

A Robust Optimization Approach to Dynamic Pricing and Inventory Control with no Backorders

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Abstract

In this paper, we present a robust optimization formulation for dealing with demand uncertainty in a dynamic pricing and inventory control problem for a make-to-stock manufacturing system. We consider a multi-product capacitated, dynamic setting. We introduce a demand-based fluid model where the demand is a linear function of the price, the inventory cost is linear, the production cost is an increasing strictly convex function of the production rate and all coefficients are time-dependent. A key part of the model is that no backorders are allowed. We show that the robust formulation is of the same order of complexity as the nominal problem and demonstrate how to adapt the nominal (deterministic) solution algorithm to the robust problem.

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1 Introduction

1.1 Motivation

Fluid models provide a powerful tool for understanding the behavior of systems where the dynamic aspect plays an important role. In recent years, there has been a lot of research in an attempt to provide a deeper understanding of fluid models from a theoretical as well as an application point of view. In particular, an attractive feature of these models is that they provide good scheduling, production and inventory policies in a variety of settings. Furthermore, they approximate well the underlying stochasticity of problems in a deterministic way. Fluid models arise in applications as diverse as routing, communication, queueing, supply chain and transportation systems.

The overall goal of this research is to introduce and study a robust optimization model and its application to dynamic pricing and inventory control. In [1] we study the problem when demand is deterministic and introduce a solution algorithm for computing the optimal pricing and production policy over the time horizon for all products. In this paper we introduce demand uncertainty and use ideas from robust optimization in order to obtain the robust counterpart of the nominal problem. Furthermore, we provide a solution algorithm for solving the robust formulation and illustrate it is of the same difficulty as the nominal problem. To the best of our knowledge, this is the first paper introducing ideas from robust optimization in a fluid model for dynamic pricing.

1.2 Some Related Literature and Contributions

In this section, we will briefly review some literature on fluid models, demand uncertainty and the use of robust optimization to address it. The reader should refer to [1] for additional references on related literature on dynamic pricing, inventory management and the maximum principle for optimality.

A large part of the literature has focused on the solution of linear fluid models (see for example Anderson [2],[3], [4], Pullan [23], [24], Tyndall [28]). This part of the literature shows existence of an optimal solution with piecewise constant controls. Pullan in particular showed strong duality and designed a class of algorithms. However, when fluid models are nonlinear, the dynamic together with the nonlinear aspect of the problem make them harder to analyze. Nonlinear fluid models are particularly useful for dynamic pricing and inventory management in supply chains. Examples of supply chain industries where fluid models of the type we discuss in this paper are relevant, include industries with a high volume of throughput and data on costs and demand that change a lot. The hardware as well as the semiconductor industries are such examples.

The key contribution of this paper is addressing demand uncertainty through robust optimization. The problem of demand uncertainty has motivated a significant amount of literature in the field of Revenue Management and Pricing. A number of different approaches have been introduced to model this uncertainty. Zabel [31] considers two models of uncertain demand: a multiplicative model ($d_t = \eta_t u(p_t)$) and an additive model ($d_t = u(p_t) + \eta_t$), where d_t is the demand at time t , p_t is the price at time t , $u(p) = a - \frac{a}{b}p$, i.e. a downward sloping linear demand curve, and η_t is assumed to be either exponentially or uniformly distributed with $E[\eta_t] > 0$. Young [30] and Federgruen and Heching [18] generalize the demand model to be of the form $d_t = \gamma_t(p)\epsilon_t + \delta_t(p)$, where γ and δ have first derivatives non positive and ϵ_t is a random term with a finite mean. Gallego and van Ryzin [19], [20] as well as Bitran and Mondschein [15] assume that demand follows a Poisson process with a deterministic intensity that depends on price and time. Raman and Chatterjee [25] model the stochasticity of the demand by introducing an additive model where the random noise is a continuous time Wiener process.

For additional details and references, see review papers such as [14], [17] and [29].

Robust optimization seeks an optimal solution of a problem when its data is uncertain. A robust optimization formulation was first considered by Soyster [27] in the case of a linear optimization problem where the data were uncertain within a convex set. He addresses uncertainty by taking a worst-case approach. Nevertheless, such an approach decreases the performance of the solution significantly. Ben-Tal

and Nemirovski [9] studied robust convex optimization. They show that some polynomial-time algorithms allow to efficiently solve exactly or approximatively some of these problems. In [10] they show that the robust counterpart of a linear program with data within ellipsoidal uncertainty sets is a conic quadratic program which is solvable in polynomial time. Numerical examples of this approach can be found in [11]. El-Ghaoui et al [8], [16] consider semidefinite robust optimization problems. Bertsimas and Sim [12] studied the tradeoff between robustness of a solution to a linear programming problem and the sub-optimality of the solution. Bertsimas and Thiele [13] apply robust optimization principles to supply chain management.

The contributions of this paper are the following:

1. We introduce robust optimization ideas to a fluid model. To the best of our knowledge, this is the first paper introducing ideas of robust optimization in a fluid model setting.
2. We study a continuous time model for a joint pricing and inventory control problem that allows no backorders and incorporates demand uncertainty. We use ideas from robust optimization in order to introduce a model of demand uncertainty without assuming any probability distribution on the random component of the demand.
3. We reformulate the robust optimization model as a deterministic model of the same order of complexity as the nominal problem.
4. We demonstrate how to adapt the solution algorithm, introduced in [1] for the nominal problem, to the robust problem and show it is of the same order of difficulty.

1.3 Structure of the Paper

In Section 2, we review the nominal problem and describe a model of demand uncertainty. In Section 3, we present the robust formulation and the demand model. In Section 4, we derive an equivalent formulation for the robust counterpart problem. In Section 5, we briefly explain how to adapt the algorithm described in [1] in order to solve the robust counterpart. We give some numerical results in Section 6, and we conclude in Section 7.

2 Notations and Definitions

Inputs

- $[0, T]$: time horizon;
- N : number of products;
- $K(t)$: shared production capacity rate at time t (non negative);
- I_i^0 : initial non negative inventory level for product i ;
- $h_i(t)$: holding cost of one unit of product i at time t ;
- $f_i(\cdot)$: production cost function for product i with respect to the production rate;
- $\alpha_i(t), \beta_i(t)$: coefficients used for product i at time t in the linear relationship between price and demand: $d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t)$.

Outputs

- $p_i(t)$: price of one unit of product i at time t (control variable);
- $u_i(t)$: production flow rate of product i at time t (control variable);
- $I_i(t)$: inventory level (number of units) of product i at time t (state variable).

Definitions

We denote by $I(\cdot), p(\cdot), u(\cdot), \alpha(\cdot), \beta(\cdot)$ the vectors with respective components

$$I_i(\cdot), p_i(\cdot), u_i(\cdot), \alpha_i(\cdot), \beta_i(\cdot), \quad i = 1, \dots, N.$$

$I^*(\cdot), p^*(\cdot), u^*(\cdot)$ will denote the optimal solution.

We define:

- Constrained (resp. unconstrained) interval: interval of time where the inventory level equals zero (resp. is positive);
- Constrained (resp. unconstrained) product: product belonging to a constrained (resp. unconstrained) interval;
- Active (resp. inactive) product: product with a positive (resp. equal to zero) production rate.

We notice that for any pricing and production policy (and in particular for an optimal policy), the inventory level will lie in a sequence of intervals, where the inventory level is successively positive and equal to zero. A constrained interval starts at an *entry time* and finishes at an *exit time*, i.e. the time the inventory level becomes again positive.

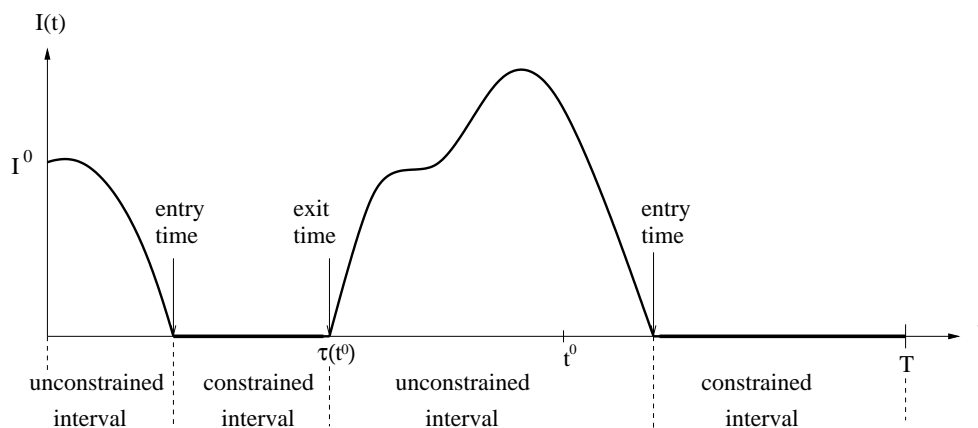


Figure 1: Example of optimal inventory level evolution

3 The Nominal Problem and a Demand Uncertainty Model

We consider the nominal problem P

$$\begin{aligned}
\max \quad & \sum_{i=1}^N \int_0^T \left(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \right) dt \\
\text{s.t.} \quad & \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
& I_i(t) \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
& p_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} \\
& u_i(t), p_i(t) \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
& \sum_{i=1}^N u_i(t) \leq K(t), \quad \forall t \in [0, T] \\
& I_i(0) = I_i^0, \quad i = 1, \dots, N.
\end{aligned}$$

We introduce an additive model of demand uncertainty as follows:

$$d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t) + \epsilon_i(t),$$

where $\epsilon_i(t)$ is the uncertain parameter.

This model essentially assumes an additive demand model, i.e. that there is uncertainty on the demand parameters $\alpha_i(\cdot)$, $i = 1, \dots, N$.

In particular, we model this uncertainty as follows. We denote $\alpha_i(\cdot)$ the nominal function, $\tilde{\alpha}_i(\cdot)$ the realization, and we suppose that the realization belongs in an interval centered around the nominal function with half-length $\hat{\alpha}_i(\cdot)$. In order to model the fact that the realization is unlikely to be at its worst-case value (i.e. the furthest away from the nominal value) all the time, we introduce a ‘‘budget of uncertainty function’’ $\Gamma_i(\cdot)$ taking values in $[0, T]$. We assume that function $\Gamma_i(\cdot)$ is an increasing function of time. This function allows us to adjust the tradeoff between the level of conservatism sought for the robust solution and its performance. We assume it is non-decreasing in order to allow the aggregate error over time to increase. Moreover, we assume that $\dot{\Gamma}_i(t) \leq 1 \quad \forall i, t$, in order to ensure that the function does not grow faster than new variables are added.

We suppose that

$$0 < \hat{\alpha}_i(\cdot) < \frac{1}{2}\alpha_i(\cdot) \quad \forall i$$

and write the constraints as follows:

$$\begin{aligned} |\tilde{\alpha}_i(t) - \alpha_i(t)| &\leq \hat{\alpha}_i(t) \quad \forall t, i \\ \int_0^t \frac{|\tilde{\alpha}_i(s) - \alpha_i(s)|}{\hat{\alpha}_i(s)} ds &\leq \Gamma_i(t) \quad \forall t, i. \end{aligned}$$

Denoting

$$z_i(t) \equiv \frac{\tilde{\alpha}_i(t) - \alpha_i(t)}{\hat{\alpha}_i(t)}$$

the scaled variation of parameter $\tilde{\alpha}_i(t)$, it follows that

$$\begin{aligned} z_i(t) &\in [-1, 1] \quad \forall t, i \\ \int_0^t |z_i(s)| ds &\leq \Gamma_i(t) \quad \forall t, i. \end{aligned} \tag{1}$$

(2)

Clearly, we have

$$\int_0^t |z_i(s)| ds \leq t.$$

This implies that if for a time t , $\Gamma_i(t) \geq t$, then constraint (1) is unnecessary as it is always satisfied in that case due to the bounds imposed on $z_i(\cdot)$ by definition. As a result, in this case at the optimal solution of the robust optimization problem, $|z_i(\cdot)|$ will be equal to 1 on $[0, t]$ as this corresponds to the worst case scenario on that interval and it is allowed through the budget of uncertainty constraint. In particular, the exact value of $\Gamma_i(t)$ (if it is greater than or equal to t) does not matter. This discussion motivates us to introduce the notion of the *effective* budget of uncertainty through $\min\{t, \Gamma_i(t)\}$. Furthermore, in order to measure the global uncertainty of the problem, that is, in order to introduce a single quantity used as a metric representative of the overall budget of uncertainty, we introduce the cumulative effective budget of uncertainty defined by $\int_0^T \min\{t, \Gamma_i(t)\} dt$.

