{\omega}{2\pi}$.

<u>Example</u> mechanical oscillator

$$ma = -kx \qquad (x \text{ from equilibrium})$$

$$\frac{mv^2}{2} + \frac{kx^2}{2} = E$$

Frequency

Try: $x = x_0 \cos(\omega t)$

$$v = \frac{dx}{dt} = -\omega x_0 \sin(\omega t)$$

$$\frac{mv^2}{2} = \frac{m\omega^2 x_0^2}{2} \sin^2 \omega t \qquad \frac{kx^2}{2} = \frac{k x_0^2}{2} \cdot \cos^2 \omega t$$

To match, have to have $\omega^2 = k/m$

3. Waves

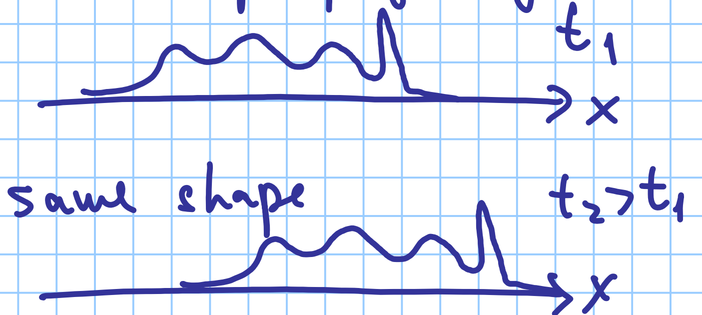
Perturbation propagates through a medium with certain velocity

Generally, function of both position and time
(x) (t)

arbitrary functions

$$u(x,t) = f(x - vt) + g(x + vt)$$

propagating to the right (increasing x)



Displacement

$$\Delta x = v \cdot \Delta t = v(t_2 - t_1)$$

Example * Waves on the surface of water

* Waves on a string with tension $v = \sqrt{\frac{T}{m/\ell}}$
* Sound waves in a gas $v^2 = \left(\frac{dp}{d\rho}\right)_S = \gamma \frac{RT}{\mu}$

Air at normal T

$v = 330\ m/s$

$\gamma = \frac{C_p}{C_V}$ – adiabatic exp.

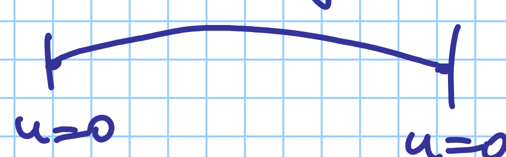
* Electromagnetic waves $v = c/n$ c: speed of light in vacuum

n – refraction index

$c = 3 \cdot 10^8\ m/s$

Example Lightning vs thunder time lag...

String: transverse wave



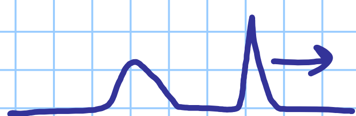
$u = 0$ at the end $(x = L)$

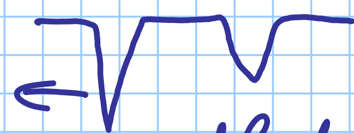
Thus, the solution must be

$$u(x,t) = \underset{\text{orig. wave}}{f(x-vt)} + \underset{\text{reflected wave)}}{g(x+vt)}$$

$u(L,t)=0$ $\qquad f(L-vt)+g(L+vt)=0$

$L+vt = y$ $\qquad \boxed{g(y) = -f(2L-y)}$





reflected wave: same shape
opposite amplitude...

Two ends - have to consider multiple reflections
like in a mirror gallery...

