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The Pricing of Time-Varying Beta[1]

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Abstract: We generalize an asset pricing model based on the Arbitrage Pricing Theory (APT) allowing beta to be time-varying. Making beta a random variable adds flexibility to the model because permits a non-linear relation between individual returns and the set of factors, and accounts for the effect of possible omitted variables. We integrate the conditional APT with a general linear stochastic process for beta. We analyze the behavior of the conditional expected return, the conditional variance and conditional covariance of individual asset returns as functions of the conditional moments of beta. On considering time-varying betas we introduce another source of uncertainty (risk) independent of the factors. We need to disentangle if this extra risk is systematic or non-systematic. To this end, we introduce a modified conditional APT model that rationalizes why the time variation of beta may represent extra systematic risk. For a sample of individual stocks, we test the hypothesis of time-varying beta and the feasibility of the modified conditional APT. We present a test for time-varying beta based on the conditional second moments of returns. We find that there is strong evidence against constancy of betas in favor of a random coefficient model, and that the time variation of beta is due to non-systematic behavior of the firms and investors should be able to diversify this risk away.

Key Words: APT, beta, (G)ARCH, Systematic Risk.

JEL Classification System-Numbers: C32, G12.

1 Introduction and Summary

Asset pricing theories postulate that the expected excess return to an asset must be a function of its riskiness. In Finance, risk is often measured by an asset's beta. In the Capital Asset Pricing Model (CAPM) (Sharpe 1964), the expected excess return to asset i is equal to the expected excess return to

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the market portfolio times beta, where beta is a standardized measure of the covariance of the return to asset i with respect to the market return. In the Arbitrage Pricing Theory (APT) (Ross 1976), the expected excess return to asset i is the sum of the risk premia associated with a set of factors times beta, where beta measures the sensitivity of the individual return with respect to each factor. These theoretical models help the investor to discriminate among a cross-section of assets according to their risk that is measured by their beta. In their original formulation, these models assume that: 1. beta is constant over time, 2. there is a linear relation between the individual asset return and the market portfolio (CAPM) or between the individual return and the set of factors (APT), and 3. there is a unique source of risk, the market portfolio in the CAPM, or if they allow for more than one source of risk as in the case of the APT, the theory is not specific to which sources should be included, hence any choice is bound to have omitted relevant sources of risk.

The constancy of beta over time is a very restrictive assumption. In fact, this assumption has been proven to be wrong by different authors, among them, Fabozzi and Francis (1978), Olhson and Rosenberg (1982), Bos and Newbold (1984), Collins, Ledolter and Rayburn (1987), and Brooks, Faff and Lee (1992). If beta is time-varying, it implies a break of the linear relationship between the individual return and the market portfolio (or the set of factors). Furthermore, the time variation of beta may reflect the influence of omitted variables in the CAPM or in the APT, as it has been shown in Leusner, Akhavein and Swamy (1996).

The first contribution of this study is the generalization of an APT model to include time-varying betas. Time-varying betas will account for non-linearities as well as for mispecification of the sources of risk. There are two methodological approaches to the modelling of beta. Leusner, Akhavein and Swamy (1996) chose a structural approach where beta is specified as a function of a set of variables such as firm size, ratio of book-to-market equity, dividend price ratio, default premium and the yield on the 10-year Treasury bill minus the 1-year Treasury bill rate. Our approach is a time-series approach. We assume that the past history of beta summarizes *all* possible sources of risk. We postulate a general linear stochastic process for the time behavior of beta and we superimpose it on an asset pricing model of the APT type. We analyze the behavior of the conditional expected return, the conditional variance and conditional covariance of individual asset returns. If beta were constant, analyzing variance and covariance of individual asset returns will not provide additional information other than of that contained in the conditional mean. However, if beta is time-varying, the conditional probability distribution of returns depends on the conditional probability distributions of the betas. In other words, beta is a random variable and its statistical moments must drive the statistical moments of returns. We show that the critical moments of beta are its conditional expectation and its conditional non-central second moment. The former affects the conditional expected return and the conditional covariance of the return with a factor and the latter drives the conditional variance of returns. On discovering these relations, we raise some econometrics issues that affect the current literature in which regression equations have conditional variances and covariances as regressand and regressors.

The second contribution of this paper is focused on what constitutes systematic risk in an APT model with time varying betas. The starting point of analysis in an APT model is the existence of factor(s) that represent systematic risk. When beta is a random variable, we introduce another source of uncertainty independent of the factor(s). The time variation of beta may account for other economic variables different from the factors. These variables may carry time-varying systematic components or time-varying idiosyncrasies or a mixture of time-varying systematic and idiosyncratic components. If the time variation of beta is systematic, it represents risk that cannot be diversified away and consequently it needs to be rewarded. On the other hand if the time-variation is purely idiosyncratic, it can be diversified away and it will not be rewarded. We propose a modified conditional APT that extracts the systematic component from the time variation of beta and assigns a factor to it. If there is not systematic component in the time variation of beta, the proposed factor should not have any power in explaining the movements in conditional expected returns.