An uncertainty set \mathcal{F} is the set of vectors $\tilde{\alpha} = (\tilde{\alpha}_1(\cdot), \dots, \tilde{\alpha}_N(\cdot))$ satisfying the constraints above. Therefore, the budget of uncertainty prevents a given product i to have a parameter α_i very different from its nominal value for an excessive part of the time horizon. However, we do not impose a budget constraint across products, i.e. in terms of number of products that may have simultaneously a demand parameter different from its nominal value.

4 A Robust Optimization Approach

4.1 A General Robust Optimization Model

In general, a robust optimization formulation seeks a solution that is feasible for any realization of the data within the uncertainty set and maximizes the realized objective function. In the case of problem P , the robust counterpart we described can be written as

$$\begin{aligned}
 & \max_{p,u,\gamma} \quad \gamma \\
 \text{s.t. } & \forall \tilde{\alpha} \in \mathcal{F}, \quad \gamma \leq \int_0^T \sum_{i=1}^N \left(p_i(\tilde{\alpha}_i - \beta_i p_i) - f_i(u_i) - h_i \tilde{I}_i \right) dt \\
 & \dot{\tilde{I}}_i = u_i - \tilde{\alpha}_i + \beta_i p_i, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
 & \tilde{I}_i \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
 & p_i \leq \frac{\tilde{\alpha}_i}{\beta_i}, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
 & u_i, p_i \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N \\
 & \sum_{i=1}^N u_i \leq K, \quad \forall t \in [0, T] \\
 & \tilde{I}_i(0) = I_i^0, \quad i = 1, \dots, N.
 \end{aligned} \tag{3}$$

We observe that we may write the inventory level at time t as follows:

$$\begin{aligned}
 \tilde{I}_i(t) &= I_i^0 + \int_0^t (u_i(s) - \alpha_i(s) - z_i(s)\hat{\alpha}_i(s) + \beta_i(s)p_i(s)) ds \\
 &= I_i(t) - \int_0^t z_i(s)\hat{\alpha}_i(s) ds
 \end{aligned}$$

where

$$I_i(t) = I_i^0 + \int_0^t (u_i(s) - \alpha_i(s) + \beta_i(s)p_i(s)) ds$$

is the nominal inventory level.

The constraints must be satisfied for all $z(\cdot)$ such that for all i and t ,

$$\begin{aligned}
 -1 &\leq z_i(t) \leq 1 \\
 \int_0^t |z_i(s)| ds &\leq \Gamma_i(t)
 \end{aligned}$$

where $\tilde{\alpha}_i(t) = \alpha_i(t) + z_i(t)\hat{\alpha}_i(t)$.

In order to reformulate this problem, we need, for each constraint where uncertainty is involved, to determine the realization of $\tilde{\alpha}$, that is, the “worst-case scenario” (for example for constraint (3), the realization that minimizes the right hand side). Then we will be guaranteed that the constraint is satisfied for any realization.

We may rewrite constraint (3) as follows:

$$\begin{aligned}
 \gamma &\leq \int_0^T \sum_{i=1}^N \left(p_i(\alpha_i - \beta_i p_i) - f_i(u_i) - h_i I_i + p_i \hat{\alpha}_i z_i + h_i \int_0^t z_i \hat{\alpha}_i ds \right) dt \\
 \Leftrightarrow \gamma &\leq C + \int_0^T \sum_{i=1}^N \left(p_i(t) \hat{\alpha}_i(t) z_i(t) + h_i(t) \int_0^t z_i(s) \hat{\alpha}_i(s) ds \right) dt
 \end{aligned}$$

for all $z(\cdot)$ such that for all i and t ,

$$\begin{aligned} -1 &\leq z_i(t) \leq 1 \\ \int_0^t |z_i(s)| ds &\leq \Gamma_i(t), \end{aligned}$$

where

$$C = \int_0^T \sum_{i=1}^N \left(p_i(\alpha_i - \beta_i p_i) - f_i(u_i) - h_i I_i \right) dt$$

corresponds to the nominal objective function and is independent of z .

We seek the feasible realization of z that minimizes the right-hand side in the above inequality. We notice that we may consider each product separately; in other words we minimize each term of the sum across products. We thus need to find, for all i , the feasible realization of z_i that solves

$$\min_{z_i} \int_0^T \left(p_i(t) \hat{\alpha}_i(t) z_i(t) + h_i(t) \int_0^t z_i(s) \hat{\alpha}_i(s) ds \right) dt.$$

Since p_i , $\hat{\alpha}_i$ and h_i are positive valued, it is easy to see that in the optimal solution, $z_i \leq 0$. Therefore, after transformation of variables, we may rewrite the constraints on the new variable z_i as follows

$$\begin{aligned} 0 &\leq z_i(t) \leq 1 \quad \forall t \\ \int_0^t z_i(s) ds &\leq \Gamma_i(t) \quad \forall t \end{aligned}$$

(note that we abuse notations to avoid confusion) and rewrite the minimization problem as

$$\begin{aligned} &\min_{z_i} \int_0^T \left(-p_i(t) \hat{\alpha}_i(t) z_i(t) + h_i(t) \int_0^t (-z_i(s) \hat{\alpha}_i(s)) ds \right) dt \\ \Leftrightarrow &-\max_{z_i} \int_0^T p_i(t) \hat{\alpha}_i(t) z_i(t) dt + \int_0^T \left(h_i(t) \int_0^t z_i(s) \hat{\alpha}_i(s) ds \right) dt \\ \Leftrightarrow &-\max_{z_i} \int_0^T p_i(t) \hat{\alpha}_i(t) z_i(t) dt + \int_{t=0}^T \left(\int_{s=0}^t h_i(t) z_i(s) \hat{\alpha}_i(s) ds \right) dt \\ \Leftrightarrow &-\max_{z_i} \int_0^T p_i(t) \hat{\alpha}_i(t) z_i(t) dt + \int_{s=0}^T \left(\int_{t=s}^T h_i(t) z_i(s) \hat{\alpha}_i(s) dt \right) ds \\ \Leftrightarrow &-\max_{z_i} \int_0^T p_i(t) \hat{\alpha}_i(t) z_i(t) dt + \int_{s=0}^T H_i(s) z_i(s) \hat{\alpha}_i(s) ds \\ \Leftrightarrow &-\max_{z_i} \int_0^T (p_i(t) + H_i(t)) z_i(t) \hat{\alpha}_i(t) dt, \end{aligned}$$

where $H_i(t) \equiv \int_t^T h_i(s) ds$.

Therefore, we obtain the equivalent subproblem

$$\begin{aligned} &-\max_{z_i} \int_0^T (p_i(t) + H_i(t)) z_i(t) \hat{\alpha}_i(t) dt \\ \text{s.t.} & \quad 0 \leq z_i(t) \leq 1 \quad \forall t \\ & \quad \int_0^t z_i(s) ds \leq \Gamma_i(t) \quad \forall t. \end{aligned}$$

We notice at this point that this subproblem depends on the control variable $p_i(\cdot)$, and therefore, cannot be directly solved. This dependency comes from the fact that the revenue term is non linear in both the

control variable p_i and the uncertain parameter α_i . We could write the dual of this separated continuous linear program and include it as part of the overall robust counterpart formulation, but its solution cannot be derived independently from the rest of the problem. Consequently, the complexity of the resulting robust formulation increases and the new formulation is difficult to solve. The new robust optimization problem has now a higher order of complexity than the nominal problem.

We notice that if pricing was not a decision, or if demand was external (i.e., not depending on pricing), the general model of uncertainty (i.e. with realized revenues) would yield a tractable formulation (see for example [13] for a discretized version of the problem). This discussion motivates the model we introduce and study in the remainder of this paper.

4.2 A Simplified Robust Optimization Model

The remarks in the previous section motivate a slightly different formulation when defining the robust optimization model we consider. For reasons of tractability, in the remainder of the paper, we consider a robust approach that maximizes an objective function consisting of *nominal* revenues minus realized costs. In other words, we will take the mean value in the revenues term (i.e., price times demand) of the objective function. However, we still consider demand uncertainty in the cost terms of the objective function (i.e., we consider realized costs) and in the feasibility constraints. That is, we consider “expected”, i.e. nominal revenues (not in the probabilistic sense, but at the center of the interval of variation). This modelling simplification allows us to simplify the approach and make the model tractable.

As a side remark, later in the paper, we will show that a model considering the nominal cost as opposed to the realized cost in the objective function, would lead to the same formulation.

For this model, the robust counterpart of problem P can be written as follows (we omit the time argument):

$$\begin{aligned} \max_{p, u, \gamma} \quad & \gamma \\ \text{s.t. } \forall \tilde{\alpha} \in \mathcal{F}, \quad & \gamma \leq \int_0^T \sum_{i=1}^N \left(p_i(\alpha_i - \beta_i p_i) - f_i(u_i) - h_i \tilde{I}_i \right) dt \end{aligned} \quad (4)$$

$$\dot{\tilde{I}}_i = u_i - \tilde{\alpha}_i + \beta_i p_i, \quad \forall t \in [0, T] \quad i = 1, \dots, N \quad (5)$$

$$\tilde{I}_i \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N \quad (6)$$

$$p_i \leq \frac{\tilde{\alpha}_i}{\beta_i}, \quad \forall t \in [0, T] \quad i = 1, \dots, N \quad (7)$$

$$u_i, p_i \geq 0, \quad \forall t \in [0, T] \quad i = 1, \dots, N$$

$$\sum_{i=1}^N u_i \leq K, \quad \forall t \in [0, T]$$

$$\tilde{I}_i(0) = I_i^0, \quad i = 1, \dots, N. \quad (8)$$

We observe that constraints (4), (6) and (7) are the ones where uncertainty has an impact, (equation (5), along with the initial conditions (8), simply define the state variable \tilde{I} .)

Constraints (4) and (6) link time instants by involving inventory levels. Notice that these depend on all previous control decisions. In the contrast, constraint (7) is separable across time. This will have an impact in the way we derive the robust counterpart in the following sense: a constraint that does not link together time instants needs to be satisfied at each time for the worst realization, while for a constraint that links time, the worst realization may not occur at each time because of the budget constraint involving Γ_i .

We will proceed, similarly to the previous section, by determining for each constraint the worst case realization of the uncertain parameter $\tilde{\alpha}$ in order to reformulate the robust problem.

