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Supply Chain Competition with Multiple Manufacturers and Retailers*

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We study competition in a supply chain where multiple manufacturers compete in quantities to supply a set of products to multiple risk-averse retailers who compete in quantities to satisfy the uncertain consumer demand. For the symmetric supply chain, we give closed-form expressions for the unique equilibrium. We find that, provided there is a sufficiently large number of manufacturers and retailers, the supply chain efficiency (the ratio of the aggregate utility in the decentralized and centralized chains) can be raised to one by inducing the right degree of retailer differentiation. Also, risk aversion results in triple marginalization: retailers require a strictly positive margin to distribute even when they are perfectly competitive, as otherwise they are unwilling to undertake the risk associated with the uncertainty in demand. For the asymmetric supply chain, we show how numerical optimization can be used to compute the equilibria, and find that the supply chain efficiency may drop sharply with the asymmetry of either manufacturers or retailers. We also find that the introduction of asymmetric product assortment reduces the degree of competition among retailers, and thus has an effect similar to that of reducing the number of retailers. We show that, unlike in the symmetric chain, the asymmetric chain efficiency depends on product differentiation and risk aversion, because of the interaction between these features and the asymmetry of manufacturers and retailers.

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1. Introduction

We study a model of competition in supply chains that captures several features that have been ignored in much of the existing literature: simultaneous competition among manufacturers and

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retailers, product and retailer differentiation, and retailer risk aversion. Our objective is to understand how these features affect the decentralized supply chain equilibrium and its efficiency, defined as the ratio of the aggregate utility of manufacturers and retailers in the decentralized and centralized chains.

It is well known that aggregate profit in a decentralized supply chain with one manufacturer and one retailer may be lower than in a vertically integrated chain where a central planner makes decisions for both the manufacturer and the retailer. This is a manifestation of the *double marginalization* phenomenon, which was first identified by Spengler [1950]. Two fascinating questions related to this phenomenon have attracted much attention in the recent literature; see the review by Cachon [2003]. First, how inefficient is the decentralized supply chain? Second, are there any contractual conditions between the manufacturers and the retailers that *coordinate* the supply chain?

The supply chain with one manufacturer and one retailer has been extensively studied. Pasternack [1985], for instance, studies the supply chain where a single manufacturer serves a single retailer who faces an exogenously determined retail price and a stochastic demand. He shows that there exists a class of contracts—based on a constant wholesale price and a buy-back rate for any unsold units—that coordinate this supply chain.

The supply chain where a single manufacturer supplies multiple competing retailers has also been studied. Bernstein and Federgruen [2005] consider one manufacturer and multiple retailers who compete by choosing their retail prices. They assume that the demand faced by each retailer is stochastic with a distribution that depends on the retail prices of *all* retailers. They show that there exists a class of *nonlinear* price-discount sharing contracts that coordinate the supply chain. Cachon and Lariviere [2005] identify a class of revenue-sharing contracts that coordinate the supply chain with one manufacturer and competing retailers. These contracts require that the retailers share a fixed proportion of their revenues with the manufacturers, the wholesale price is set equal to a fixed proportion of the manufacturing cost, and every retailer pays an amount equal to the externality its order quantity imposes on its fellow competing retailers. Netessine and Zhang [2005] consider a supply chain with one manufacturer and competing retailers who face an exogenously determined retail price and a stochastic demand whose distribution depends on the order quantities of all retailers. They compare the situations where the retailers compete on substitute versus complement products, and find that while competition on complements exacerbates the double marginalization effect, competition on substitutes alleviates it.[1]

We consider a model of competition in the supply chain where, in addition to multiple competing retailers, there are also *multiple competing manufacturers*. This is a first step towards bridging the gap between much of the academic supply chain literature, which focuses on one-manufacturer-one-

[1] Other papers in the literature study supply chains with one manufacturer and multiple retailers. Lee and Whang [2002] study how a secondary market where retailers trade inventory alters the equilibrium. Li [2002] studies the incentives for firms to share information vertically. Tyagi [1999] considers the impact of downstream entry on supply chain performance.

retailer, or, at most, one-manufacturer-many-retailer supply chains, and the much more complex supply chains encountered in practice; see, for instance, Netessine [2009]. Moreover, our model also captures other features that have been ignored in much of the existing literature: product and retailer differentiation, and retailer risk aversion. Specifically, our model considers multiple manufacturers who compete in quantities to supply a set of differentiated products to multiple differentiated risk-averse retailers who compete in quantities to satisfy uncertain consumer demand. Each *manufacturer* chooses its supply quantity for each of the products in order to maximize its profit, anticipating the reaction of the retailers (that is, their order quantities) and anticipating also the wholesale prices resulting from the market clearing conditions. Each *retailer* chooses its order quantity for the wholesale market for each of the products in order to maximize its expected utility from sales in the retail market. Consumer demand is characterized by a stochastic linear inverse demand function that captures both product and retailer differentiation and models uncertainty via a multiplicative noise.

Our model is closely related to that proposed by Corbett and Karmarkar [2001] (herein C&K), who consider simultaneous quantity competition at multiple tiers in the supply chain. There are several important differences between our model and their model. On the one hand, our model is more general than the C&K model in two respects. First, while the C&K model assumes that there is a unique homogeneous product and there is no retailer differentiation, our model allows for both product and retailer differentiation. Second, while the C&K model assumes the demand is deterministic, our model assumes that demand is stochastic and the retailers may be risk averse. On the other hand, our model is more specific than the C&K model in one dimension: while the C&K model considers the general case where there is competition at multiple tiers in the supply chain, we restrict our model to the case with only two tiers. Another important difference between our paper and the C&K paper is the objective: while C&K study entry decisions in a supply chain, our work focuses on how the decentralized equilibrium and its efficiency depend on the structure and characteristics of the supply chain.