The third contribution of this paper consists of testing the maintained hypothesis of time-varying beta and the likelihood of the proposed modified conditional APT. A common feature of the studies dealing with time-varying coefficients is that are generally based on state-space models estimated with Kalman filter techniques. We offer an alternative test based on conditional second moments. The test takes advantage of the first contribution of this paper that explains the behavior of the conditional covariance and variance of returns. We construct a system of two seemingly unrelated regressions. The regressand of the first equation is the conditional variance of the individual stock return, and the regressand of the second equation is the conditional covariance of the return with a factor, that is the market portfolio. In both equations, the regressor is the conditional variance of the market portfolio. The test consists of imposing a non-linear restriction across these two equations. Under the null hypothesis of constant beta, the non-linear restriction amounts to test that the variance of beta is equal to zero. This type of statistical testing has been well documented by Gallant (1987). For the sample under study we are able to strongly reject the constancy of beta. Consequently, we need to address the question on the validity of the proposed modified APT, which postulates that there are systematic components on the variance of beta. The risk implied by the time-variation of beta is a controversial issue. Empirical studies offer different answers. Chen and Keown (1981) claim that random diversification of portfolios reduces to tiny amounts the extra risk induced by time-varying beta. In Schwert and Seguin (1990) time-varying beta represents risk to be rewarded. In Ferson and Harvey (1991), the proportion of the variance of returns due to variation in beta is very small compared to the variation in the corresponding risk premium. For the sample under study, we reject the modified APT implying that the time variation of beta represents unsystematic risk which can be diversified away.

The organization of the paper is the following. In section 2.1, we present an APT model with time-varying betas, summarizing the first contribution of this paper in Proposition 1. In section 2.2, the modified APT model is introduced. In section 3.1, we perform the empirical testing for constancy of beta, and in section 3.2, the testing of the modified APT model is addressed.

2 Models with Time-Varying Beta

In this section, we compare a dynamic APT model with constant betas to a dynamic APT model with time-varying betas. Let us assume, without loss of generality, that there is one dynamic factor that drives the returns. Let R_{it} be the excess return to asset i; μ_{it} the conditional expected excess return to asset i; f_t the dynamic factor with mean zero; b_i the sensitivity of asset i to the factor; and ε_{it} the idiosyncratic noise, with mean zero and constant variance over time. We can write

$$R_{it} = \mu_{it} + b_i f_t + \varepsilon_{it}$$

Following an argument as in Ross (1976), the conditional expected return will approximately be

$$\mu_{it} = b_i \lambda_t$$

where λ_t is the time-varying risk premium of the factor f_t. The conditional variance of asset i is

$$var_{t-1}(R_{it}) = b_i^2 var_{t-1}(f_t) + var(\varepsilon_{it})$$

that is split into a systematic or non-diversifiable component, $b_i^2 var_{t-1}(f_t)$, and a non-systematic or diversifiable component, $var(\varepsilon_{it})$. The covariance of asset i with the factor f_t is

$$cov_{t-1}(R_{it}, f_t) = b_i var_{t-1}(f_t)$$

In this model the beta, b_i, fully characterizes the asset i. The conditional mean and the systematic part of the conditional variance depend on b_i. Furthermore, b_i is the traditional ratio of the conditional covariance between the individual return and the factor divided by the conditional variance of the factor. The unique attribute of asset i is b_i, and analyzing the conditional mean of returns will suffice to estimate b_i.

2.1 A General Stochastic Process for b_{it}

Now, let us consider a one-dynamic factor model with time-varying betas b_{it}.

$$R_{it} = \mu_{it} + b_{it} f_t + \varepsilon_{it} \tag{1}$$

Let us assume that $\{b_{it}\}$ is a stationary stochastic process. According to the Wold's decomposition, a general process for b_{it} can be written as

$$b_{it} = b_i + \sum_{j=0}^{\infty} \psi_j a_{i,t-j} \tag{2}$$

where $\psi_0 = 1$ and a_{it} is a white noise process, statistically independent of the factor f_t, and uncorrelated with the idiosyncratic noise ε_{it},

$$E(a_{it}) = 0\ , \quad E(a_{it}^2) = \sigma_{ai}^2\ , \quad E(a_{it} f_t) = E(a_{it} f_t^2) = 0\ , \quad E(a_{it}\varepsilon_{it}) = 0 \tag{3}$$

The following proposition characterizes the conditional moments of R_{it}.

Proposition 1: Under (1), (2), and (3), the first and second conditional moments of R_{it} are

1-(i) $\mu_{it} = E_{t-1}(b_{it})\lambda_t + \lambda_0$

1-(ii) $var_{t-1}(R_{it}) = E_{t-1}(b_{it}^2) var_{t-1}(f_t) + var(\varepsilon_{it})$

1-(iii) $cov_{t-1}(R_{it}, f_t) = E_{t-1}(b_{it}) var_{t-1}(f_t)$

Proof:

1-(i). The formal proof follows the same steps as in Ross (1976). Here we present an heuristic argument.