4.3 An Equivalent Robust Formulation

We start by considering constraint (7) for a given product i and at a given time t . Clearly, the worst case is obtained when the numerator is the smallest, i.e. $z_i(t) = -1$. It may be seen that for any given time t and index i , it is possible to find a vector of functions z such that $z_i(t) = -1$, and $\tilde{\alpha} \in \mathcal{F}$. As a result, in the robust counterpart, constraint (7) is written as

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i, t.$$

In constraint (6), we seek for the realization of z_i that minimizes $\tilde{I}_i(t)$, or equivalently, that minimizes

$$- \int_0^t z_i(s) \hat{\alpha}_i(s) ds.$$

Clearly in the optimal solution, $z_i \geq 0$, so we can rewrite this subproblem as follows, for each product i and at any given time t :

$$\begin{aligned} \min_{z_i} \quad & - \int_0^t z_i(s) \hat{\alpha}_i(s) ds \\ \text{s.t.} \quad & \int_0^t z_i(s) ds \leq \Gamma_i(t) \\ & 0 \leq z_i(s) \leq 1 \quad \forall s \in [0, t], \end{aligned}$$

where the decision variable is the function z_i over $[0, t]$. Equivalently,

$$\begin{aligned} - \max_{z_i} \quad & \int_0^t z_i(s) \hat{\alpha}_i(s) ds \\ \text{s.t.} \quad & \int_0^t z_i(s) ds \leq \Gamma_i(t) \\ & 0 \leq z_i(s) \leq 1 \quad \forall s \in [0, t]. \end{aligned}$$

Note that t is fixed. For a given t , we want to find the realization of z_i on $[0, t]$ which makes $\tilde{I}_i(t)$ the smallest.

This is a particular instance of a continuous linear program. This class of problems was introduced by Bellman [6], [7] to model some economic processes. A dual formulation for this class of problems was studied by Tyndall [28]. Some results by Tyndall also establish strong duality under some regularity assumptions on the data of the problem. Using these results, we obtain strong duality with the dual problem given by:

$$\begin{aligned} - \min_{\omega_i(t), r_i(\cdot, t)} \quad & \omega_i(t) \Gamma_i(t) + \int_0^t r_i(s, t) ds \\ \text{s.t.} \quad & \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t] \\ & \omega_i(t) \geq 0 \\ & r_i(s, t) \geq 0 \quad \forall s \in [0, t] \end{aligned}$$

or equivalently

$$\begin{aligned} \max_{\omega_i(t), r_i(\cdot, t)} \quad & -\omega_i(t) \Gamma_i(t) - \int_0^t r_i(s, t) ds \\ \text{s.t.} \quad & \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t] \\ & \omega_i(t) \geq 0 \\ & r_i(s, t) \geq 0 \quad \forall s \in [0, t]. \end{aligned} \tag{9}$$

We notice that in this case the primal (and thus the dual) subproblem takes as inputs only the known parameters $\Gamma_i(\cdot)$ and $\hat{\alpha}_i(\cdot)$.

Let's now rewrite constraint (4) (omitting the time argument to ease the reading):

$$\begin{aligned}\gamma &\leq \int_0^T \sum_{i=1}^N \left(p_i(\alpha_i - \beta_i p_i) - f_i(u_i) - h_i I_i + h_i \int_0^t z_i \hat{\alpha}_i ds \right) dt \\ \Leftrightarrow \gamma &\leq C + \int_0^T \sum_{i=1}^N \left(h_i(t) \int_0^t z_i(s) \hat{\alpha}_i(s) ds \right) dt\end{aligned}$$

for all $z(\cdot)$ such that for all i and t ,

$$\begin{aligned}-1 &\leq z_i(t) \leq 1 \\ \int_0^t |z_i(s)| ds &\leq \Gamma_i(t),\end{aligned}$$

where

$$C = \int_0^T \sum_{i=1}^N \left(p_i(\alpha_i - \beta_i p_i) - f_i(u_i) - h_i I_i \right) dt.$$

We seek the feasible realization of z that minimizes the right-hand side in the above inequality. It follows that the constraint will be satisfied for any feasible realization. As previously, we separate this problem across products. We thus need to find, for all i , the feasible realization of z_i that solves

$$\min_{z_i} \int_0^T \left(h_i(t) \int_0^t z_i(s) \hat{\alpha}_i(s) ds \right) dt.$$

It is easy to see that in the optimal solution, $z_i \leq 0$. Therefore, after a variable transformation, we may rewrite the constraints on the new variable z_i as follows

$$\begin{aligned}0 &\leq z_i(t) \leq 1 \quad \forall t \\ \int_0^t z_i(s) ds &\leq \Gamma_i(t) \quad \forall t\end{aligned}$$

and rewrite the minimization problem as

$$\begin{aligned}\min_{z_i} &\int_0^T \left(h_i(t) \int_0^t (-z_i(s) \hat{\alpha}_i(s)) ds \right) dt \\ \Leftrightarrow -\max_{z_i} &\int_0^T H_i(t) z_i(t) \hat{\alpha}_i(t) dt,\end{aligned}$$

where $H_i(t) \equiv \int_t^T h_i(s) ds$, by using the same technique as in the previous section. Therefore, we obtain the equivalent subproblem

$$\begin{aligned}-\max_{z_i} &\int_0^T H_i(t) z_i(t) \hat{\alpha}_i(t) dt \\ \text{s.t.} &0 \leq z_i(t) \leq 1 \quad \forall t \\ &\int_0^t z_i(s) ds \leq \Gamma_i(t) \quad \forall t.\end{aligned}$$

This is an instance of a separated continuous linear program. This class of problems was introduced by Anderson [2] to study a job shop scheduling problem. Pullan [24] studied duality properties for these problems. Using these results, we can write the dual as:

$$\begin{aligned}
\max_{\Pi_i, q_i} \quad & - \int_0^T \Gamma_i(t) d\Pi_i(t) - \int_0^T q_i(t) dt \\
\text{s.t.} \quad & -\Pi_i(t) + q_i(t) \geq H_i(t) \hat{\alpha}_i(t) \quad \forall t \\
& q_i(t) \geq 0 \quad \forall t \\
& \Pi_i(T) = 0 \\
& \Pi_i \text{ non-decreasing.}
\end{aligned} \tag{10}$$

Pullan showed that if the functions $(H_i \cdot \hat{\alpha}_i)(\cdot)$ and $\Gamma_i(\cdot)$ are piecewise analytic, for the optimal solutions z_i lying in the space of measurable bounded functions, strong duality holds. In what follows, we will assume that these assumptions hold.

This model is interesting and useful in our analysis since both the primal and the dual problems are independent of the price (i.e. independent of the control variable of the initial problem). This remark will allow us to greatly simplify the robust counterpart problem.

In the robust problem, strong duality allows us to replace the minimization problems (primal subproblems) by their respective dual maximization subproblems for each constraint. Moreover, at the optimum, the maximum will be realized, therefore, we can simply replace the maximization subproblems by their objective functions and integrate the constraints on the dual variables into the constraints of the robust counterpart.

Therefore, after moving the first constraint back to the objective function, we obtain the following.

Theorem 1. *The robust optimization formulation for problem P is:*

$$\begin{aligned}
\max_{p, u, \Pi, q, \omega, r} \quad & \int_0^T \sum_i \left(p_i(t) (\alpha_i(t) - \beta_i(t) p_i(t)) - f_i(u_i(t)) - h_i(t) I_i(t) - q_i(t) \right) dt - \int_0^T \sum_i \Gamma_i(t) d\Pi_i(t) \\
\text{s.t.} \quad & \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t) p_i(t) \quad \forall i \quad \forall t \in [0, T] \\
& I_i(t) \geq \omega_i(t) \Gamma_i(t) + \int_0^t r_i(s, t) ds \quad \forall i \quad \forall t \in [0, T] \\
& -\Pi_i(t) + q_i(t) \geq H_i(t) \hat{\alpha}_i(t) \quad \forall i \quad \forall t \in [0, T] \\
& \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall i \quad \forall s \in [0, t] \quad \forall t \in [0, T] \\
& \omega_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\
& r_i(s, t) \geq 0 \quad \forall i \quad \forall s \in [0, t] \quad \forall t \in [0, T] \\
& q_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\
& \Pi_i(T) = 0 \quad \forall i \\
& \Pi_i \text{ non-decreasing} \quad \forall i \\
& p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \quad \forall t \in [0, T] \\
& p_i(t), u_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\
& \sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0, T] \\
& I_i(0) = I_i^0 \quad \forall i \\
& H_i(t) = \int_t^T h_i(s) ds \quad \forall i \quad \forall t \in [0, T]
\end{aligned} \tag{11}$$

4.4 Simplification of the Robust Formulation

We notice that $\hat{\alpha}_i$, $h_i(\cdot)$ and $\Gamma_i(\cdot)$, $i = 1 \dots, N$ are data, therefore, each of the two dual subproblems (9), (10) takes only known parameters as inputs. Let's suppose we have at our disposal an efficient algorithm for solving continuous linear programs and therefore, we are able to solve these dual subproblems. Let's denote ω^* , r^* the optimal solutions of problem (9).

We observe that when we plug the optimal value of problem (10) into the robust formulation (11), it appears as a constant in the objective function. Moreover, the constraints of the dual subproblem (10) are no longer necessary since they are independent of the remaining state and control variables.

Remark: For this reason, an approach maximizing the nominal objective function, i.e. considering both revenues and costs as nominal, but preserving the demand uncertainty impact on the inventory level in the no-backorder constraint, would lead to the same robust optimization formulation.

As a result, we obtain the following robust optimization formulation:

$$\max_{p,u} \quad \sum_i \left[\int_0^T \left(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \right) dt \right] \quad (12)$$

$$\text{s.t.} \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) \quad \forall i \quad \forall t \in [0, T] \quad (13)$$

$$I_i(t) \geq J_i(t) \quad \forall i \quad \forall t \in [0, T] \quad (14)$$

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \quad \forall t \in [0, T] \quad (15)$$

$$p_i(t), u_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T]$$

$$\sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0, T]$$

$$I_i(0) = I_i^0 \quad \forall i$$

$$J_i(t) = \omega_i^*(t)\Gamma_i(t) + \int_0^t r_i^*(s, t) ds \quad \forall i \quad \forall t \in [0, T].$$

In this formulation, the uncertainty of demand has an effect only on the no backorder constraint and the upper limit on prices. This makes sense intuitively since when the inventory costs are uncertain, given no additional information, it seems sensible for the system to optimize using as objective the “expected” cost, i.e. the costs obtained using the nominal demand.

The uncertainty of demand translates into protection levels for the prices and the inventory levels (see (14), (15)) that are stronger than in the nominal case. That is, protection levels ensure that the inventory remains above level $J_i > 0$, and prices below the limit $\frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} < \frac{\alpha_i(t)}{\beta_i(t)}$. As a result, even with some variation in the demand - within the introduced uncertainty constraints - the inventory level will remain positive, and prices will remain below their upper bound. These protection levels depend on the budget of uncertainty Γ and on the length $\hat{\alpha}$ of the interval of variation for the demand parameter α . They are determined through the solution of the dual subproblem (9).

The following theorem follows after a change of variable $\bar{I}_i = I_i - J_i$.