Our contribution is fourfold. Our first contribution is to propose a novel model of competition in the supply chain that captures simultaneous competition among manufacturers and retailers, product and retailer differentiation, and retailer risk aversion. Our model is a stylized economic model and hence some of its assumptions are unlikely to hold in a strict sense in real-world supply chains, but we think that the insights from our model, just like those from other related stylized models such as the one considered by Corbett and Karmarkar [2001], can help us improve our understanding of the effect of competition in real-world supply chains. Moreover, there are a few real-world situations that are close to our model, and that may help to motivate our research. This is the case of supply chains manufacturing and distributing certain standardized commodities such as printer paper, light bulbs, or construction materials. For instance, multiple UK retailers such as Euroffice and Ukofficedirect offer multiple *standardized* copier paper products (e.g., white A4 paper with weights of 80, 90, 100, and 120 grams per square meter) manufactured by multiple manufacturers such as HP, Canon, and E4. Arguably, most consumers would consider the products from the different manufacturers identical. Moreover, there are many other situations where most consumers would not consider the products from different manufacturers identical, but they would consider them very similar, and hence the insights from our model may still be useful. This category includes supply chains for products such as toothpaste, mineral water, or milk. For instance, Evian, Volvic, and Vittel all produce still and sparkling mineral water in plastic bottles of identical capacity, and multiple UK retailers such as Ocado and Tesco distribute the nearly identical products from all three producers.

Our second contribution is to study (analytically) the symmetric supply chain. We show that under mild assumptions there exists a unique equilibrium in the symmetric chain, we give closed-form expressions for the equilibrium, and we perform comparative statics. When studying the effect of *increasing the number of retailers*, we show that the results given by Corbett and Karmarkar [2001] hold also for the more general case with product differentiation, stochastic demand, and risk averse retailers. Specifically, we show that in the symmetric supply chain with either two manufacturers and three nondifferentiated retailers, or three manufacturers and two nondifferentiated retailers, the decentralized chain is efficient. We then show that adding *retailer differentiation* to the model yields an interesting generalization of this result. Specifically, we show that, provided that there is a sufficiently large number of manufacturers and retailers, efficiency one can always be achieved by inducing the right degree of retailer differentiation in the symmetric supply chain (for instance, by using customer loyalty programs). *Retailer risk aversion* leads to *triple marginalization*: even perfectly competitive retailers require a strictly positive margin to participate in the retail market, as otherwise they would not be willing to undertake the risk associated with uncertain demand. Finally, we note that neither risk aversion nor product differentiation affect the efficiency of the *symmetric* supply chain because in the symmetric chain all manufacturers produce (and all retailers distribute) each product in the same quantity, and hence the relative effect of risk aversion and product differentiation on the utility of the decentralized and centralized chains is the same.

Our third contribution is to study (numerically) the asymmetric supply chain. We first show how the equilibrium conditions for the asymmetric chain can be reformulated as a so-called *equilibrium problem with equilibrium constraints*. We then use numerical methods to study the properties of the equilibrium for the asymmetric case. Our first insight is that the efficiency of the decentralized supply chain may drop rapidly with the asymmetry of either manufacturers or retailers. The reason for this is that while in the centralized supply chain only the cheapest manufacturer produces and only the least risk averse retailer distributes, in the decentralized chain even the most expensive manufacturers may produce and the most risk-averse retailers may distribute because of competition. Our second insight is that, not surprisingly, the asymmetry of product assortments at different retailers leads to a decrease in the degree of competition among retailers, and thus it has a similar effect on the equilibrium quantities as a reduction in the number of retailers. Our third insight is to show that the efficiency of the decentralized *asymmetric* chain depends on both retailer risk aversion and product differentiation because of the interaction between these features and the asymmetry of manufacturers and retailers.

Our fourth contribution is to show that there exists a class of revenue sharing contracts—similar to those studied by Cachon and Lariviere [2005] for the case with one manufacturer and competing retailers—that may be used to coordinate the symmetric supply chain with multiple manufacturers and retailers. Our purpose for studying these contracts is theoretical: to improve our understanding of the nature of competition in a supply chain with multiple manufacturers and retailers. We do not advocate, however, the practical implementation of these complex contracts.

Our work is related to other papers in the literature. Perakis and Roels [2007] study competition under price-only contracts, where it is well known that coordination cannot be achieved, and so they focus on studying the efficiency of a decentralized supply chain with one manufacturer and multiple retailers. They also consider the case with multiple manufacturers and one retailer, but they do not consider the case where there is simultaneous competition between multiple manufacturers and multiple retailers. Choi [1991] studies price competition between two manufacturers that produce substitute products. He compares the situation where a single retailer distributes the products of both manufacturers, to that where each manufacturer uses an exclusive retailer. Cachon and Kok [2009] consider two manufacturers who compete to supply a single retailer, and study the relative merits of the wholesale-price, quantity discount, and two-part tariff contracts. They conclude that, in the presence of competing manufacturers, a retailer may favor the quantity discount and two-part tariff contracts to the wholesale-price contract, a result that contrasts with the results obtained in the literature for the case with a single manufacturer. Our work complements that of Choi [1991] and Cachon and Kok [2009] by considering also competition among retailers, in addition to competition among manufacturers. Finally, Majumder and Srinivasan [2008] study network supply chains, which are large interconnected supply chains that are not limited to one or two tiers, but rather may be composed by multiple interlinked tiers. They study the effect of contract leadership and leader position on the supply chain efficiency. They also study competition between supply chain networks. Their work differs substantially from ours because we focus on two-tier chains, where there is competition in both tiers, while Majumder and Srinivasan [2008] focus on general supply chain networks, where there may be multiple tiers, and competition occurs between separate networks.