The goal is to form a riskless arbitrage portfolio of assets, R_{pt}. Let the individual returns follow model (1). An arbitrage portfolio is constructed with N assets, each one with weight w_i,

$$R_{pt} = \sum_{i=1}^{N} w_i R_{it} = \sum_{i=1}^{N} w_i \mu_{it} + \sum_{i=1}^{N} w_i b_{it} f_t + \sum_{i=1}^{N} w_i \varepsilon_{it}$$

If b_{it} follows the process described by (2) then,

$$R_{pt} = \sum_{i=1}^{N} w_i \mu_{it} + \sum_{i=1}^{N} w_i \left(b_i + \sum_{j=1}^{\infty} \psi_j a_{i,t-j} \right) f_t + \sum_{i=1}^{N} w_i a_{it} f_t + \sum_{i=1}^{N} w_i \varepsilon_{it}$$

If N is large enough, we can invoke the law of large numbers and $\sum_i w_i a_{it}$ and $\sum_i w_i \varepsilon_{it}$ will approach zero asymptotically. Because this portfolio is an arbitrage portfolio which does not require a change in wealth,

$$\sum_{i=1}^{N} w_i = 0 \tag{4}$$

and because we require the portfolio to be riskless

$$\sum_{i=1}^{N} w_i(b_i + \sum_{j=1}^{\infty} \psi_j a_{i,t-j}) = \sum_{i=1}^{N} w_i E_{t-1}(b_{it}) = 0 \tag{5}$$

Since the portfolio is riskless and it does not involve new wealth, the return should be zero

$$R_{pt} = \sum_{i=1}^{N} w_i \mu_{it} = 0 \tag{6}$$

Putting together (4), (5), and (6), we can conclude that, at time t, the conditional mean μ_{it} is spanned by the vectors 1 and $E_{t-1}(b_{it})$

$$\mu_{it} = \lambda_0 + E_{t-1}(b_{it})\lambda_t$$

1-(ii). From (1)

$$var_{t-1}(R_{it}) = var_{t-1}(b_{it}f_t) + var(\varepsilon_{it}) + 2\,cov_{t-1}(b_{it}f_t, \varepsilon_{it})$$

Because of independence between a_{it} and f_t,

$$var_{t-1}(b_{it}f_t) = E_{t-1}(b_{it}^2)var_{t-1}(f_t)$$

where

$$E_{t-1}(b_{it}^2) = var_{t-1}(b_{it}) + (E_{t-1}(b_{it}))^2 = \sigma_a^2 + \left(b_i + \sum_{j=1}^{\infty} \psi_j a_{i,t-j}\right)^2$$

Furthermore

$$cov_{t-1}(b_{it}f_t, \varepsilon_{it}) = 0$$

because f_t and ε_{it} are uncorrelated.

1-(iii). Multiply both sides of (1) by f_t and apply conditional expectation.

Proposition 1 explores the dynamics of a given asset. Three remarks are in order.

First, note that if the betas are time-varying and we estimate a model with constant betas, the error term must be heteroscedastic.[2] From (2) we calculate

[2] For a survey on the econometrics of time varying parameters models see Raj and Ullah (1981).

the conditional expected value of b_{it}

$$E_{t-1}(b_{it}) = b_i + \sum_{j=1}^{\infty} \psi_j a_{i,t-j}$$

and we substitute it in *1-(i)*,

$$\mu_{it} = b_i \lambda_t + \sum_{j=1}^{\infty} \psi_j a_{i,t-j} \lambda_t + \lambda_0$$

We see that the relation between the expected return and the risk premium associated with f_t is non-linear due to the term $a_{i,t-j}\lambda_t$. If the estimation of μ_{it} proceeds as if beta were constant, that is considering just $b_i\lambda_t + \lambda_0$, the error term must account for the non-linear part $\sum_{j=1}^{\infty} \psi_j a_{i,t-j}\lambda_t$. Standard diagnostic tests in the error term will show that heteroscedasticity is present.

Second, some articles as Schwert and Seguin (1990) or as Ng, Engle and Rothschild (1992) estimated regression models with conditional variances and covariances as regressand and regressors under the assumption of constant betas. These type of regressions should follow formulations as *1-(ii)* and *1-(iii)* where the regression coefficients $E_{t-1}(b_{it}^2)$ and $E_{t-1}(b_{it})$ and generally time-varying. Assuming constant betas implies that regressions of type *1-(ii)* and *1-(iii)* should have constant regression coefficients. It is necessary to test for time-varying betas before proceeding with the estimation of regressions *1-(ii)* and *1-(iii)*.

Third, comparisons of the conditional moments of returns are based on the conditional moments of their beta. Since the conditional variance of the factor, $var_{t-1}(f_t)$, and the risk premium on the factor, λ_t, are common features for all assets, the characterization of the return to asset i depends on the moments of b_{it}. If we consider the conditional distribution of R_{it}, we observe that the conditional mean of returns is a function of the conditional first moment of b_{it} *(1-i)*, the conditional variance of R_{it} is a function of the conditional non-central second moment of b_{it} *(1-ii)*, and the traditional ratio (systematic risk) of conditional covariance between the individual return and the factor to conditional variance of the factor is equal to the conditional expected value of b_{it} *(1-iii)*. This is an important difference with respect to models with constant betas. In *1-(i)*, the systematic component of μ_{it} is governed by $E_{t-1}(b_{it})$, and in *1-(ii)* the systematic component of $var_{t-1}(R_{it})$ depends on $E_{t-1}(b_{it}^2)$. APT models with time-varying betas may account for two potential sources of risk. Two assets may have the same expected value of b_{it} but different $E_{t-1}(b_{it}^2)$. For instance, consider two assets i and j such that asset i has a time-varying beta and asset j has a constant beta. Furthermore, let us assume that $E_{t-1}(b_{it}) = b_j$. Given the definition of variance, we have

$$var_{t-1}(b_{it}) = E_{t-1}(b_{it}^2) - (E_{t-1}(b_{it}))^2 = E_{t-1}(b_{it}^2) - b_j^2 \geq 0$$

According to *1-(ii)* and since $E_{t-1}(b_{it}^2) \geq b_j^2$, assets with time-varying betas are more sensitive to changes in the variance of the factor than assets with constant betas. The return to asset i is more uncertain than that of asset j. This example presents an interesting question. The time variation of beta induces some uncertainty that is not due to the factor. It may represent some other economic variables and it is our task to disentangle if these variables convey systematic or non-systematic risk. The next section offers some insight to this problem.