Alternative: try harmonic waves

$$u(x,t) = A\sin(kx-\omega t+\phi_1) + B\sin(kx+\omega t+\phi_2)$$

$$\left.\begin{aligned} f(y) &= A\sin(ky+\phi_1) \\ g(y) &= B\sin(ky+\phi_2) \end{aligned}\right\} \Rightarrow \quad v = \omega/k$$

Separate x and t:

$$A\sin(kx-\omega t+\phi_1) = A\sin kx\cos(\omega t-\phi_1) - A\cos kx\sin(\omega t-\phi_1)$$

$$B\sin(kx+\omega t+\phi_2) = B\sin kx\cos(\omega t+\phi_2) + B\cos kx\sin(\omega t+\phi_2)$$

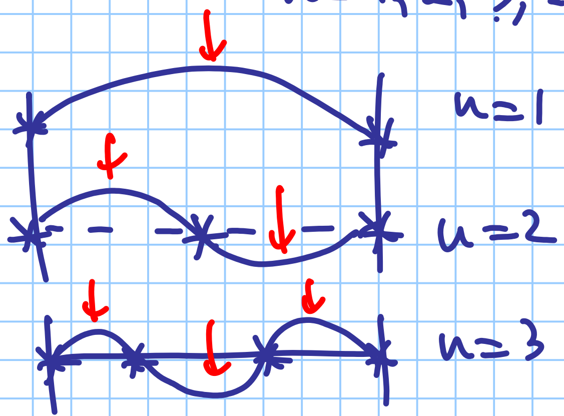
sum $=0$ at $x=0 \Rightarrow \quad \phi_2 = -\phi_1 \equiv \phi$ $\quad$ so that $\cos(kx)=0$

$A=B$

$$u(x,t) = 2A\sin kx\cdot\cos(\omega t+\phi)$$

$u = 0$ at $x = L \Rightarrow \sin(kL) = 0 \Rightarrow k = \frac{\pi n}{L}$

$n = 1, 2, 3, \ldots$ standing waves



$n=1$ – principal mode

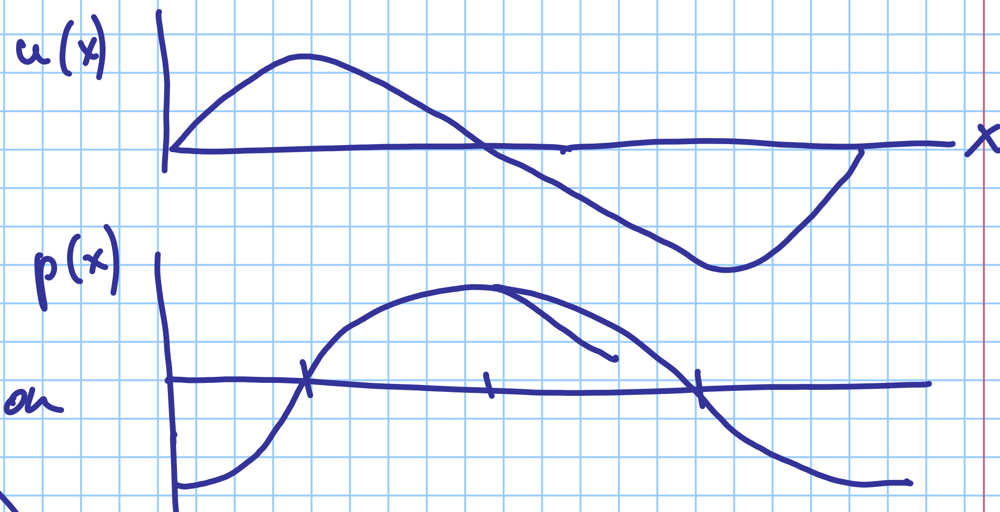
$*$ = node

$\downarrow$ = antinode

Organ pipes (longitudinal)