The rest of this paper is organized as follows. Section 2 states the model. Section 3 gives closed-form expressions for the unique equilibrium in the symmetric chain. Section 4 performs comparative statics on the symmetric chain equilibrium. Section 5 studies (numerically) the asymmetric chain. Section 6 concludes. The electronic companion to this paper contains four appendices. Appendix A gives a premier on complementarity problems, Appendix B gives some auxiliary results, Appendix C gives the proofs for all results in the manuscript, and Appendix D discusses the coordinating revenue sharing contracts.

2. A model of competition among manufacturers and retailers

Our model considers M manufacturers that compete to supply P products to N risk-averse retailers who compete to satisfy uncertain consumer demand. Each *manufacturer* chooses its supply quantity for each of the P products in order to maximize its profit, anticipating the reaction of the retailers (that is, their order quantities) and anticipating also the wholesale prices resulting from the market clearing conditions. Each *retailer* chooses its wholesale market order quantity for each of the P products in order to maximize its expected utility from retail sales, where the expectation is taken with respect to the distribution of the retail demand function. The remainder of this section gives a detailed description of our model.

2.1. The retail market demand

We assume retail demand satisfies the following stochastic linear inverse demand function:

$$\hat{p} = (\hat{a} - \hat{B}\hat{x})\epsilon,$$

where $\hat{p} \in \mathbb{R}^{NP}$ is the aggregate price vector, given by $\hat{p} = (p_1, p_2, \ldots, p_N)$, where p_j is the vector of the jth retailer prices with $p_j = (p_{j1}, p_{j2}, \ldots, p_{jP})$, p_{jk} is the price of the jth retailer for the kth product, and ϵ is a positive scalar random variable with mean one and finite standard deviation σ. The vector $\hat{a} \in \mathbb{R}^{NP}$, where $\hat{a} = (a_1, a_2, \ldots, a_N)$ and $a_j = (a_{j1}, a_{j2}, \ldots, a_{jP})$, gives the prices that the consumers would be willing to pay if the retail market supply was zero and the random variable $\epsilon = 1$. The matrix $\hat{B} \in \mathbb{R}^{NP \times NP}$ is the matrix of inverse demand sensitivities. The vector $\hat{x} \in \mathbb{R}^{NP}$ is the aggregate retailer order vector, given by $\hat{x} = (x_1, x_2, \ldots, x_N)$, where x_j is the jth retailer order vector $x_j = (x_{j1}, x_{j2}, \ldots, x_{jP})$.

Linear demand models have been widely used both in the Economics literature (see, for instance, Singh and Vives [1984] and Häckner [2003]) as well as in the Operations Management literature (Goyal and Netessine [2007], Farahat and Perakis [2008a], and Farahat and Perakis [2008b]). In fact, they have been considered in the context of supply chain competition. Cachon and Lariviere [2005] mention that the results for their model with competing retailers can be applied for the particular case of “Cournot competition with deterministic linear demand”, and they give as an example a deterministic linear inverse demand function for a single homogeneous product with retailer differentiation. The model studied by Corbett and Karmarkar [2001] assumes a deterministic linear inverse demand function for a single homogeneous product without retailer differentiation. Our linear inverse demand model extends the model in Corbett and Karmarkar [2001] in two respects: first, our model captures demand uncertainty, and second, our model allows both product and retailer differentiation.

We model *demand uncertainty* in a multiplicative form, through the scalar random variable ϵ, which appears in the linear inverse demand function multiplying the term $\hat{a} - \hat{B}\hat{x}$. We assume that ϵ is positive and has a mean one and standard deviation σ. This implies that $E[\hat{p}] = \hat{a} - \hat{B}\hat{x}$ is the *expected* inverse demand function, and the realized value of ϵ determines how far the realized demand function is from the expected demand function. Bernstein and Federgruen [2005] also model demand uncertainty using a multiplicative form, although their results do not require linearity of the demand function.

Our demand model captures *product and retailer differentiation* through the structure of the matrix of inverse demand sensitivities $\hat{B}$. The matrix $\hat{B}$ can be written as:

$$\hat{B} = \begin{pmatrix} H_1 & G_{12} & \dots & G_{1N} \\ G_{21} & H_2 & \dots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \dots & H_N \end{pmatrix},$$

and hence we have $p_j = (a_j - H_j x_j - \sum_{l \neq j} G_{jl} x_l)\epsilon$. *Product differentiation* is captured in our model if $(H_j)_{k_1 k_1} \neq (H_j)_{k_1 k_2}$; that is, if, at the jth retailer, the sensitivity of the k_1-th product price to the k_1-th product supply, is different from the sensitivity of the k_1-th product price to the k_2-th product supply, or if $(H_j)_{k_1 k_1} \neq (H_j)_{k_2 k_1}$; that is, when at a given retailer, a change in the supply of a given product has a different impact on the price of that same product and on the price of a *different* product. *Retailer differentiation* is captured in our model if $(H_{j_1})_{k_1 k_2} \neq (G_{j_1 j_2})_{k_1 k_2}$; that is, when the sensitivity of the k_1-th product price at the j_1-th retailer to the k_2-th product supply at the j_1-th retailer is different from the sensitivity of the k_1-th product price to the k_2-th product supply at the j_2-th retailer, or if $(H_{j_1})_{k_1 k_2} \neq (G_{j_2 j_1})_{k_1 k_2}$; that is, a change in the supply of a product at a given retailer affects differently the price of the product at the same retailer and the price of the product at another retailer.