2.2 A Particular Process for b_{it} and a Modified APT

Among the possible linear models that equation (2) describes, there is one that deserves particular attention. This is the Hildreth-Houck random coefficient model (Hildreth and Houck 1968)

$$b_{it} = b_i + a_{it}$$

where a_{it} is white noise. This model has been the most successful in explaining the time behavior of beta. Bos and Newbold (1984) find a strong rejection of constancy of betas. They tested a random coefficient model against a first-order autoregressive stochastic parameter model and they could not find strong evidence against the random coefficient model. They thought that their asymptotic test statistic was lacking power in finite samples. Brooks, Faff and Lee (1992) analyzed the same hypothesis with an approximate point-optimal invariant test (King 1987), that has good small-sample properties, and they validated the Bos and Newbold results. Collins, Ledolter and Rayburn (1987) tested a random coefficient model against a more general alternative, ARMA(1,1) and they found that for individual stocks, the evidence against the random coefficient model is weak. In portfolios, there is more evidence of autocorrelated beta but we suspect that this may be due to the cross-correlation among the individual stocks of the portfolio. Their analysis relies in regression models with time-varying coefficients and likelihood ratio tests.

For a random coefficient model, note that there is no distinction between the conditional and unconditional moments of b_{it},

$$E_{t-1}(b_{it}) = E(b_{it}) = b_i \qquad E_{t-1}(b_{it}^2) = E(b_{it}^2) = b_i^2 + \sigma_{ai}^2$$

Under the random coefficient model, we compare two possible scenarios that give rise to two models of expected returns. In the first scenario, we assume that the time variation of beta does not represent any extra risk, in the second scenario, we assume that the time variation of beta implies extra risk and we present a new model: a modified conditional APT, which incorporates an extra factor to account for the risk induced by the time variation of beta.

Model 1. Time-Varying Beta Does Not Represent Extra Risk
For a random coefficient model for beta and according to Proposition 1, the conditional distribution of returns is

$$\begin{aligned} &\mu_{it} = b_i\lambda_t + \lambda_0 \\ &var_{t-1}(R_{it}) = (b_i^2 + \sigma_{ai}^2)var_{t-1}(f_t) + var(\varepsilon_{it}) \\ &cov_{t-1}(R_{it}, f_t) = b_i var_{t-1}(f_t) \end{aligned} \tag{7}$$

The conditional expected return depends on the mean of b_{it}, which is a constant; and the systematic part of the conditional variance depends on the non-central second moment of b_{it}, which is also a constant. The decomposition of the conditional variance of the return says that there is a portion of the variability of the return that is driven by the variance of the factor, that is $(b_i^2 + \sigma_{ai}^2)var_{t-1}(f_t)$ and another portion that is driven by the idiosyncratic risk of the firm, that is $var(\varepsilon_{it})$. It is clear that the term $b_i^2 var_{t-1}(f_t)$ constitutes systematic risk. However, the term $\sigma_{ai}^2 var_{t-1}(f_t)$ does not constitute systematic risk according to the reading of equation (7), that says that the expected returns depend on the *constant* component of b_{it} which is b_i. The only systematic risk, specific to asset i, in this model is b_i. Further reading of the set of equations (7) says that b_i is the conditional covariance between the individual return and the factor divided by the conditional variance of the factor. Any time-variation of beta that is contained in a_{it} should be understood as diversifiable risk and it will not be priced.

Model 2. Time-Varying Beta Does Represent Extra Risk
To motivate the second possible scenario, consider two assets i and j such that $b_i = b_j$ but asset i has a time-varying beta that follows the random coefficient model and asset j has a constant beta. Let us assume that the idiosyncratic component is the same for both assets. Under these assumptions and according to Model 1, we have

$$\mu_{it} = \mu_{jt} \qquad var_{t-1}(R_{it}) > var_{t-1}(R_{jt}) \qquad cov_{t-1}(R_{it}, f_t) = cov_{t-1}(R_{jt}, f_t)$$

These two assets have the same expected return but asset i has higher variance. If the investors perceive the asset with higher variance as systematically riskier, given that they are receiving the same expected return, they will switch their investment from asset i to asset j. The higher demand of asset j will press its price upwards, decreasing its return, and the lower demand of asset i will depress its price and it will increase its return. Consequently, $\mu_{it} > \mu_{jt}$. This represents a contradiction with our set of assumptions, concluding that equation (7) and Model 1 are unsustainable. Hence, if assets with higher variance are perceived as riskier then there must be a risk premium associated with the time variation of beta. The investors are demanding higher compensation in order to hold asset i.