Theorem 2. *Assuming that r_i^* is piecewise differentiable with respect to its second argument, and that ω_i^* and Γ_i are piecewise differentiable, we obtain the following equivalent robust formulation for problem*

P :

$$\begin{aligned}
& \max_{p,u} \quad \sum_i \left[\int_0^T \left(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)\bar{I}_i(t) \right) dt \right] \\
& \text{s.t.} \quad \dot{\bar{I}}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) - D_i(t) \quad \forall i \quad \forall t \in [0, T] \\
& \quad \bar{I}_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\
& \quad p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \quad \forall t \in [0, T] \\
& \quad p_i(t), u_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\
& \quad \sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0, T] \\
& \quad \bar{I}_i(0) = \bar{I}_i^0 \quad \forall i
\end{aligned}$$

where $\bar{I}_i^0 = I_i^0 - \omega_i^*(0)\Gamma_i(0)$ and

$$D_i(t) = \dot{\omega}_i^*(t)\Gamma_i(t) + \omega_i^*(t)\dot{\Gamma}_i(t) + r_i^*(t, t) + \int_0^t \frac{\partial r_i^*}{\partial t}(s, t) ds.$$

This problem is very similar to the original problem, in terms of type and number of constraints and variables. However, it differs in the fact that we cannot introduce a parameter $\bar{\alpha}_i(t)$ such that we have:

- $\dot{I}_i(t) = u_i(t) - \bar{\alpha}_i(t) + \beta_i(t)p_i(t)$ be the evolution of inventory levels
- $\frac{\bar{\alpha}_i(t)}{\beta_i(t)}$ be the upper limit of the price $p_i(t)$
- the revenue term of the objective function be of the form $p_i(t)(\bar{\alpha}_i(t) - \beta_i(t)p_i(t))$.

Therefore, a straightforward application of the algorithm we propose in [1] is not possible. In what follows we propose a modified algorithm solving the nominal problem that will also solve this new robust optimization reformulation. The discussion that follows also allows us to illustrate that solving the robust optimization model is of the same difficulty as the nominal one.

4.5 Preliminary Result

Proposition 1. *The function $J_i(\cdot)$ is non decreasing, i.e. the function $D_i(\cdot)$ is non negative for all i .*

Proof. We consider the dual problems (P_t) and (P_{t+dt}) respectively at times t and $t + dt$:

$$\begin{aligned}
& \max_{\omega_i(t), r_i(\cdot, t)} \quad -\omega_i(t)\Gamma_i(t) - \int_0^t r_i(s, t) ds \\
& \text{s.t.} \quad \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t] \\
& \quad \omega_i(t) \geq 0 \\
& \quad r_i(s, t) \geq 0 \quad \forall s \in [0, t],
\end{aligned}$$

$$\begin{aligned}
& \max_{\omega_i(t+dt), r_i(\cdot, t+dt)} \quad -\omega_i(t+dt)\Gamma_i(t+dt) - \int_0^{t+dt} r_i(s, t+dt) ds \\
& \text{s.t.} \quad \omega_i(t+dt) + r_i(s, t+dt) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t+dt] \\
& \quad \omega_i(t+dt) \geq 0 \\
& \quad r_i(s, t+dt) \geq 0 \quad \forall s \in [0, t+dt].
\end{aligned}$$

We denote by $(\omega_i^*(t), r_i^*(\cdot, t))$, $(\omega_i^*(t+dt), r_i^*(\cdot, t+dt))$ the respective optimal solutions. It is clear that $(\omega_i^*(t+dt), r_i^*(\cdot, t+dt))$ is feasible for (P_t) , therefore, we have

$$-\omega_i^*(t)\Gamma_i(t) - \int_0^t r_i^*(s, t)ds \geq -\omega_i^*(t+dt)\Gamma_i(t) - \int_0^t r_i^*(s, t+dt)ds$$

or equivalently

$$\omega_i^*(t+dt)\Gamma_i(t) + \int_0^t r_i^*(s, t+dt)ds \geq \omega_i^*(t)\Gamma_i(t) + \int_0^t r_i^*(s, t)ds.$$

As a result, we observe that

$$\begin{aligned} J_i(t+dt) &= \omega_i^*(t+dt)\Gamma_i(t+dt) + \int_0^{t+dt} r_i^*(s, t+dt)ds \\ &= \omega_i^*(t+dt)\Gamma_i(t) + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_0^t r_i^*(s, t+dt)ds + \int_t^{t+dt} r_i^*(s, t+dt)ds \\ &\geq \omega_i^*(t)\Gamma_i(t) + \int_0^t r_i^*(s, t)ds + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_t^{t+dt} r_i^*(s, t+dt)ds \\ &= J_i(t) + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_t^{t+dt} r_i^*(s, t+dt)ds. \end{aligned}$$

Since $\dot{\Gamma}_i(t) \geq 0$ by assumption, and for feasibility of (P_{t+dt}) , $\omega_i^*(t+dt) \geq 0$, $r_i^*(s, t+dt) \geq 0 \forall s \in [t, t+dt]$, we obtain that

$$J_i(t+dt) \geq J_i(t).$$

□

Next we show that the robust optimization formulation can be solved using a method which is of the same difficulty as the method in [1] for solving the original nominal problem.

5 Solution of the Robust Formulation

Consider the following problem:

$$\begin{aligned} \max_{p, u} \quad & \sum_i \left[\int_0^T \left(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \right) dt \right] \\ \text{s.t.} \quad & \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) - D_i(t) \quad \forall i \quad \forall t \in [0, T] \\ & I_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\ & p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \quad \forall t \in [0, T] \\ & p_i(t), u_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T] \\ & \sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0, T] \\ & I_i(0) = \bar{I}_i^0 \quad \forall i \end{aligned}$$

where $0 \leq \hat{\alpha}_i(t) \leq \frac{1}{2}\alpha_i(t)$, $D_i(t) \geq 0 \forall i, t$ are assumed to be known.

Assumption 1. For all products i , $\alpha_i(t)$, $\hat{\alpha}_i(t)$, $\beta_i(t)$, $h_i(t)$, $D_i(t)$ as well as $K(t)$ are positive, continuous functions of time. Furthermore, $\hat{\alpha}_i(t) \leq \frac{1}{2}\alpha_i(t) \forall i, t$.

Assumption 2. For product i , $i = 1, \dots, N$, function f_i is strictly convex, non-negative and increasing.

Assumption 3. $f'_i(0) < \frac{\alpha_i(t)}{\beta_i(t)}$, $i = 1, \dots, N \quad \forall t \in [0, T]$ (and thus $f'_i(0) < \frac{\alpha_i(t) + 2D_i(t)}{\beta_i(t)}$, $i = 1, \dots, N \quad \forall t \in [0, T]$).

Assumption 4. The following inequality holds at all times t

$$i \text{ s.t. } 0 > \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} - f'_i(\hat{\alpha}_i(t) + D_i(t)) \quad \left(\hat{\alpha}_i(t) + D_i(t) \right) \leq K(t).$$

This last assumption ensure that the production capacity level is sufficiently large to guarantee that the minimum inventory level constraints can be satisfied, i.e. that there exists a feasible solution to this problem.

In order to solve this problem, we will use ideas from continuous dynamic programming and the Maximum Principle. More specifically, we first write the Lagrangian and the necessary conditions for optimality involving the decision variables, Lagrange multipliers, and adjoint variables. Then we derive a method to solve simultaneously all the necessary conditions, i.e. determine the multipliers, adjoint variables, and decision variables that solve the system of conditions. We do so by first writing the optimal policy as a function of the multipliers and adjoint variables. Then we assume that some characteristics of the system are observable and determine the optimal solution under these assumptions. Finally, we relax this assumption.

Appendix A provides the details of this algorithm. For the sake of brevity, we do not give the details of the proof that this algorithm finds the optimal solution, but the derivation is very similar to [1]. One difference is that constrained products never idle in the robust problem, while in the nominal problem they either (a) idle or (b) are produced in order to satisfy the demand. We do recognize two possible states however, one (a') in which they are produced in order to keep the inventory level at zero and allow for demand uncertainty (the rate depends on D_i and $\hat{\alpha}_i$ only), and a second one (b') where they are produced in order to satisfy the demand with uncertainty. This discussion leads to conclude that the new algorithm for solving the robust optimization formulation is of the same order of complexity as the algorithm for solving the nominal problem.

It should be noted that considering a continuous time system makes the problem solution significantly more complex than in a discrete time setting, in which the problem would be a quadratic program under linear constraints. For such a model there exists readily available software for determining a solution. A continuous time approach has the advantage of not introducing any approximation to the real setting: it provides the exact solution of the system. When taking a discrete time approach, one has to decide what a reasonable time step should be, and to allow prices changes and production only at those time, while in reality a supplier may have more flexibility. In order to avoid being too restrictive, the time step needs to be very small, and if the time horizon is large the size of the problem may become exceedingly large. Moreover, we believe that a similar approach can be applied to problems in areas other than dynamic pricing and inventory control, where the evolution of the system evolves dynamically and justify a continuous time approach. We believe that the techniques presented here may be helpful to those areas as well.

6 Numerical Results

6.1 Choice of Parameters

In this section, we consider a numerical example for two products on a time horizon $[0, 10]$ that is similar to the example we considered in [1]. In this paper we also introduce demand uncertainty. Our goal is to understand the relationship between the optimal objective value and the budget of uncertainty $\Gamma_i(\cdot)$. As a result, we will consider only one demand scenario and demand uncertainty model, and a capacity level that is constant at 75% of the maximum of the cumulative production rate achieved in the nominal

	β	h	γ	I^0	$\alpha(t)$	$\hat{\alpha}(t)$
product 1	1	1	10	10	$30 + 10t - t^2$	$0.1t + 0.2$
product 2	1	2	20	10	$30 + 10t - t^2$	$0.1t + 0.2$

Table 1: Data chosen as input in the numerical implementation

	scenario 1	scenario 2	scenario 3	scenario 4	scenario 5	scenario 6
$\Gamma_i(t), i = 1, 2$	$1 + 0.8t$	$1 + 0.5t$	$1 + 0.2t$	$0.5 + 0.8t$	$0.5 + 0.5t$	$0.5 + 0.2t$
$\int_0^T \min\{t, \Gamma_i(t)\} dt$	47.5	34	19.375	44.375	29.75	14.84375

Table 2: Multiple scenarios of budget of uncertainty

case under not capacity constraint: $K(t) = 4.9014$. This guarantees that the capacity constraint is tight for most of the time horizon. We consider the same production cost structure where the cost $f_i(\cdot)$ is a quadratic function of the production rate.

$\hat{\alpha}_i(\cdot)$ which represents the half-length of the allowed range for parameter $\alpha_i(\cdot)$, must satisfy $0 < \hat{\alpha}_i(\cdot) < \frac{1}{2}\alpha_i(\cdot)$. For ease of computations in this example, we consider input parameters $\hat{\alpha}_i(\cdot)$ that are linear functions of the time (nevertheless, the linearity assumption is not necessary):

$$\hat{\alpha}_i(t) = a_i t + b_i,$$

where $a_i, b_i \geq 0$. Indeed, it is reasonable to consider that in practice, the accuracy of a forecast for the demand is non increasing on the time horizon, i.e. that the length of the interval of feasible outcomes is non decreasing, hence $a_i \geq 0$.

We choose $a_i, b_i, i = 1, 2$ such that, the uncertainty on α_i is rather small initially, when the forecast should be rather accurate, and is about 4% of its nominal value at the end of the time horizon.

The input data are summarized in the following table.

Notice that $f_i(u_i) = \frac{\gamma_i}{2} u_i^2$.

The choice of $\Gamma_i(\cdot)$ satisfies $0 \leq \dot{\Gamma}_i(t) \leq 1$. In this example, we first consider these input parameters $\Gamma_i(\cdot)$ to be linear functions of the time:

$$\Gamma_i(t) = g_i t + c_i,$$

where $g_i, c_i \geq 0, g_i < 1$.

Indeed, it can be seen that as soon as the graph of $\Gamma_i(\cdot)$ is above the 45° line with the horizontal axis, its actual value does not matter and $|z_i(t)|$ takes the value 1, meaning that the realized $\tilde{\alpha}_i$ lies at the extreme of its allowed range (worst case scenario). To avoid this from happening on much of the time horizon, we choose $g_i < 1$.