2.2. The retailers

The profit of the jth retailer can be written as

$$-x_j^T v + x_j^T (a_j - \hat{B}_{j\bullet} \hat{x})\epsilon,$$

where $v = (v_1, v_2, \dots, v_P)$ is the vector of wholesale prices and $\hat{B}_{j\bullet} = (G_{j1}, \cdots, H_j, \cdots, G_{jN})$. The first term in the jth retailer profit is the wholesale cost of its order quantity ($x_j^T v$), and the second term is the retailer's revenue as a function of its own order quantity and the order quantities of the other retailers; that is, the scalar product of the order vector x_j and the retail market clearing price $(a_j - \hat{B}_{j\bullet} \hat{x})\epsilon$. Note that the only uncertain component in the jth retailer profit is the random variable ϵ, which determines the realized demand function, while the wholesale price v is deterministic, and the retailer order vector x_j is a decision that retailers have to make before the uncertainty about the retail demand function is resolved.[2]

[2] Note, however, that the uncertainty in the retail demand function does not result in demand-supply mismatches because the retail price is set to clear the retail market. We choose to model demand uncertainty in this manner because we focus on how demand uncertainty and retailer risk aversion affect the equilibrium in the decentralized chain, rather than on inventory management strategies geared to address demand-supply mismatches.

We assume that each retailer is risk averse and its preferences are represented by a mean-standard-deviation utility function. Specifically, we assume that given a wholesale price v and the order quantities of the other retailers, the jth retailer chooses its order quantity x_j to maximize a combination of the mean and standard deviation of its profit:

$$\max_{x_j \geq 0} \quad -x_j^T v + x_j^T(a_j - \hat{B}_{j\bullet}\hat{x}) - \gamma_j x_j^T(a_j - \hat{B}_{j\bullet}\hat{x})\sigma, \tag{1}$$

where the first two terms in the jth retailer objective function are the jth retailer *mean* profit, and the third term is the *standard deviation* of the jth retailer profit multiplied by its risk aversion parameter γ_j. Summarizing, the jth retailer chooses an order quantity that leads to high mean and low standard deviation of profits. The tradeoff between the mean and the standard deviation can be adjusted by changing the risk aversion parameter.

A few comments are in order. First, note that most of the existing supply chain literature assumes retailers are risk neutral. This is indeed the case for the three papers that are more closely related to our work: Corbett and Karmarkar [2001], Bernstein and Federgruen [2005], and Cachon and Lariviere [2005]. We think the risk neutrality assumption makes sense in the context of these papers because they study single-product models, which potentially capture only a fraction of the overall retailer profits. We think the risk aversion assumption makes more sense in the context of our model, which considers multiple products and may represent the entire retailer product portfolio, and hence may be used to capture all of the retailer profits. In other words, it makes more sense to assume that the retailers will be risk averse when making decisions about their entire product portfolio than when making decisions about a single product. We must note, however, that our model can also be used to study the case where the retailers are risk neutral by setting the retailer risk-aversion parameter equal to zero.[3]

Second, we use the mean-standard-deviation utility function because of its widespread use in industry and academics. Ever since Markowitz [1952] presented his seminal work on portfolio selection[4], the mean-standard-deviation utility function has been one of the most popular approaches to model risk aversion because of its simplicity and tractability. This is particularly evident in the context of our model where we can rewrite the jth retailer decision simply as:

$$\max_{x_j \geq 0} \quad -x_j^T v + \eta_j x_j^T(a_j - \hat{B}_{j\bullet}\hat{x}), \tag{2}$$

[3] There are a few papers that consider risk aversion in one-manufacturer-one-retailer supply chains. The seminal paper by Eeckhoudt et al. [1995] studies the risk averse newsboy. Agrawal and Seshadri [2000] extend their framework to include, in addition, pricing decisions. Gan et al. [2005] investigate how a supply chain involving a risk-neutral supplier and a downside-risk-averse retailer can be coordinated with a supply contract. Mieghem [2007] extends the model to a newsvendor network.

[4] Note that Markowitz originally used a mean-variance utility function, but it is easy to show that there is a one-to-one correspondence between the mean-standard-deviation utility and mean-variance utility. To see this, note that it is easy to show using penalty method theory that the third term in the mean-standard-deviation utility function (1) can be replaced by a constraint on the profit standard deviation $(x_j^T(a_j - \hat{B}_{j\bullet}\hat{x})\sigma < \delta)$, for a certain threshold δ. Also, this constraint is equivalent to the following constraint on the profit variance $(x_j^T(a_j - \hat{B}_{j\bullet}\hat{x})\sigma)^2 < \delta^2$.

where $\eta_j = 1 - \gamma_j \sigma$. Note that the jth retailer would optimally choose a zero order unless $\eta_j > 0$, and hence to avoid trivial equilibria herein we assume $\eta_j > 0$. The parameter η_j captures the intensity of the impact of demand uncertainty on the utility of the jth retailer. If the risk-aversion parameter and the uncertainty in the demand function are relatively large, then η_j will be close to zero, and the impact of the demand uncertainty on the jth retailer utility will be large. If, on the other hand, the risk-aversion parameter and the uncertainty in the demand function are small, then η_j will be close to one and the impact of uncertainty will be small. Finally, if the jth retailer is risk-neutral ($\gamma_j = 0$) or the demand is deterministic ($\sigma = 0$), then $\eta_j = 1$ and the uncertainty has no impact on the jth retailer's decisions.

Third, we assume there is no manufacturer differentiation; that is, we assume all manufacturers sell each of the P products at the same wholesale market price. We make this assumption for tractability, but note nonetheless that assuming there is no manufacturer differentiation is more realistic than assuming there is no retailer differentiation. The reason for this is that order quantities in wholesale markets are typically much larger than those in retail markets, and hence one may expect that retailers would be less willing to pay different prices for the same product from different manufacturers than consumers, who may be willing to pay a higher price for the same product to obtain it from a retailer that is more conveniently located than others.