We design a modified APT model which considers the time variation of beta as systematic risk. Suppose that the stochastic disturbance a_{it} in the random coefficient model for b_{it} can be decomposed as follows $a_{it} = \sigma_{ai}\xi_t$, where ξ_t is i.i.d. with mean zero and variance one. The return to asset i can be written as

$$\begin{aligned} R_{it} &= \mu_{it} + b_i f_t + a_{it} f_t + \varepsilon_{it} = \mu_{it} + b_i f_t + \sigma_{ai}\xi_t f_t + \varepsilon_{it} \\ &= \mu_{it} + b_i f_t + \sigma_{ai} g_t + \varepsilon_{it} \end{aligned} \tag{8}$$

where $g_t \equiv \xi_t f_t$. This is a two-factor model where f_t and g_t are the factors. Following the same kind of arguments as in the previous section, we can write the conditional distribution of returns as

$$\begin{aligned} &\mu_{it} = b_i\lambda_t + \sigma_{ai}\eta_t + \lambda_0 \\ &var_{t-1}(R_{it}) = (b_i^2 + \sigma_{ai}^2)var_{t-1}(f_t) + var(\varepsilon_{it}) \\ &cov_{t-1}(R_{it}, f_t) = b_i var_{t-1}(f_t) \\ &cov_{t-1}(R_{it}, g_t) = \sigma_{ai} var_{t-1}(g_t) \end{aligned} \tag{9}$$

where η_t is the risk premium associated with the new factor g_t. This model is consistent with the second scenario, where $\mu_{it} > \mu_{jt}$. Investors expect higher return to asset i because carries an extra risk premium due to the factor g_t. In fact, a reading of the first and the last equation in (9) says that asset i is exposed to extra systematic risk and its sensitivity to it is summarized by σ_{ai}. Furthermore, σ_{ai} is equal to the covariance between the return to asset i and the factor g_t divided by the variance of the factor g_t.

Comparing both models, in particular equation (7) with equation (9), it is easy to see that the time-variation of beta will be priced if and only if $\sigma_{ai}\eta_t$ is statistically significant in explaining conditional expected returns. In order to test the validity of the modified APT model we proceed as follows. Consider the first equation in (9) to which we apply unconditional expectation. We have

$$E(\mu_{it}) = b_i E(\lambda_t) + \sigma_{ai} E(\eta_t) + \lambda_0 \tag{10}$$

note that $E(\mu_{it}) = E(R_{it})$ because the expectation of the conditional expectation of a random variable is its unconditional expectation. Equation (10) offers the grounds to perform a cross-section regression, where the dependent variable is the unconditional expected return, $E(R_{it})$ and the regressors are b_i and σ_{ai}. The hypothesis of interest is $H_0 : E(\eta_t) = 0$. A rejection of this null implies that the time variation of beta is systematic risk and that Model 2 should be accepted. The contrary implies that the time variation of beta is diversifiable risk and that Model 1 should be accepted. This test is performed in section 3.2.

3 Empirical Study

This empirical study is based on the results of Proposition 1. We pursue two objectives: first, to introduce a test for the constancy of betas and the feasibility of the random coefficient model, and secondly, to offer some insights into the pricing of the time variation of beta, according to the test described in section 2.2.

Schwert and Seguin (1990) empirically validated a single factor model of heteroscedasticity in stock returns, where the variance of the market portfolio was driving the variance and covariance of individual stocks. According to their results, the factor is the return to the market portfolio, R_{mt}. We proceed in our empirical study with $f_t = R_{mt}$. We need to extract the conditional variances and covariances of the individual returns and the market portfolio. The sample used in this study is the same as that in González-Rivera (1996). The sample consists of the stock returns of the United States computer companies recorded in the Center for Research in Security Prices (CRSP) tapes of the University of Chicago, under the Standard Industry Classification (SIC) codes of 3573, 3574, 7372 and 7379. The frequency of the data is weekly, running from July 7, 1962 to December 29, 1987. The NYSE value-weighted index is used as a proxy for the market portfolio.

The extraction of the conditional variances and covariances is performed through the design of a bivariate GARCH model, where each stock in the sample is paired with the market portfolio. The system is

$$R_t = d + \varepsilon_t$$

$$\varepsilon_t | \mathscr{I}_{t-1} \sim N(0, H_t)$$

$$H_t = C'C + A'\varepsilon_{t-1}\varepsilon'_{t-1}A + B'H_{t-1}B \tag{11}$$

where d is a 2×1 vector of constants, C, A, and B are 2×2 parameter matrices,

$$R_t = \begin{pmatrix} R_{it} \\ R_{mt} \end{pmatrix} \qquad \varepsilon_t = \begin{pmatrix} \varepsilon_{it} \\ \varepsilon_{mt} \end{pmatrix}$$

$\mathscr{I}_{t-1}$ is the information set containing information on the vector R_t up to time $t-1$, and the elements of H_t are $h_{it} \equiv var_{t-1}(R_{it})$, $h_{imt} \equiv cov_{t-1}(R_{it}, R_{mt})$, and $h_{mt} \equiv var_{t-1}(R_{mt})$. Since H_t is a variance-covariance matrix, we need to ensure that is positive definite. Engle and Kroner (1995) showed that the specification in (11) guarantees that H_t is positive definite. The model is estimated using quasi-maximum likelihood method (QMLE). The likelihood function is written under the assumption of conditional normality.

$$l_t = -\frac{N}{2} log\, 2\pi - \frac{1}{2}\, log\, |H_t| - \frac{1}{2}\varepsilon'_t H_t^{-1} \varepsilon_t$$

$$L(\theta) = \sum_t l_t(\theta)$$

where l_t is the log-likelihood of observation t. Under standard conditions, the QMLE estimator is consistent and asymptotically normally distributed with variance covariance matrix $\mathscr{A}^{-1}\mathscr{B}\mathscr{A}^{-1}$, where $\mathscr{A}$ is minus the expectation of the Hessian and $\mathscr{B}$ is the expectation of the outer product of the score (Bollerslev and Wooldridge 1992).