In order to study the effect of the budget uncertainty on the optimal objective value (i.e. performance), we will consider multiple scenario in which only the parameter $\Gamma_i(\cdot)$ varies, and in which it varies in the following way: on the one hand we let the value at time 0 change but keep a constant slope, and on the other hand we keep a constant slope, and change the value at time 0. In all scenarios we assign the same budget of uncertainty to both products.

We will compute the cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma_i(t)\} dt$ as a measure of the global uncertainty in each scenario.

6.2 Closed-Form Solution for the Sub-Problem (9)

In order to implement our algorithm, we need to compute the derivative of the inventory safety level $D_i(\cdot)$ and the modified initial inventory level \bar{I}_i^0 . The reader can refer to Appendix B for a derivations

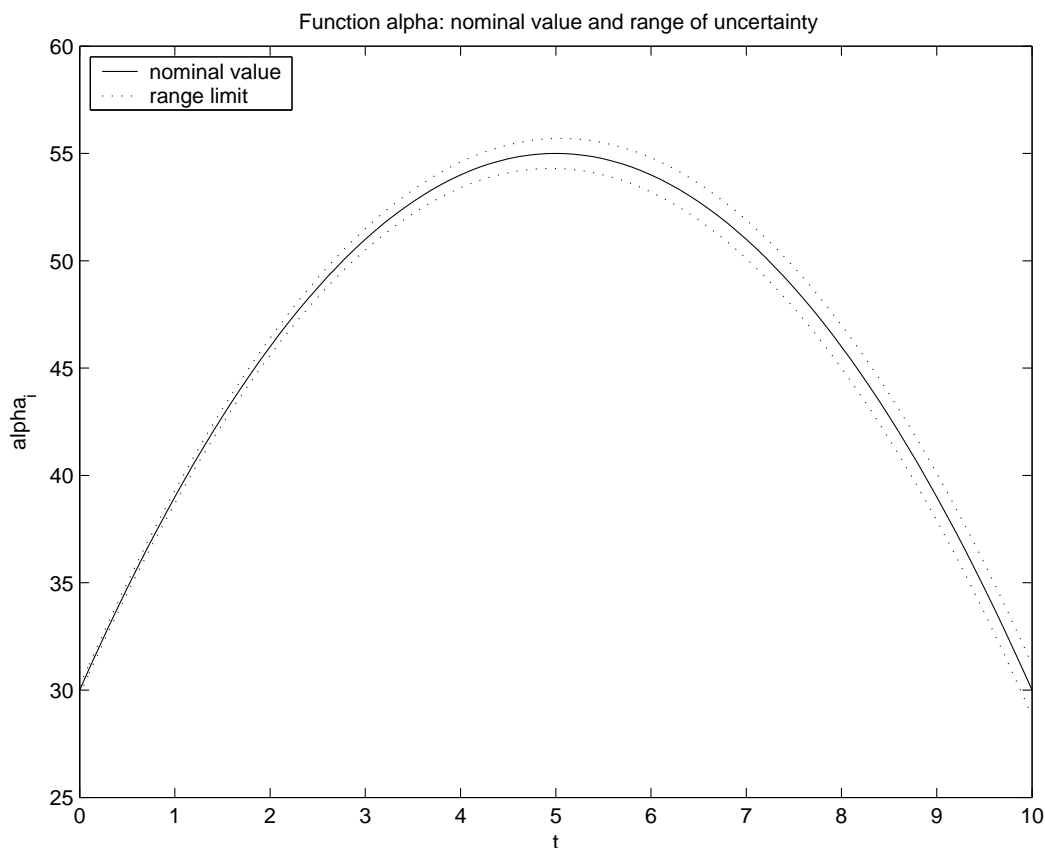


Figure 2: Demand uncertainty

of the following results.

$$\bar{I}_i^0 = I_i^0$$

and

$$D_i(t) = \begin{cases} at + b & \text{if } 0 \leq t < \frac{c}{1-g} \\ a(1-g)(gt + c) + g(at + b) & \text{if } \frac{c}{1-g} < t \leq T. \end{cases}$$

It is also easy to verify that in all scenarios we have at all times

$$\sum_{i=1, 2} (\hat{\alpha}_i(t) + D_i(t)) < K(t)$$

which implies that Assumption 5 holds.

6.3 Results

Using these inputs, we run the algorithm and obtain the production rates, prices, and inventory levels under the optimal policy for the robust formulation we described in (12). Recall that in this formulation, the inventory levels are constrained to remain above a safety level $J_i(t) \geq 0$.

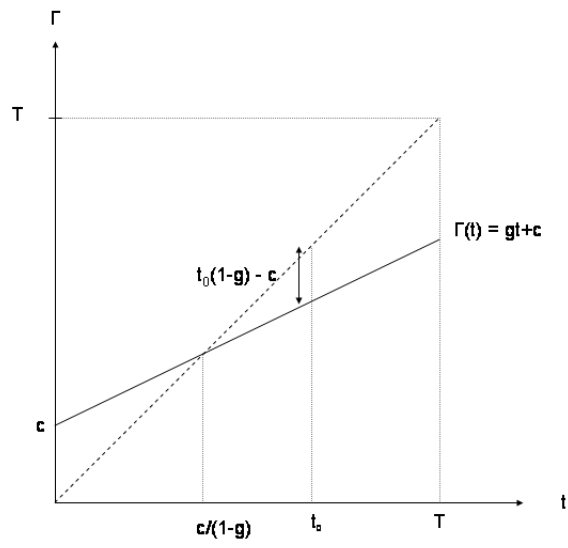


Figure 3: Choice of budget uncertainty function $\Gamma(\cdot)$.