2.3. The manufacturers

The ith manufacturer chooses its supply quantity to maximize its profit from sales in the wholesale market, assuming the rest of the manufacturers keep their supply quantities fixed, and anticipating the market clearing wholesale market price as well as the retailer order quantities. In game-theoretic terms, each manufacturer acts as a Cournot[5] player with respect to the other manufacturers, but as a Stackelberg player with respect to the retailers. Mathematically, the ith manufacturer faces the following decision:

$$\max_{y_i \geq 0, v} \quad v^T y_i - c_i^T y_i \tag{3}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} y_k = \sum_{j=1}^{N} x_j(v), \tag{4}$$

where M is the number of manufacturers, $c_i \in \mathbb{R}^P$ is the ith manufacturer unit cost, and $x_j(v) : \mathbb{R}^P \to \mathbb{R}^P$ is the jth retailer order quantity for a given wholesale price v—we will give conditions

[5] We assume in our model that there is no manufacturer differentiation, and hence, we also assume manufacturers compete in quantities as price competition among manufacturers would lead to a trivial equilibrium where wholesale prices would be equal to the manufacturing cost of the cheapest manufacturer. Our demand model allows, however, for retailer differentiation, and hence we could consider both quantity as well as price competition among retailers. We choose to model quantity competing retailers because we are interested (among others) in the important particular case where there is no retailer differentiation, and in this particular case, price competition would lead to a trivial equilibrium where the utility of every retailer is zero. Corbett and Karmarkar [2001] also assume quantity competition at all tiers in their model.

later under which the functions $x_j(v)$ are well defined. Note that the objective function (3) is the ith manufacturer's profit, whereas constraint (4) represents the wholesale market clearing conditions.[6]

Finally, note that in our model all the uncertainty takes place in the retail market, and that the wholesale market is deterministic. The reason for this is that in our model all wholesale market related decisions are made before the uncertainty in the retail demand function is realized. As a result, the objective function of the manufacturer is simply its profit, which is deterministic, and thus the manufacturer risk aversion does not play a role in the decentralized chain.

3. The symmetric supply chain

In this section we focus on the case where the supply chain is symmetric with respect to all manufacturers and retailers. Specifically, we assume all manufacturers have identical costs, all retailers have identical risk-aversion parameters, and the linear inverse demand function is symmetric with respect to all retailers. These conditions are rigorously stated in the following assumption.

Assumption 3.1 (Symmetric model). The manufacturer costs satisfy $c_i = c$ for all i, the retailer risk-aversion parameters satisfy $\gamma_j = \gamma$ for all j, the inverse demand intercepts satisfy $a_j = a$ for all j, and the matrix of inverse demand sensitivities satisfies $H_j = H$ for all j and $G_{j_1 j_2} = G$ for all $j_1 \neq j_2$.

3.1. The centralized solution in closed form

We first consider the case where there is a central planner whose objective is to maximize the aggregate expected utility of all manufacturers and retailers. We show that under mild assumptions there exists a unique optimal retailer supply schedule in the centralized supply chain, and we give a closed-form expression for the optimal retailer supply schedule.

Let the matrix $D = (H + (N-1)G)/N$. We will make use of the following assumptions.

Assumption 3.2. The matrix $\hat{B}$ is positive definite.

Assumption 3.3. The following inequality holds $D^{-1}(a - c/\eta) > 0$, where $\eta = 1 - \gamma\sigma$.

The following theorem gives closed-form expressions for the optimal retailer supply schedule.

Theorem 3.4. Let Assumptions 3.1, 3.2, and 3.3 hold, then

1. there is a unique retailer supply schedule that maximizes the expected utility of the central planner,

[6] In the mathematical formulation of the problem, both the supply quantity y_i and the wholesale market price v appear as optimization variables, but note that only the supply quantity y_i is a free decision of the ith manufacturer. The wholesale market price is implicitly determined by the wholesale market clearing constraint in Equation (4), as a function of the supply quantities y_k. Moreover, the exact same wholesale market clearing condition (4) is imposed on the decision problems of all manufacturers, and hence they all face the same wholesale market price.

2. the supply schedule is such that every retailer orders the same quantity vector x and the total supply is

$$\text{Total supply} = Nx = \frac{1}{2}D^{-1}\Big(a - \frac{c}{\eta}\Big), \tag{5}$$

3. the expected retail price at every retailer is $E[p] = (1/2)(a + c/\eta)$,
4. the centralized supply chain total expected utility is $\text{CTU} = (\eta/4)(a - c/\eta)^T D^{-1}(a - c/\eta)$,
5. the centralized expected consumer surplus is $\text{CS} = (1/8)(a - c/\eta)^T D^{-1}(a - c/\eta)$.

Assumption 3.3 implies that the order quantities given by equation (5) are strictly positive for all products. Basically, the assumption requires that the vector of prices at zero supply a, the manufacturing cost c, and the parameter η are such that it is profitable to supply a positive amount of all products in the centralized chain. This is not a very strong assumption because it holds provided we remove from the market all "unprofitable" products.

Note that Assumption 3.2 does not hold for the case without retailer differentiation; that is, for the case where $H = G$ and thus $D = H$. The following proposition, however, shows that although the optimal retailer supply schedule may not be unique in the case without retailer differentiation, the schedule given in Theorem 3.4 is one of the multiple centralized schedules that have identical total supply, expected retail price, and total expected utility, provided H is positive definite.

Assumption 3.5. The matrix H is positive definite.

Proposition 3.6. Let Assumptions 3.1, 3.3, and 3.5 hold. In addition, assume there is no retailer differentiation ($H = G$), then

1. the following retailer supply schedule maximizes the expected utility of the central planner: every retailer orders the same quantity vector x and the total demand is $Nx = (1/2)H^{-1}(a - c/\eta)$,
2. the expected retail price at every retailer is $E[p] = (1/2)(a + c/\eta)$,
3. the centralized supply chain total expected utility is $CTU = \eta\theta/4$, where $\theta = (a - c/\eta)^T H^{-1}(a - c/\eta)$,
4. and the centralized expected consumer surplus is $CS = \theta/8$.

3.2. The decentralized equilibrium in closed form

We now focus on the decentralized supply chain where M manufacturers and N retailers compete to maximize their individual utilities.

3.2.1. Assumptions. We need the following assumption to prove the existence and uniqueness of equilibrium in the symmetric decentralized supply chain.