3.1 Constancy of Beta

In this section we propose a test for constancy of beta based on the conditional second moments of individual returns. The test takes advantage of the results *1-(ii)* and *1-(iii)* of Proposition 1, exploiting the restrictions imposed in the conditional variance and covariance of individual returns by time-varying betas. According to the APT model with time-varying betas where the beta follows a random coefficient model (equations (7)), the conditional variance and covariance of individual returns are driven by the conditional variance of the market portfolio in the following fashion

$$h_{it} = (b_i^2 + \sigma_{ai}^2)h_{mt} + \sigma_{\varepsilon i}^2$$
$$h_{imt} = b_i h_{mt}$$

This behavior suggests that a test for constancy of beta can be performed comparing the coefficients of h_{mt}. We write the above equations in a regression framework as

$$h_{it} = \gamma_{01i} + \gamma_{1i}h_{mt} + v_t \qquad (12)$$
$$h_{imt} = \gamma_{02i} + \gamma_{2i}h_{mt} + v_t$$

The test for constancy of beta consists of a non-linear cross-equation restriction of the following form

$$H_0 : \gamma_{1i} = \gamma_{2i}^2 \qquad H_1 : \gamma_{1i} > \gamma_{2i}^2 \qquad (13)$$

The null hypothesis contains model (12) under constant betas. A similar way to write the null in (13) could be $\sigma_{ai}^2 = 0$. The alternative hypothesis contains the model (12) under the hypothesis of a random coefficient model for beta.

Model (12) constitutes a system of two equations where the conditional variance of the market portfolio is a common regressor. Both equations of model (12) are estimated jointly because the restriction contained in the null hypothesis (13) imposes a common parameter across equations.

The test for constancy of beta is based in a minimum distance criterion (Gallant 1987). It compares two objective functions: Q_0 (under the null) and

Q_1 (under the alternative hypothesis). The test is asymptotic and is distributed as a χ^2 with as many degrees of freedom as restrictions are tested. The test is

$$T = n(Q_0 - Q_1) \tag{14}$$

The objective function (Q) to minimize is

$$Q = r'(S_{ols}^{-1} \otimes I)r/n$$

where r is a vector of stacked residuals for the two equations, S is an estimator of the variance-covariance matrix of the errors across equations, and n is number of observations.

Table I displays the results of the estimation of system (12) and the testing of the hypothesis (13) for 81 stocks. Columns (1) and (3) contain estimates of γ_1 and γ_2 and columns (2) and (4) contains their respective t-test for the null hypothesis $\gamma_1 = 0$ and $\gamma_2 = 0$. Column (5) contains the T test (14) for the hypothesis (13). The T test is χ^2-distributed with one degree of freedom. Given that the regressor h_{mt} has to be estimated before we proceed with the estimation of the system (12), a generated regressor problem may occur (Pagan 1984) and the OLS standard errors of the estimates of the parameters of the model may be biased downwards. Nevertheless, the actual setting corresponds to the conditions stated in theorem 3 of Pagan (1984) and for the cases in which we are interested, the asymptotic t-statistics are valid. In general, very little is known about the generated regressor problem caused by GARCH estimates of conditional variances.

The empirical results for our sample of individual stocks are in agreement with the findings of Schwert and Seguin (1990) for size-ranked portfolios. The variance of the market portfolio is a very strong factor driving the variances of individual returns. Only two stocks have statistically non-significant $\hat{\gamma}_1$ and one stock has a non-significant $\hat{\gamma}_2$ at the conventional 5% significance level. The test T rejects the null hypothesis $\gamma_{1i} = \gamma_{2i}^2$ for 64 stocks at the statistical level of 5%, hence we reject the constancy of betas in favor of the random coefficient model.

An indirect check on the validity of the random coefficient model can be designed through the comparison of two estimators of $b_i = E(b_{it})$. On one hand, we have the estimator $\hat{\gamma}_{2i}$ obtained from the system of equations (12). On the other, we have obtained the quantities h_{imt} and h_{mt} from the bivariate GARCH system (11). We construct an estimator of beta as $\tilde{b}_{it} = h_{imt}/h_{mt}$. We average $\tilde{b}_{it}$ over time to give us the mean of b_{it}. We run a cross-sectional regression of the type $\hat{\gamma}_{2i} = a + bE(\tilde{b}_{it})$. If $\hat{\gamma}_{2i}$ and $E(\tilde{b}_{it})$ are both estimators of b_i, it must be the case that $a = 0$ and $b = 1$. The estimates are $\hat{a} = -0.65$ with a $t = -1.65$ and $\hat{b} = 1.38$ with a $t = 1.51$ for the respective null hypothesis. These tests fail to reject the null hypotheses $a = 0$ and $b = 1$. It is likely that the t-statistics are biased upwards, but even so, it makes our null hypotheses more likely to be true.