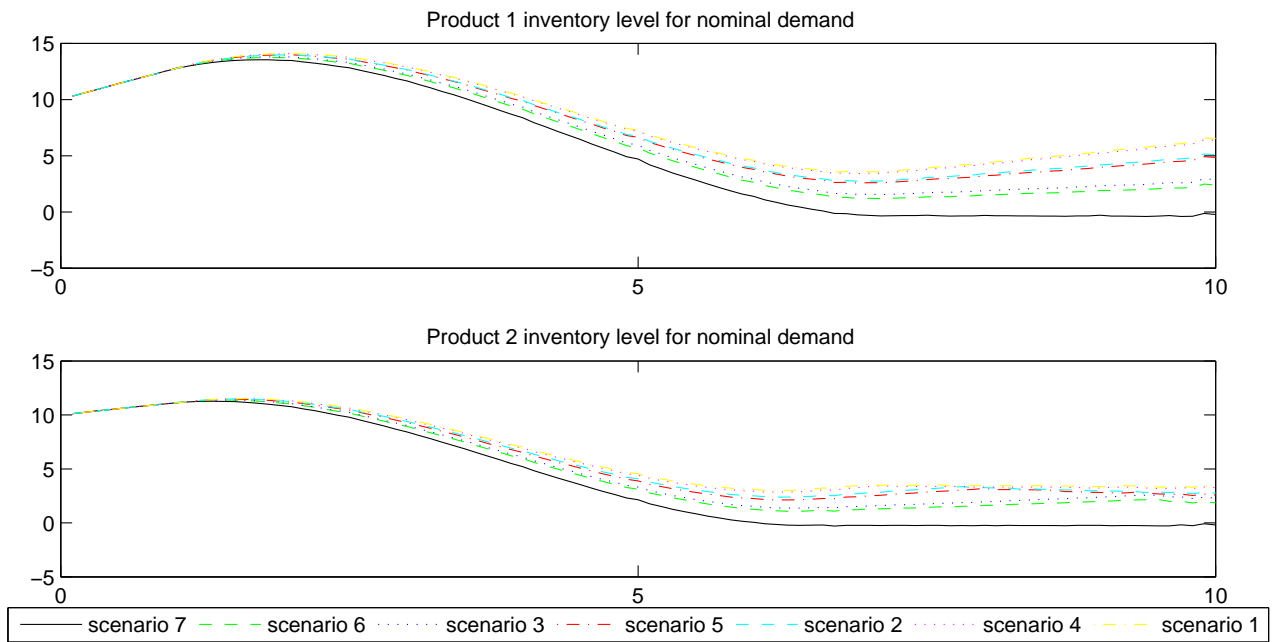


Figure 4: Optimal inventory levels over time results

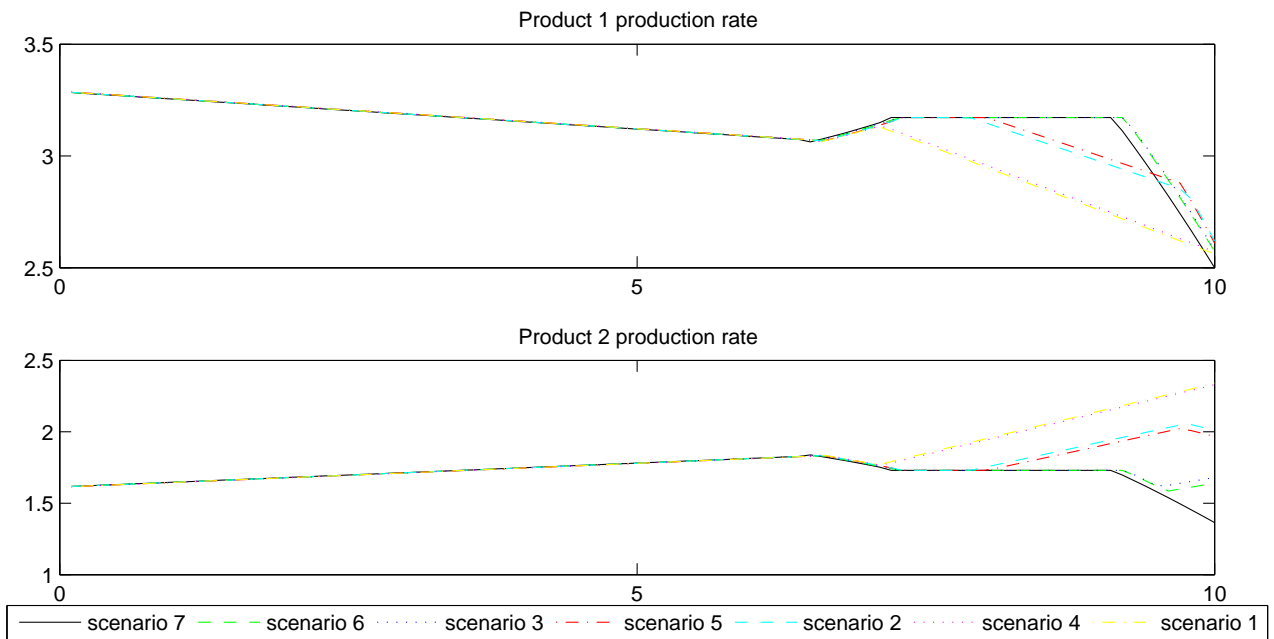


Figure 5: Optimal production rates over time results

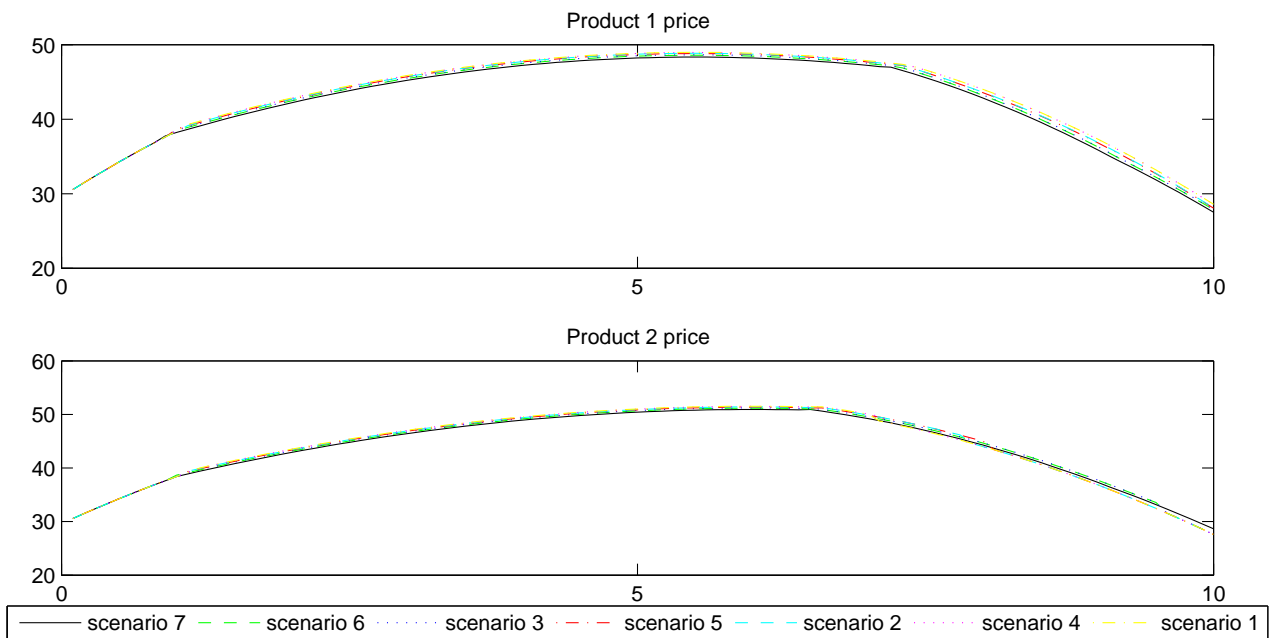


Figure 6: Optimal prices over time results

Similar to the nominal problem, the system starts by building up inventory in anticipation of the demand peak that occurs in the middle of the time horizon. Then the inventory levels are kept at the minimum level - $J_i(t)$ in the robust case - for the remaining time. Notice that prices have the same shape as the demand curves.

We notice that in all scenarios, the capacity constraint is tight except at the very end of the time horizon. Since product 2 is cheaper to produce and to hold, its production rate is maintained at a lower value than for product 1 as the products are unconstrained. In that stage, the budget of uncertainty has no significant influence on the production rates. This makes sense because we saw in formulating the problem that uncertainty played a role in the no backorder constraint and the upper bound on prices. As a result, while the inventory level is positive, the uncertainty does not matter.

In the constrained phase, the safety level $J_i(t)$ for the inventory increases as the budget of uncertainty increases, which is consistent with our interpretation that the more demand uncertainty we impose to the system, the larger should be the minimum value of the inventory level guaranteeing no backorder for any realization of the demand. We observe that in that phase, as the cumulative budget of uncertainty increases, the production rate for product 2 increases more and more because maintaining the level of inventory for that product at $J_2(t)$ requires to produce more and more, given that production rate for product 2 starts from a lower value than production rate for product 1.

Finally, we notice that prices slightly increase as the cumulative budget of uncertainty increases, reflecting the increasing difficulty of satisfying all constraints as the uncertainty increases.

These numerical results suggest that the cumulative effective budget of uncertainty might be a relevant metric for measuring the global uncertainty. This is further confirmed by the following computation on the objective value.

We compute in each scenario the optimal objective value and the cumulative effective budget of uncertainty (see Table 2).

	Budget of uncertainty $\Gamma_i(t), i = 1, 2$	Cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma_i(t)\} dt$	Objective value
scenario 1	$1+0.8t$	47.5	1693.5
scenario 2	$1+0.5t$	34	1791.5
scenario 3	$1+0.2t$	19.4	1928.7
scenario 4	$0.5+0.8t$	44.4	1707.7
scenario 5	$0.5+0.5t$	29.8	1812.7
scenario 6	$0.5+0.2t$	14.8	1972.3
scenario 7	0	0	2198.4

Table 3: Objective value results

Table 2 and Figure 7 seem to suggest that the cumulative effective budget of uncertainty may be a reasonable way of measuring the uncertainty in the problem. Notice that as the uncertainty increases, the optimal objective value decreases. This illustrates the trade-off between optimality (high optimal objective value) and robustness (high level of uncertainty).

7 Conclusion

In this paper, we have proposed and studied a robust optimization approach for incorporating demand uncertainty in a fluid model of dynamic pricing and inventory control. Using ideas from robust optimization, we reformulate the problem as a deterministic problem of a similar form as the original nominal formulation. Our approach does not make assumptions on the probabilistic distribution of the demand

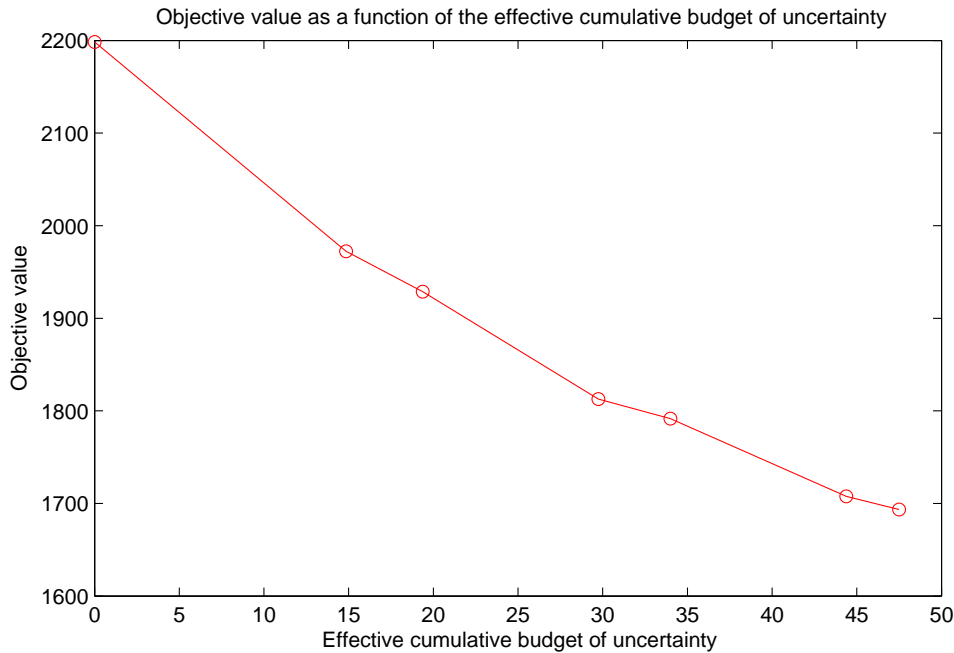


Figure 7: Trade-off between robustness and performance

but rather assumes an additive demand model whose uncertain parameter lies in a given interval, and a budget of uncertainty. Furthermore, we were able to adapt the algorithm for solving the deterministic problem and show that the adapted algorithm is no more complex than the original one. We implemented this algorithm on a numerical example that illustrates the trade-off between robustness and performance.

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A Appendix: Algorithm

Input:

T time horizon;
 N number of products (products are indexed by i);

and for all i :

$\alpha_i(t)$, $\beta_i(t)$ such that $d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t)$;
 $f_i(\cdot)$ production cost function for product i ,
takes the production rate as argument;
strictly convex, increasing, non-negative function such that $f'_i(0) \leq \frac{\alpha_i(t) + 2D_i(t)}{\beta_i(t)}$,
 $h_i(t)$ holding cost per unit of product i ;
 $\psi_i(t)$ such that $f'_i{}^{-1}(\psi_i(t)) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}\psi_i(t) - D_i(t) = 0$.
 $\phi_i(t, \eta)$ such that $\phi_i(t, \eta) = F_{t,i}^{-1}(\eta)$ where $F_{t,i}(X) = X - f'_i\left(\frac{\alpha_i(t) - \beta_i(t)X}{2} + D_i(t)\right)$;
 $\tilde{\psi}_i(t) = \begin{cases} \psi(t) & \text{if } f'_i(\hat{\alpha}_i(t) + D_i(t)) \leq \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} \\ f'_i(\hat{\alpha}_i(t) + D_i(t)) & \text{otherwise.} \end{cases}$
 $\tilde{\phi}_i(t) = \begin{cases} \phi_i(t, \eta(t)) & \text{if } \eta(t) \leq \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} - f'_i(\hat{\alpha}_i(t) + D_i(t)) \\ \eta(t) + f'_i(\hat{\alpha}_i(t) + D_i(t)) & \text{otherwise.} \end{cases}$
 I_i^0 initial level of inventory for product i .

It is important to notice that the definition of $\tilde{\phi}_i(t)$ involves $\eta(t)$. To make this clearer we will use also the notation $\tilde{\phi}_i(t) \equiv \tilde{\phi}_i(t, \eta(t))$.

Output:

$u_i(t)$, $p_i(t)$, $i = 1, \dots, N$, $\forall t \in [0, T]$.

Notations:

\bar{I}_i^0	critical value of the initial inventory above which it is optimal not to produce product i on the entire time horizon;
$\eta(t)$	Lagrange multiplier for the capacity constraint;
$\rho_i(t)$	Lagrange multiplier for the constraint on the non negativity of $I_i(t)$;
$q_i(t)$	adjoint variable for the dynamic constraint for product i ;
τ	first time the capacity constraint becomes tight
$\gamma_i(t)$	binary variable equal to 1 when product i has a positive initial inventory level lower than \bar{I}_i^0 and is on the first unconstrained interval at time t , equal to 0 otherwise;
$\delta_i(t)$	binary variable equal to 1 when product i is on a constrained interval at time t , equal to 0 otherwise;
t_i^0	first entry time on a constrained interval for product i ;
t_i^2	subsequent entry time on a constrained interval for product i ;
t_i^1	exit time from a constrained interval for product i ;
\mathcal{I}	set of active constrained products;
\mathcal{I}'	set of active unconstrained products;

Remarks:

- We observe that $\psi_i(t) = \phi_i(t, 0)$;
- We may derive

$$\frac{d\phi_i}{dt}(t, \eta(t)) = \frac{\partial\phi_i}{\partial t}(t, \eta(t)) + \dot{\eta}(t) \frac{\partial\phi_i}{\partial \eta}(t, \eta(t))$$

Algorithm for 1 product

Initialization

1. Let

$$H(t) = - \int_t^T h(s) ds.$$

2. Let

$$z(t) = \begin{cases} -\alpha(t) - D(t) & \text{if } H(t) < -\frac{\alpha(t)}{\beta(t)} \\ \frac{1}{2} \left(-\alpha(t) + \beta(t)H(t) \right) - D(t) & \text{if } H(t) \geq -\frac{\alpha(t)}{\beta(t)} \end{cases}.$$

3. Let

$$\bar{I} = - \int_0^T z(s) ds.$$

- If $I^0 \geq \bar{I}$, go to 4.
- If $0 < I^0 < \bar{I}$, go to 5.
- If $I^0 = 0$, go to 9.

4. Large initial inventory level

Let

$$\begin{aligned} q(t) &= H(t) \quad \forall t \in [0, T]; \\ \rho(t) &= 0 \quad \forall t \in [0, T]; \\ \delta(t) &= 0 \quad \forall t \in [0, T]; \\ \gamma(t) &= 1 \quad \forall t \in [0, T]. \end{aligned}$$

Go to 10.