Assumption 3.7. The matrix $\hat{B} + \hat{H}$ is positive definite, where

$$\hat{B} = \begin{pmatrix} H & G & \dots & G \\ G & H & \dots & G \\ \vdots & \vdots & \ddots & \vdots \\ G & G & \dots & H \end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} H & 0 & \dots & 0 \\ 0 & H & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H \end{pmatrix}.$$

Note that Assumption 3.7 is weaker than Assumption 3.2 required to show the existence and uniqueness of a centralized optimal supply schedule.[7]

3.2.2. Retailers equilibrium. The following proposition shows that the retailer equilibrium is unique, symmetric, and can be found as the solution to a linear complementarity problem (LCP); see Appendix A for a premier on complementarity problems, and Cottle et al. [2009] for an in-depth treatment.

Proposition 3.8. Let Assumptions 3.1 and 3.7 hold, then

1. there exists a unique retailers equilibrium,
2. the retailers equilibrium is symmetric,
3. the order quantity of each retailer is the unique solution to the following LCP:

$$0 \leq \frac{v}{\eta} - a + (2H + (N-1)G)x \perp x \geq 0, \tag{6}$$

where given $s, t \in \mathbb{R}^n$, the expression $0 \leq s \perp t \geq 0$ means that for each component $k = 1, 2, \ldots, n$, we must have that either $s_k = 0$ or $t_k = 0$, and both s_k and t_k must be nonnegative.

Let the matrix $B = (2H + (N-1)G)/(N+1)$. To give a closed-form expression for the retailer order quantities in equilibrium, we need to make one more assumption.

Assumption 3.9. The following condition holds at equilibrium $B^{-1}(a - v/\eta) > 0$.

Proposition 3.10. Let Assumptions 3.1, 3.7, and 3.9 hold. Then, there exists a unique retailer equilibrium at which every retailer orders the following quantity:

$$x = \frac{1}{N+1} B^{-1}\left(a - \frac{v}{\eta}\right). \tag{7}$$

Assumption 3.9 implies that the order quantities given by equation (7) are strictly positive for all products. Basically, the assumption requires that the vector of prices at zero supply a, the wholesale price v, and the parameter η are such that it is profitable to supply and sell a positive amount of all products. This is not a very strong assumption because it holds provided we remove from the market all "unprofitable" products.

3.2.3. Manufacturer-retailer equilibrium. The following theorem gives closed-form expressions for the equilibrium quantities in the symmetric supply chain.

Theorem 3.11. Let Assumptions 3.1, 3.7, and 3.9 hold, then there exists a unique manufacturer retailer equilibrium, at equilibrium every manufacturer supplies the same quantity y, every retailer orders the same quantity x, and the total supply is

$$\text{Total supply} = My = Nx = \frac{MN}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right).$$

[7] To see this, note that if $\hat{B}$ is positive definite, then any of its diagonal blocks must be positive definite, and hence, H must be positive definite. Positive definiteness of H implies that $\hat{H}$ is positive definite, and hence, $\hat{B} + \hat{H}$ is also positive definite.

Moreover, the wholesale price, expected retail price, manufacturer profit, retailer utility, aggregate utility, and consumer surplus are given by the closed-form expressions in Theorem C.1 in Appendix C (page 38).

An important particular case is the case without retailer differentiation; that is, $H = G$ and thus $B = D$. Proposition 3.12 shows that the closed-form expressions in Theorem C.1 can be further simplified for this particular case, and the first column (labeled "General (M, N)") in Table 1 gives the simplified expressions.

Proposition 3.12. Let the assumptions of Theorem 3.11 hold, and assume also that there is no retailer differentiation (that is, $H = G$), then there exists a unique manufacturer retailer equilibrium, at which every manufacturer supplies the same quantity, and every retailer orders the same quantity. Moreover, the equilibrium quantities for this case are as stated in the first column (labeled "General (M, N)") in Table 1.

The following proposition gives the closed-form expression for the supply chain efficiency for the case without retailer differentiation. Moreover, it also establishes the monotonicity properties of the equilibrium quantities and the supply chain efficiency as functions of the number of manufacturers and retailers for the supply chain without retailer differentiation. Table 2 summarizes these monotonicity properties.

Proposition 3.13. Let the assumptions in Propositions 3.6 and 3.12 hold, then the supply chain efficiency is

$$\text{efficiency} = \frac{4MN(M+N+1)}{(M+1)^2(N+1)^2}. \tag{8}$$

If, in addition, $a - c/\eta > 0$, then the monotonicity properties of the equilibrium quantities and the supply chain efficiency as functions of the number of manufacturers and retailers are as stated in Table 2.

Note that the assumption that $a > c/\eta$ is not very strong, because it only requires that the vector of prices the customers would be willing to pay for the products assuming the aggregate supply was zero for all products and $\epsilon = 1$ (that is, a) is larger than a multiple of the manufacturing cost vector c.[8]

4. Discussion of the symmetric supply chain

To gain understanding about the decentralized supply chain, we now discuss how the decentralized equilibrium quantities for the symmetric chain depend on the number of manufacturers and retailers, the degree of product and retailer differentiation, and the retailer risk aversion.

[8] Also, this assumption is only needed to show the monotonicity properties of the wholesale market price and the expected retail market price, while all other monotonicity results (including those of the supply chain efficiency) hold even if this assumption does not hold.

4.1. Number of manufacturers and retailers

We focus on the case where there is no retailer differentiation because in this case we can carry out a fair comparison between supply chains with different number of retailers. The reason for this is that in the absence of retailer differentiation, the aggregate demand function (aggregated over the multiple retailers) depends only on the retail price and not on the number of retailers. This result is rigorously established in the following proposition.

PROPOSITION 4.1. Let Assumptions 3.1 and 3.5 hold, and assume also that there is no retailer differentiation, then the aggregate demand function depends only on the retail price and not on the number of retailers in the supply chain.