Table I. $h_{it} = \gamma_{01i} + \gamma_{1i} h_{mt} + v_t \quad h_{imt} = \gamma_{02i} + \gamma_{2i} h_{mt} + \nu_t;$ T test for $H_0 : \gamma_{1i} = \gamma_{2i}^2$

Stock	(1) $\hat{\gamma}_{1i}$	(2) $t(\hat{\gamma}_{1i})$	(3) $\hat{\gamma}_{2i}$	(4) $t(\hat{\gamma}_{2i})$	(5) T test
1	5.12	8.6	1.63	19.5	24.4
2	5.76	14.7	1.09	12.8	164
3	7.92	38.2	2.63	71.9	387
4	9.29	11.8	1.93	18.0	69.3
5	8.98	19.0	1.10	15.0	306
6	14.24	18.7	2.43	21.8	147.3
7	4.49	5.0	2.26	19.8	0.6*
8	10.43	7.5	1.30	5.4	55.6
9	5.12	11.1	1.39	16.3	68.4
10	5.47	48.5	1.50	32.4	432
11	11.56	23.0	2.87	46.4	126
12	9.60	23.1	2.28	51.4	197
13	0.17	1.5*	−0.17	−5.5	1.5*
14	1.21	6.3	0.62	14.7	26.0
15	6.61	43.5	2.37	75.6	42
16	5.17	15.5	2.04	57.4	18.8
17	3.90	6.9	1.84	18.0	1.4*
18	4.49	7.7	1.42	23.7	25.3
19	26.15	5.8	0.95	4.7	26
20	10.70	9.0	3.12	17.2	1.2*
21	1.81	19.1	0.96	32.2	58.7
22	5.51	34.0	2.29	58.6	2.9*
23	4.83	19.1	0.76	9.8	237.6
24	3.05	13.0	0.99	13.3	50.1
25	2.56	57.1	1.51	173.0	37
26	8.71	37.9	2.68	59.7	268
27	8.28	25.6	2.06	42.1	313.3
28	15.13	3.7	1.02	7.7	13.0
29	0.93	0.8*	0.62	3.4	0.2*
30	4.01	26.0	1.76	90.5	65
31	13.35	16.4	3.20	33.1	38.4
32	7.50	28.7	1.57	36.3	475
33	7.35	25.1	2.31	54.3	170
34	2.90	30.0	1.59	89.0	18
35	34.12	6.7	5.31	13.6	1.4*
36	1.65	12.7	0.74	10.6	77.1
37	9.51	31.0	1.97	47.3	406
38	9.28	7.5	0.38	2.4	55.4
39	12.58	5.1	0.56	2.4	26
40	2.03	17.0	0.89	18.9	98.6
41	9.41	14.3	1.81	17.1	90.1
42	8.94	12.6	2.05	8.9	22
43	4.65	22.9	1.71	50.6	81
44	11.15	13.2	1.37	16.1	137
45	5.68	15	1.89	19.5	57.2
46	3.18	8.1	1.56	21.0	1.5*
47	0.96	90.3	0.91	128.2	105
48	5.25	12.9	1.10	15.5	121.1

Table I (continued)

Stock	(1) $\hat{\gamma}_{1i}$	(2) $t(\hat{\gamma}_{1i})$	(3) $\hat{\gamma}_{2i}$	(4) $t(\hat{\gamma}_{2i})$	(5) T test
49	1.16	20.9	1.02	67.5	8.6
50	8.16	19.9	1.66	26.3	162.4
51	1.40	73.5	1.10	185.0	343
52	6.41	31	1.56	34.7	513
53	3.28	4.9	0.94	9.8	14.3
54	2.47	17.8	0.24	5.1	291
55	10.98	25.7	0.55	5.6	494
56	4.86	18.4	1.64	52.2	106
57	4.20	87.5	2.06	220.7	2.8*
58	17.17	29.1	2.15	49.3	514
59	14.55	8.5	3.09	17.8	22.7
60	1.61	11.8	1.25	65.6	0.1*
61	2.58	21.9	1.04	43.9	179
62	17.08	11.2	−0.05	−0.3*	106
63	5.83	19.3	2.07	35.6	34.1
64	1.15	2.9	0.78	18.8	2*
65	4.04	3.2	2.06	18	0.04*
66	302.16	2.2	−5.20	−3.84	4.7
67	1.60	20	1.27	85.7	0*
68	0.86	2.9	0.60	11.7	2.6*
69	5.08	6.7	0.88	19.6	33
70	31.84	5.8	−4.42	−10.5	6.3
71	5.33	9.1	−0.11	−2.1	82
72	4.15	8.0	1.58	28.3	18.4
73	−0.16	−0.1*	0.49	5.6	0.09*
74	2.12	4.3	1.35	42.7	0*
75	5.13	33	1.93	42.8	47.8
76	2.32	69.4	1.46	135.3	101
77	10.34	17.5	2.61	19	8.6
78	3.38	34.3	1.58	92.1	196
79	1.14	14.6	1.16	44.9	18
80	2.95	5.1	1.43	14.5	4.17
81	−2.86	−1*	−0.42	−4.9	1.2*

* Statistically non-significant at the 5% level. $\chi_1^2(5\%) = 3.84$

3.2 Pricing of Time-Varying Betas

In this section we apply the test proposed in section 2.2. Since beta is time-varying for the sample under study, we need to assess if the time variation of beta represents systematic risk or, to the contrary, represents non-systematic risk. Recall equation (10),

$$E(R_{it}) = b_i E(\lambda_t) + \sigma_{ai} E(\eta_t) + \lambda_0$$

Equation (10) is a cross-sectional model in which the regressors are b_i and σ_{ai}. The hypothesis of interest is $H_0 : E(\eta_t) = 0$. If we reject the null hypothesis, the time variation of beta represents systematic risk. A failure to reject the null implies that there is not sufficient statistical evidence to validate the modified APT model and the extra risk induced by the time variation of beta is diversifiable.