5. Small initial inventory level (First unconstrained interval)

Define

$$\begin{aligned} q(t) &= q^0 + \int_0^t h(s) ds \\ y(t) &= \begin{cases} -\alpha(t) - D(t) & \text{if } q(t) < -\frac{\alpha(t)}{\beta(t)} \\ \frac{1}{2} \left(-\alpha(t) + \beta(t)q(t) \right) - D(t) & \text{if } -\frac{\alpha(t)}{\beta(t)} \leq q(t) \leq \min\{f'(0); \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)}\} \\ f'^{-1}\left(q(t)\right) - \frac{\alpha(t)}{2} + \frac{\beta(t)}{2}q(t) - D(t) & \text{if } f'(0) \leq q(t) \leq \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)} \\ -\hat{\alpha}(t) - D(t) & \text{if } \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)} \leq q(t) \leq f'(0) \\ f'^{-1}\left(q(t)\right) - \hat{\alpha}(t) - D(t) & \text{if } q(t) > \max\{\frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)}; f'(0)\}. \end{cases} \end{aligned}$$

Solve for q^0 and $t_0 \geq 0$ (smallest feasible solution) the following nonlinear system:

$$\begin{cases} \tilde{\psi}(t_0) = q(t_0) \\ \dot{\tilde{\psi}}(t_0) \leq h(t_0) \\ \int_0^{t_0} y(t) dt = -I^0 \end{cases}$$

6. Let $q(t)$ as given above, $\gamma(t) = 1$, $\delta(t) = 0$ and $\rho(t) = 0$ on $[0, t_0]$, Go to 7.

7. (Next intervals)

Define

$$\begin{aligned} q(t) &= \tilde{\psi}(t_1) + \int_{t_1}^t h(s) ds \\ \bar{y}(t) &= \begin{cases} -\alpha(t) - D(t) & \text{if } q(t) < -\frac{\alpha(t)}{\beta(t)} \\ \frac{1}{2} \left(-\alpha(t) + \beta(t)q(t) \right) - D(t) & \text{if } -\frac{\alpha(t)}{\beta(t)} \leq q(t) \leq \min\{f'(0); \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)}\} \\ f'^{-1}\left(q(t)\right) - \frac{\alpha(t)}{2} + \frac{\beta(t)}{2}q(t) - D(t) & \text{if } f'(0) \leq q(t) \leq \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)} \\ -\hat{\alpha}(t) - D(t) & \text{if } \frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)} \leq q(t) \leq f'(0) \\ f'^{-1}\left(q(t)\right) - \hat{\alpha}(t) - D(t) & \text{if } q(t) > \max\{\frac{\alpha(t)-2\hat{\alpha}(t)}{\beta(t)}; f'(0)\}. \end{cases} \end{aligned}$$

Attempt to solve for $t_1 \geq t_0$ and $t_2 > t_1$ (smallest feasible solutions) the following nonlinear system:

$$\begin{cases} \tilde{\psi}(t_2) = q(t_2) \\ \dot{\tilde{\psi}}(t_2) \leq h(t_2) \\ \int_{t_1}^{t_2} \bar{y}(t) dt = 0 \end{cases}$$

If there is no solution, let $t_1 = t_2 = T$ and go to 10. If there is a solution, go to 8

8. Let

$$\begin{aligned}
(q + \rho)(t) &= \tilde{\psi}(t) \quad \forall t \in [t_0, t_1] \\
\gamma(t) &= 0 \quad \forall t \in [t_0, t_1] \\
\delta(t) &= 1 \quad \forall t \in [t_0, t_1] \\
\rho(t) &= 0 \quad \forall t \in [t_1, t_2] \\
q(t) &\quad \text{as described in step 7} \quad \forall t \in [t_1, t_2] \\
\gamma(t) &= 0 \quad \forall t \in [t_1, t_2] \\
\delta(t) &= 0 \quad \forall t \in [t_1, t_2]
\end{aligned}$$

Do $t_0 \leftarrow t_2$; go to 7.

9. **Zero initial inventory level**

Let $t_0 = 0$; go to 7.

10. **Final step** Let

$$p(t) = \begin{cases} 0 & \text{if } q(t) + \rho(t) < -\frac{\alpha(t)}{\beta(t)} \\ \frac{1}{2} \left(q(t) + \rho(t) + \frac{\alpha(t)}{\beta(t)} \right) & \text{if } -\frac{\alpha(t)}{\beta(t)} \leq q(t) + \rho(t) \leq \frac{\alpha(t) - 2\hat{\alpha}(t)}{\beta(t)} \\ \frac{\alpha(t) - \hat{\alpha}(t)}{\beta(t)} & \text{if } q(t) + \rho(t) > \frac{\alpha(t) - 2\hat{\alpha}(t)}{\beta(t)} \end{cases} \quad \forall t \in [0, T]$$

$$u(t) = \begin{cases} 0 & \text{if } q(t) + \rho(t) \leq f'(0) \\ f'^{-1}(q(t) + \rho(t)) & \text{if } q(t) + \rho(t) > f'(0). \end{cases} \quad \forall t \in [0, T]$$

Algorithm for multiple products:

Initialization

1. Do the algorithm for 1 product above, for each of the N products. Output δ_i , γ_i , u_i . Remove all products for which $I_i^0 > \bar{I}_i$; update N .

2. Let

$$\tau = \min \left\{ \min \left\{ t : \sum_{i=1}^N u_i(t) = K(t) \right\}; T \right\}$$

3. If $\tau = T$, stop.

Otherwise, let

$$S_1 \equiv \{i : \gamma_i(\tau) = 1, \delta_i(\tau) = 0\}$$

$$S_2 \equiv \{i : \gamma_i(\tau) = 0, \delta_i(\tau) = 0\}$$

$$S_3 \equiv \{i : \delta_i(\tau) = 1\}.$$

Parameters τ , q_i^0 , t_i^0 , $i \in S_1$ and the current t_i^1 , t_i^2 , $i \in S_2 \cup S_3$ need to be updated simultaneously with the computation of $\eta(t)$ for $t \geq \tau$.

4. We determine $\eta(t)$ along with τ , q_i^0 , t_i^0 , $i \in S_1$ and t_i^1 , t_i^2 , $i \in S_2 \cup S_3$ where t_i^0 , t_i^2 are the smallest solutions such that all of the following holds:

- $\forall i \in S_1$, we have

- $q_i(t) = q_i^0 + \int_0^t h_i(s) ds$, $\rho_i(t) = 0 \quad \forall t \in [0, t_i^0]$,
- $q_i(t_i^0) = \bar{\phi}_i(t_i^0, \eta(t_i^0))$,
- $\frac{d\bar{\phi}_i}{dt}(t_i^0) \leq h_i(t_i^0)$ and
- $\int_0^{t_i^0} y_i(t) dt = -I_i^0$ where

$$y_i(t) = \begin{cases} -\alpha_i(t) - D_i(t) & \text{if } q(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2} \left(-\alpha_i(t) + \beta_i(t)q_i(t) \right) - D_i(t) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) \leq \min\{f'_i(0) + \eta(t); \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)}\} \\ f'_i{}^{-1}(q_i(t) - \eta(t)) & \\ -\frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) - D_i(t) & \text{if } f'_i(0) + \eta(t) \leq q_i(t) \leq \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} \\ -\hat{\alpha}_i(t) - D_i(t) & \text{if } \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} \leq q_i(t) \leq f'_i(0) + \eta(t) \\ f'_i{}^{-1}(q_i(t) - \eta(t)) - \hat{\alpha}_i(t) - D_i(t) & \text{if } q_i(t) > \max\{\frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)}; f'_i(0) + \eta(t)\}. \end{cases}$$

- $\forall i \in S_2$, we have

- $q_i(t) = \tilde{\psi}_i(t_i^1) + \int_{t_i^1}^t h_i(s) ds$, $\rho_i(t) = 0$ on $[t_i^1, t_i^2]$,
- $q_i(t_i^2) = \bar{\phi}_i(t_i^2, \eta(t_i^2))$,
- $\frac{d\bar{\phi}_i}{dt}(t_i^2) \leq h_i(t_i^2)$ and
- $\int_{t_i^1}^{t_i^2} y_i(t) dt = 0$

- $\forall i \in S_3$, we have

- $q_i(t) + \rho_i(t) = \tilde{\psi}_i(t)$ on $[t_i^0, \tau]$,
- $q_i(t) + \rho_i(t) = \bar{\phi}_i(t, \eta(t))$ on $[\tau, t_i^1]$,
- $q_i(t) = \bar{\phi}_i(t_i^1, \eta(t_i^1)) + \int_{t_i^1}^t h_i(s) ds$,
- $\rho_i(t) = 0$ on $[t_i^1, t_i^2]$
- $q_i(t_i^2) = \bar{\phi}_i(t_i^2, \eta(t_i^2))$,
- $\frac{d\bar{\phi}_i}{dt}(t_i^2) \leq h_i(t_i^2)$ and
- $\int_{t_i^1}^{t_i^2} y_i(t) dt = 0$

- $\eta(t) = 0$ on $[0, \tau]$,

- $\sum_{i=1}^N u_i(t)$ reaches $K(t)$ for the first time at time τ , where $u_i(t) = \max\{0, f'_i{}^{-1}(q_i(t) + \rho_i(t))\}$

- η is computed by the following algorithm:

- Reorder the indices from 1 to N by increasing value of

$$\{q_i(\tau) + \rho_i(\tau) - f'_i(0), i : \delta_i(\tau) = 0\} \cup \left\{ \frac{\alpha_i(\tau) - 2\hat{\alpha}_i(\tau)}{\beta_i(\tau)} - f'_i(\hat{\alpha}_i(\tau) + D_i(\tau)), i : \delta_i(\tau) = 1 \right\}.$$

$$(\text{Define } q_0(\tau) + \rho_0(\tau) - f'_0(0) = \frac{\alpha_0(\tau) - 2\hat{\alpha}_0(\tau)}{\beta_0(\tau)} - f'_0(\hat{\alpha}_0(\tau) + D_0(\tau)) \equiv 0).$$

- Let

$$\mathcal{I} = \{i : \delta_i(\tau) = 1, \frac{\alpha_i(\tau) - 2\hat{\alpha}_i(\tau)}{\beta_i(\tau)} - f'_i(\hat{\alpha}_i(\tau) + D_i(\tau)) \geq 0\}$$

$$\bar{\mathcal{I}} = \{i : \delta_i(\tau) = 1, \frac{\alpha_i(\tau) - 2\hat{\alpha}_i(\tau)}{\beta_i(\tau)} - f'_i(\hat{\alpha}_i(\tau) + D_i(\tau)) < 0\}$$

$$\mathcal{I}' = \{i : \delta_i(\tau) = 0, q_i(\tau) - f'_i(0) > 0\}$$

$$i_1 = \min \mathcal{I} \quad (\infty \text{ if } \mathcal{I} = \emptyset)$$

$$i'_1 = \min \mathcal{I}' \quad (\infty \text{ if } \mathcal{I}' = \emptyset)$$

$$k = \min\{i_1, i'_1\}$$

(c) Find $\eta(t)$ such that

$$\begin{aligned} & \sum_{\mathcal{I}} \left(\frac{1}{2} (\alpha_i(t) - \beta_i(t) \phi_i(t, \eta(t))) + D_i(t) \right) + \sum_{\bar{\mathcal{I}}} (\hat{\alpha}_i(t) + D_i(t)) \\ & + \sum_{\mathcal{I}'} f_i'^{-1}(q_i(t) - \eta(t)) = K(t) \end{aligned} \quad (16)$$

- If $\eta(\tau) \in [\max\{0; q_{k-1}(\theta) + \rho_{k-1}(\tau) - f_{k-1}'(0)\}; q_k(\tau) + \rho_k(\tau) - f_k'(0)] \cap [\max\{0; \frac{\alpha_{k-1}(\tau) - 2\hat{\alpha}_{k-1}(\tau)}{\beta_{k-1}(\tau)} - f_{k-1}'(\hat{\alpha}_{k-1}(\tau) + D_{k-1}(\tau)); \frac{\alpha_k(\tau) - 2\hat{\alpha}_k(\tau)}{\beta_k} - f_k'(\hat{\alpha}_k(\tau) + D_k(\tau))\}]$, $\eta(\tau) \in [\max\{0; q_{k-1}(\tau) + \rho_{k-1}(\tau) - f_{k-1}'(0)\}; q_k(\tau) + \rho_k(\tau) - f_k'(0)]$, go to 4d.
- Otherwise, if $k = i_1$, do $\mathcal{I} \leftarrow \mathcal{I} \setminus \{i_1\}$; $\bar{\mathcal{I}} \leftarrow \bar{\mathcal{I}} \cup \{i_1\}$; $k = \min\{i_1, i_1'\}$; go to 4c.
- Otherwise, if $k = i_1'$, do $\mathcal{I}' \leftarrow \mathcal{I}' \setminus \{i_1'\}$; $k = \min\{i_1, i_1'\}$; go to 4c.