For the remainder of this section we assume that the assumptions in Propositions 3.6, 3.12, and 3.13 hold. Tables 1 and 2 compile the results from these three propositions.

4.1.1. One manufacturer and one retailer. We first compare the decentralized equilibrium for the case with one manufacturer and one retailer with the centralized solution. We observe that, as a result of double marginalization, the aggregate supply in the decentralized supply chain with one manufacturer and one retailer is half of the supply in the associated centralized chain. Also, assuming $a > c/\eta$, the expected retail price is higher in the decentralized chain than in the centralized chain. In addition, the expected aggregate utility in the decentralized chain with one manufacturer and one retailer is 25% lower than that in the centralized supply chain. We also note that the manufacturer profit is higher (specifically, double) than the expected retailer utility—the explanation for this is that the manufacturer has the advantage of being the leader in the Stackelberg game; see [Vives 1999, Section 7.4]. Finally, we note that the expected consumer surplus in the decentralized chain is one fourth of that in the centralized chain; that is, double marginalization has a strong negative impact on consumer surplus.

4.1.2. M **manufacturers and** N **retailers.** We use the results in Table 2, which are proved in Proposition 3.13, to analyze the dependence of the decentralized equilibrium on the number of manufacturers and retailers.

The aggregate supply increases when either the number of manufacturers or retailers increases; that is, whenever the degree of competition among manufacturers or retailers increases. Also, the wholesale price decreases when the number of manufacturers increases, but it does not depend on the number of retailers. The explanation for this is that the retailers are price takers with respect to the wholesale market price, and hence the wholesale market price depends only on the intensity of competition among the manufacturers and not among retailers. The *expected retail price* decreases whenever we increase either the number of manufacturers or retailers. Essentially, an increase in the number of manufacturers leads to a decrease of the wholesale price and part of this decrease is passed on by the retailers to the final consumers. Also, even for fixed number of manufacturers, an increase in the number of retailers leads to lower retail prices because of the effect of increased competition.

With respect to *manufacturer profit, expected retailer utility, and consumer surplus*, we note that an increase in the number of manufacturers leads to a decrease in the aggregate manufacturer profit and an increase in the aggregate expected retailer utility; that is, retailers benefit from increased competition among manufacturers. Likewise, an increase in the number of retailers leads to a decrease in aggregate expected retailer utility and an increase in aggregate manufacturer profit. Finally, increasing either the number of manufacturers or retailers results in an increase in the expected consumer surplus; that is, consumers always benefit from an increased degree of competition regardless of whether it is among manufacturers or retailers.

We now focus on the *supply chain efficiency.* For the case with a monopolist manufacturer, the efficiency is strictly increasing in the number of retailers, and it converges to one as the number of retailers grows large. The intuition is that as the degree of competition among retailers increases, the market power of the monopolistic manufacturer increases, and hence the decentralized equilibrium gets closer to the centralized solution. Moreover, in the limit as the number of retailers grows large, the retailers become perfect competitors, their utility vanishes, and hence the efficiency is equal to one. Note that part of the retailers utility goes to the manufacturers, and part to the consumers, whose surplus increases monotonically with the number of retailers. On the other hand, when there are six or more manufacturers, the efficiency is decreasing in the number of retailers. The interpretation for this is that, when there are six or more manufacturers, the degree of competition among manufacturers is such that the benefit to them of an increased degree of competition between the retailers is smaller than the utility loss for the retailers. Finally, when the number of manufacturers is between two and five, the efficiency is strictly concave in the number of retailers, and achieves a unique maximum. This maximum efficiency is equal to one for the cases with two manufacturers and three retailers and three manufacturers and two retailers—basically, these are situations where there is a perfect balance between the degrees of vertical and horizontal competition and hence the decentralized chain is efficient. We observe identical symmetric insights for the case where we keep the number of retailers fixed and change the number of manufacturers. Finally, from column 6 in Table 1, we observe that in the supply chain with a large number of both manufacturers and retailers, the aggregate supply in the decentralized chain is larger than that in the centralized chain, and the aggregate expected utility is zero. This is a situation where both the manufacturers and the retailers face a situation of perfect competition, and hence both the manufacturer profit and the retailer utility vanish. Also, note that for this case, expected consumer surplus is very high (compared to the cases with a small number of manufacturers and retailers) because all supply chain value is captured by the consumers.

4.2. Retailer differentiation

Our analysis in the previous section confirms that the results in [Corbett and Karmarkar 2001, Proposition 2] also hold in the context of our model with differentiated products, stochastic demand, and risk-averse retailers. We now show how this result can be generalized for the case with retailer differentiation.

To study the effect of retailer differentiation in isolation from the effect of product differentiation, we focus on the case where there is a single product. We also assume that $H = 1$ and we analyze how changes in the value of G between 0 (the case when retailers are completely differentiated) and 1 (the case when there is no retailer differentiation) affect the decentralized equilibrium. Note that changing the value of G is equivalent to changing the demand function, and hence it is hard to compare the decentralized equilibria corresponding to different values of G. Instead, we focus on how the efficiency of the decentralized chain depends on the degree of retailer differentiation. We define $\nu = H - G = 1 - G$ as the degree of retailer differentiation—when $\nu = 0$, there is no retailer differentiation, whereas when $\nu = 1$ retailers are completely differentiated.

Proposition 4.2. Let the assumptions in Theorems 3.4 and 3.11 hold. Assume also that the number of retailers is greater than or equal to two ($N \geq 2$), there is a single product, and the matrix $H = 1$. Then

1. if $M = 1$ and $N < \infty$, or if $N = M = 2$, the supply chain efficiency is strictly decreasing in the degree of retailer differentiation ν and is strictly smaller than one. If $M = 1$ and $N = \infty$ then the efficiency is equal to 1 for all ν.
2. if $(N, M) = (2, 3)$ or $(N, M) = (3, 2)$, the supply chain efficiency is strictly decreasing in the degree of retailer differentiation ν, and the efficiency for the case without retailer differentiation is one,
3. otherwise (if $(N-1)(M-1) > 2$), then the supply chain efficiency is strictly concave in the degree of retailer differentiation, and it reaches its unique maximum equal to one for $\nu = 1 - 2/((M-1)(N-1))$.