We have two estimators of b_i, these are $\hat{\gamma}_{2i}$ and $E(\tilde{b}_{it})$, and an estimator of σ_{ai} can be constructed as $\hat{\gamma}_{1i} - \hat{\gamma}_{2i}^2 = \hat{\sigma}_{ai}^2$. In Table II we summarize the results of the cross-sectional regression.

Models (1) and (2) approximate the b_i regressor by the estimator $\hat{\gamma}_{2i}$. The first moment of beta, b_i is able to explain about 6% of the variability in expected returns. The second moment of beta, σ_{ai} is not statistically significant at the 5% level. One of the implications of Proposition 1 is that regressions of the type presented above have to have heteroscedastic errors since the variance of the returns depend on the values of b_i and σ_{ai}. This is the reason why we present two types of t-ratios. The first one is the uncorrected OLS t-

Table II. Cross-sectional regression $E(R_{it}) = \alpha_0 + \alpha_1 b_i + \alpha_2 \sigma_{ai}$

	$b_i \approx \hat{\gamma}_{2i}$		$b_i \approx E(\tilde{b}_{it})$	
	(1)	(2)	(3)	(4)
α_0	.10E−02	.17E−02	−.3E−02	−.2E−02
	(1.0)	(1.4)	(−1.7)	(−1.2)
	(1.2)	(1.7)	(−1.4)	(−1.2)
α_1	.13E−02	.13E−02	.40E−02	.40E−02
	(2.4)	(2.4)	(3.4)	(3.4)
	(2.7)	(2.7)	(2.7)	(2.8)
α_2	–	−.40E−03	–	−.30E−03
	–	(−.9)	–	(−0.9)
	–	(−1.0)	–	(−1.0)
R^2	.07	.08	.13	.14
$Adj.\,R^2$	.06	.06	.12	.12
F	5.8	3.4	11.6	6.1
(p-value)	(0.01)	(0.03)	(0.001)	(0.003)

t-statistics underneath the coefficient estimates, the first number is the OLS t-ratio, the second number is the White's consistent t-ratio.

ratio and the second one is based on a consistent estimator of the variance-covariance matrix that is robust against heteroscedasticity (White 1980). Residual plots, not presented here, indicate the presence of heteroscedasticity. Using robust t-ratios does not change the previous conclusions. Models (3) and (4) approximate the b_i by the estimator $E(\tilde{b}_{it})$. While the statistical significance of the first moment of beta and the statistical non-significance of the second moment have been maintained, the R^2 has doubled in size. 12% of the variability of mean returns is explained by b_i. Consequently, for the sample of individual American computer stocks, the previous results seem to indicate that the modified APT model should be rejected, implying that the time variation observed in beta is due to idiosyncratic behavior of the firms and investors should be able to diversify this risk away.

4 Conclusions

The traditional asset pricing models as CAPM and APT have been criticized in three fronts: 1. they assume that beta, as a measure of risk, is constant over time; 2. there is a linear relation between the individual asset return and the market portfolio (factors) in CAPM (APT); and 3. the market portfolio (factors) may not be the only sources of risk and, consequently, there may be omitted variables in the specification of the asset pricing model.

Allowing for betas that are time-varying is enough for removing the assumption on linearity and, at the same time, accounts for the possible effect of omitted variables. In this paper, we have generalized an APT model including time-varying betas. We have treated beta as a random variable and we have inquired on the consequences of this random behavior on the formation of conditional expected returns, conditional covariance and variance of returns. We have shown that the relevant statistical moments of beta affecting the statistical moments of returns are its conditional mean and its conditional non-central second moment. This result is summarized in Proposition 1.

Under the assumption that beta is time-varying, we have designed a modified conditional APT model that rationalizes why the time variation of beta may represent extra systematic risk. We have extracted the potential systematic component from the time variation of beta and we have constructed the factor associated with this extra risk. If there is not systematic component in the time variation of beta, the proposed factor should not have any power in explaining conditional expected returns. We have offered an empirical test that assesses the validity of the modified APT model.

We have conducted an empirical study with two objectives in mind. The first objective was to test the maintained hypothesis that beta is time-varying.

We have presented an alternative test to the ones encountered in the current literature. The test is based on the results of Proposition 1 and takes advantage of the restrictions imposed in the conditional second moments of returns by the time behavior of beta. For the sample under study, we have found that there is strong evidence against constancy of beta in favor of a random coefficient model. The second objective of our empirical study was to implement a test that will shed some light on the type of risk conveyed by the time variation of beta. We found that time variation of beta does not represent any additional systematic risk. This finding is in agreement with Chen and Keown (1981) and in disagreement with Schwert and Seguin (1990).

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