Note that the sets S_1, S_2, S_3 need to be updated each time we consider a new stage.

- (d) If at a time no greater than $t_i^0, i \in S_1$, than $t_i^2, i \in S_2$, and than $t_i^1, i \in S_3$, we observe that respectively $\eta(t)$ takes negative values, or that for some product i unconstrained, $q_i(t) + \rho_i(t) - \eta(t)$ changes sign, or that for some product i constrained, $\frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} - f_i'(\hat{\alpha}_i(t) + D_i(t)) - \eta(t)$ changes sign, we set respectively η to zero or we update the sets of active unconstrained products, or we update the set of constrained products in each class, and the value of η . In the first case, we check whether the capacity constraint becomes tight again. In the last 2 cases, we iterate this step.
- (e) Compute the derivative of $\eta(t)$ using equation (16).

5. **Final step** Let

$$p_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2} \left(q_i(t) + \rho_i(t) + \frac{\alpha_i(t)}{\beta_i(t)} \right) & \text{if } -\frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} \leq q_i(t) + \rho_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} & \text{if } q_i(t) + \rho_i(t) > \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} \end{cases} \quad \forall t \in [0; T]$$

$$u_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) - \eta(t) \leq f_i'(0) \\ f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) & \text{if } q_i(t) + \rho_i(t) - \eta(t) > f_i'(0) \end{cases} \quad \forall t \in [0; T].$$

B Appendix: Numerical results: closed form solution for $D_i(\cdot)$

In this section, we use the notation and data introduced in Section 6 to derive a closed form solution for the derivative of the inventory safety level $D_i(\cdot)$.

We showed in Section 4.4 that $D_i(t)$ is obtained by solving (for all fixed t):

$$\begin{aligned} & \min_{\omega_i(t), r_i(\cdot, t)} && \omega_i(t) \Gamma_i(t) + \int_0^t r_i(s, t) ds \\ & \text{s.t.} && \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t] \\ & && \omega_i(t) \geq 0 \\ & && r_i(s, t) \geq 0 \quad \forall s \in [0, t]. \end{aligned}$$

To ease the reading, we will omit the subscript i and argument t in the following derivation. After plugging our choice of input parameters, the problem becomes:

$$\begin{aligned} \min_{\omega, r(\cdot)} \quad & \omega + x \int_0^t r(s) ds \\ \text{s.t.} \quad & \omega + r(s) \geq Q(s) \quad \forall s \in [0, t] \\ & \omega \geq 0 \\ & r(s) \geq 0 \quad \forall s \in [0, t]. \end{aligned}$$

where we denote $x \equiv \frac{1}{gt+c}$, $Q(s) = as + b$.

We notice that $Q(\cdot)$ is positive and linearly increasing with s . Since $\omega \geq 0$ and $r \geq 0$ have positive weights in the objective function, we conclude from the structure of this linear program, that at the optimal solution, either (i) $\omega = 0$ and $r(s) = Q(s) \quad \forall s$, or (ii) there exists s_0 such that $\omega = Q(s_0)$ and $r(s) = \begin{cases} 0 & \text{if } s \leq s_0 \\ Q(s) - \omega = a(s - s_0) & \text{otherwise.} \end{cases}$

In the former case (i), the objective value is equal to

$$F_1 = x \int_0^t Q(s) ds = x(a \frac{t^2}{2} + bt).$$

In the latter case (ii), the objective value is equal to

$$F_2 = \min_{s_0 \in [0, t]} Q(s_0) + x \int_{s_0}^t a(s - s_0) ds = \min_{s_0 \in [0, t]} as_0 + b + x \int_0^{t-s_0} au \, du = \min_{s_0 \in [0, t]} as_0 + b + xa \frac{(t - s_0)^2}{2}.$$

The function above is minimized at $y \equiv t - \frac{1}{x}$. We observe that $y \leq t$ and moreover $y \geq 0 \Leftrightarrow t \geq \frac{c}{1-g}$. As a result

$$F_2 = \begin{cases} at(1-g) - ac + b + \frac{a}{gt+c} \frac{(gt+c)^2}{2} = at + b - \frac{a}{2}(gt+c) & \text{if } t \geq \frac{c}{1-g} \\ b + xa \frac{t^2}{2} & \text{otherwise.} \end{cases}$$

To decide whether case (i) or case (ii) holds, we have to determine which one gives rise to the lower objective value. We compute

- for $t \geq \frac{c}{1-g}$, $F_1 - F_2 = xa \frac{t^2}{2} + xbt - at - b + \frac{a}{2x} = (xt - 1)(b + \frac{a}{2x}(xt - 1))$.
It can be seen that $xt - 1 \geq 0$ if $t \geq \frac{c}{1-g}$, so $F_1 \geq F_2$ and case (ii) holds.
- for $t < \frac{c}{1-g}$, $F_1 - F_2 = xa \frac{t^2}{2} + xbt - b - ax \frac{t^2}{2} = b(xt - 1) \leq 0$. As a result case (i) holds.

As a conclusion,

$$\begin{aligned} \omega^*(t) &= \begin{cases} 0 & \text{if } 0 \leq t < \frac{c}{1-g} \\ at(1-g) - ac + b & \text{if } \frac{c}{1-g} \leq t \leq T \end{cases} \\ r^*(s, t) &= \begin{cases} as + b & \forall s \in [0, t] \text{ if } 0 \leq t < \frac{c}{1-g} \\ 0 & \text{if } \frac{c}{1-g} \leq t \leq T \text{ and } 0 \leq s < t(1-g) - c \\ as - at(1-g) + ac & \text{if } \frac{c}{1-g} \leq t \leq T \text{ and } t(1-g) - c \leq s \leq t. \end{cases} \end{aligned}$$

We notice that ω^* has a discontinuity at $\frac{c}{1-g}$, which is the time when the budget of uncertainty becomes a real constraint (i.e. $\Gamma(t)$ starts taking values that are lower than t). We verify the assumption that r^* is piecewise differentiable with respect to its second argument, and that ω_i^* and Γ_i are piecewise differentiable.

These results may be intuitively explained in the following way. We recall that the variables we just determined are dual variables for the problem:

$$\begin{aligned} \max_z \quad & \int_0^t z(s)\hat{\alpha}(s)ds \\ \text{s.t.} \quad & \int_0^t z(s)ds \leq \Gamma(t) \quad \leftarrow \omega(t) \\ & 0 \leq z(s) \leq 1 \quad \forall s \in [0, t] \quad \leftarrow r(s, t). \end{aligned}$$

As we explained before, if $0 \leq t < \frac{c}{1-g}$, the budget of uncertainty constraint is not tight because $z(s)$ may take the larger value in its allowed range (i.e. 1) for all $s \in [0, t]$ and still that constraint is not tight. As a result the dual variable ω for that constraint should be equal to zero. Since the constraint $z \leq 1$ is tight, the dual variable r may be positive and reflects the coefficient in the objective function. Now if $t \geq \frac{c}{1-g}$, the budget of uncertainty constraint will be tight, because z may not be equal to 1 on the entire interval $[0, t]$. Since the coefficient of z in the objective function increases with time, to obtain the maximum z should be equal to zero for the shortest possible time and then remain at 1 until time t . To satisfy the budget of uncertainty constraint, that shortest possible time is equal to the difference between t and $\gamma(t)$. During that time, since $z = 0$, by complementary slackness r should be at 0.

We may now compute

$$\bar{I}_i^0 = I_i^0 - \omega_i^*(0)\Gamma_i(0) = I_i^0$$

and (except for the discontinuity point)

$$\begin{aligned} D_i(t) &= \dot{\omega}_i^*(t)\Gamma_i(t) + \omega_i^*(t)\dot{\Gamma}_i(t) + r_i^*(t, t) + \int_0^t \frac{\partial r_i^*}{\partial t}(s, t)ds \\ &= \begin{cases} 0 + 0 + at + b & \text{if } 0 \leq t < \frac{c}{1-g} \\ a(1-g)(gt+c) + g(at(1-g) - ac + b) + at - at(1-g) + ac \\ \quad + \int_{t(1-g)-c}^t -a(1-g)ds & \text{if } \frac{c}{1-g} < t \leq T \end{cases} \\ &= \begin{cases} at + b & \text{if } 0 \leq t < \frac{c}{1-g} \\ a(1-g)(gt+c) + g(at+b) + a(1-g)(gt+c) - a(1-g)(gt+c) & \text{if } \frac{c}{1-g} < t \leq T \end{cases} \\ &= \begin{cases} at + b & \text{if } 0 \leq t < \frac{c}{1-g} \\ a(1-g)(gt+c) + g(at+b) & \text{if } \frac{c}{1-g} < t \leq T. \end{cases} \end{aligned}$$

	scenario 1	scenario 2	scenario 3
$\Gamma_i(t), i = 1, 2$	$1+0.8t$	$1+0.5t$	$1+0.2t$
$\frac{c}{1-g}$	5	2	1.25
$D_i(t), t < \frac{c}{1-g}$	$0.1t+0.2$	$0.1t+0.2$	$0.1t+0.2$
$D_i(t), t \geq \frac{c}{1-g}$	$0.096t+0.18$	$0.075t+0.15$	$0.036t+0.12$
$\hat{\alpha}_i(t) + D_i(t), t < \frac{c}{1-g}$	$0.2t+0.4$	$0.2t+0.4$	$0.2t+0.4$
$\hat{\alpha}_i(t) + D_i(t), t \geq \frac{c}{1-g}$	$0.196t+0.38$	$0.175t+0.35$	$0.136t+0.32$

Table 4: Closed-form for $D_i(\cdot)$, scenarios 1, 2, 3

	scenario 4	scenario 5	scenario 6
$\Gamma_i(t), i = 1, 2$	$0.5+0.8t$	$0.5+0.5t$	$0.5+0.2t$
$\frac{c}{1-g}$	2.5	1	0.625
$D_i(t), t < \frac{c}{1-g}$	$0.1t+0.2$	$0.1t+0.2$	$0.1t+0.2$
$D_i(t), t \geq \frac{c}{1-g}$	$0.096t+0.17$	$0.075t+0.125$	$0.036t+0.08$
$\hat{\alpha}_i(t) + D_i(t), t < \frac{c}{1-g}$	$0.2t+0.4$	$0.2t+0.4$	$0.2t+0.4$
$\hat{\alpha}_i(t) + D_i(t), t \geq \frac{c}{1-g}$	$0.196t+0.37$	$0.175t+0.325$	$0.136t+0.28$

Table 5: Closed-form for $D_i(\cdot)$, scenarios 4, 5, 6