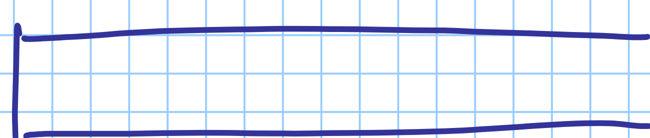
$*$ air displacement u

$*$ pressure increment p



$-\frac{du}{dx}$: compression

$$\boxed{p(x) \propto -\frac{du}{dx}}$$



closed end

$u = 0$

open end

$p = 0 \Rightarrow \frac{du}{dx} = 0$

Standing wave $\quad u = u_0 \cos(\omega t + \phi) \sin(kx)$

$u(x=0) = 0 \qquad \frac{du}{dx}(x=L) = 0 \Rightarrow \cos(kL) = 0$

$$kL = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

4. Wave superposition

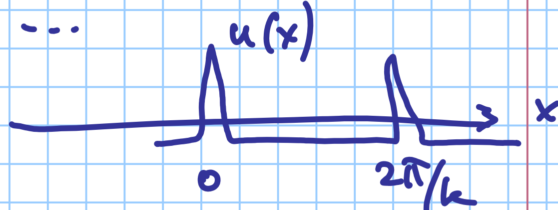
Just add the amplitudes

Consider $u = \cos(kx) + \cos(2kx) + \cos(3kx) + \ldots \cos(Nkx)$

At $x=0$ $u=N$ (add in phase)

also, at $x = \frac{2\pi}{k}, \frac{4\pi}{k}, \ldots$

Far from maxima $u = -1/2$



Contribution cancel for

$N k \Delta x \geq \pi \Rightarrow$ width of each peak

$$\frac{\pi}{Nk} = \frac{\pi}{k_{max}}$$

$$(\Delta x) \cdot (\Delta k) \geq 1/2$$

constant of order 1.

Uncertainty principle

$$(\Delta t) \cdot (\Delta \omega) \geq 1/2$$

smallest feature in real time

bandwidth

Example AM radio $u(x,t) = f(t - x/v)$

$$f(t) = A(t) \cdot \cos(\omega t + \phi)$$

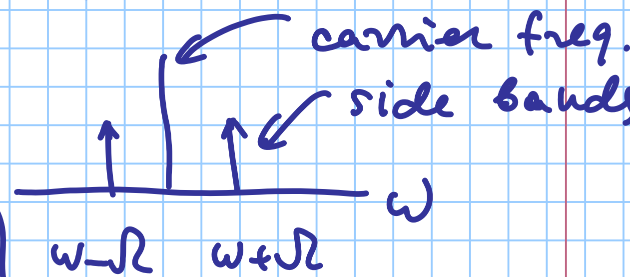
ω: carrier frequency

$A(t)$: amplitude modulation (e.g. sound)

$$A(t) = A_0 \cos(\Omega t)$$

$$\cos(\Omega t)\cos(\omega t + \phi)$$

$$= \frac{1}{2}\cos\left[(\omega+\Omega)t + \phi\right] + \frac{1}{2}\cos\left[(\omega-\Omega)t+\phi\right]$$



For speech need some 10 kHz bandwidth otherwise neighboring stations overlap

Problem: any noise is heard on AM

Solution: FM modulation

$$u(t) = A_0 \cos(\alpha(t))$$

$$\alpha(t) = \omega_0 t + \int x(t)\,dt$$

signal to be transmitted

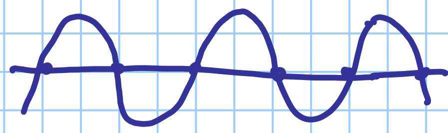
Generally, wider band needed but superior quality

Can still recover the phase even with noise present

US: 101.1 101.3 MHz

$\Delta\omega / 2\pi = 200$ kHz plenty of bandwidth.

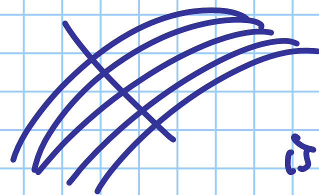
Digital broadcasting

*  Can recover as long as $f \cdot \Delta t < 1$

* sequence of 0 and 1 does not suffer that much from noise

* use error correction to gain even better fidelity

Example CD scratch 0.5 mm wide would not be heard as long as it is not along the tracks...



5. Energy - always quadratic in Amplitude

$$\Rightarrow A^2 \propto \frac{1}{r^2} \text{ for spherical waves}$$

$$A \propto \frac{1}{r}$$

Example $1.2 \frac{kWt}{m^2}$ at Earth orbit solar power

Mars ~ 1.5 AU $\Rightarrow$ a half of power

Pluto ~ 40 AU $\Rightarrow \frac{1}{1600}$ of power - not practical to use solar...