The results of this proposition are illustrated in Figure 1. One way to interpret Proposition 4.2 is that increasing retailer differentiation has an effect on efficiency equivalent to that of decreasing the number of retailers. The intuition is that both increasing retailer differentiation and decreasing the number of retailers have the same effect: reducing the intensity of competition among retailers. For instance, for the case with a single manufacturer, we know from Section 4.1, that increasing the number of retailers increases the efficiency in the absence of retailer differentiation. In the case with one manufacturer and differentiated retailers, decreasing the degree of retailer differentiation also increases efficiency. It is easy to see that increasing retailer differentiation and decreasing the number of manufacturers also have the same impact on efficiency for the cases with $(N,M)=(2,3)$ or $(N,M)=(3,2)$ or the case $(N-1)(M-1)>2$.[9]

Finally, our analysis shows that retailer differentiation yields an interesting generalization of the result by Corbett and Karmarkar [2001] that two manufacturers and three retailers (or three manufacturers and two retailers) lead to efficient decentralized chains in the case without retailer differentiation. Specifically, our result shows that if there is a sufficiently large number of manufacturers and retailers $(N-1)(M-1)>2$, efficiency one can always be achieved by choosing the right degree of retailer differentiation. This result is similar in spirit to the result by Netessine and Zhang [2005] who show, in the context of their model, that under certain conditions a supply chain where the retailers compete on substitutable products can always be perfectly coordinated by choosing the wholesale price appropriately.

4.3. Product differentiation

To study the effect of product differentiation in isolation from the effect of retailer differentiation, we focus in this section on the case without retailer differentiation, and we study how changes in the matrix H affect the decentralized equilibrium. Note that changing the matrix H is tantamount to changing the demand function, and hence it is difficult to derive insights from comparing the decentralized equilibrium for different values of H. Instead, we compare the decentralized equilibrium to the centralized one, and see how their difference depends on the product differentiation, represented by matrix H.

Proposition 4.3. Let the assumptions in Propositions 3.6 and 3.12 hold, then:

1. the decentralized and centralized aggregate supplies (S_d and S_c, respectively) satisfy $S_d/S_c = 2MN/((M+1)(N+1))$,
2. the efficiency is $4MN(M+N+1)/((M+1)^2(N+1)^2)$,
3. the decentralized wholesale and expected retail prices do *not* depend on matrix H.

Proposition 4.3 shows that although the decentralized aggregate supply and utility in the *symmetric* chain depend on product differentiation, the ratio of these quantities to their centralized counterparts does not; that is, the supply chain efficiency in a *symmetric* chain is independent of the degree of product differentiation. The explanation for this is that, in the symmetric supply chain, all retailers distribute (and all manufacturers produce) identical quantities of each product (both in the decentralized and centralized chains), and as a result the relative impact of the degree

[9] Note also that from Part 3 of Proposition 4.2, it is easy to see that the degree of retailer differentiation $\nu = 1-G$ required to achieve efficiency one is increasing in the number of manufacturers and retailers. For fixed number of manufacturers, it makes sense that the degree of differentiation required to achieve efficiency one increases with the number of retailers, as we have just argued above that decreasing the number of retailers and increasing retailer differentiation both have the same effect of reducing the degree of competition among retailers. For fixed number of retailers, this result implies that an increased degree of competition among manufacturers can be compensated, from the efficiency perspective, by increasing retailer differentiation.

of product differentiation on the supply quantities and utility in the centralized and decentralized settings is the same. In Section D, however, we will show that in the *asymmetric* supply chain, efficiency may vary with the degree of product differentiation because of the interaction between the degree of product differentiation and the asymmetry of either manufacturers or retailers.

4.4. Retailer risk aversion and demand uncertainty

The following proposition characterizes the effect of the retailers risk aversion on the decentralized supply chain equilibrium.[10]

PROPOSITION 4.4. Let the assumptions in Theorem 3.11 hold, then

1. the aggregate supply, wholesale price, and retail price in a supply chain with risk-averse retailers with parameter $\eta = 1 - \gamma\sigma$ coincide with those in a supply chain with the same demand function, but with risk neutral retailers, and with manufacturer costs $\tilde{c} = c/\eta > c$,
2. the manufacturer profit, expected retailer utility, and aggregate expected utility in a supply chain with risk averse retailers with parameter $\eta = 1 - \gamma\sigma$ are equal to those in the chain with manufacturer costs $\tilde{c} = c/\eta$ and with risk neutral retailers multiplied by the factor $\eta < 1$.

Proposition 4.4 can be interpreted as retailer risk aversion leading to an additional level of marginalization. In a decentralized chain with risk-neutral retailers, there is double marginalization: both the manufacturers and the retailers must make a strictly positive margin and this leads to lower supply and lower aggregate profits. When, in addition, the retailers are risk averse, there is an additional level of marginalization, which has an effect equivalent to increasing the manufacturing cost from c to c/η.[11] The effect of *triple marginalization* is evident in Table 1, where we note that when there is an infinite number of risk-averse retailers, the expected retail price is strictly larger than the wholesale price (by a factor of $1/\eta$). The reason for this is that, even when the number of retailers is infinite and thus retailers are perfect competitors, a retailer still requires a strictly positive margin to undertake the risk associated with demand uncertainty.

The following corollary applies the result in Proposition 4.4 to the particular case where there is no retailer differentiation and products are perfectly differentiated[12], and shows that triple

[10] Note that η depends on the product of the retailer risk aversion γ and the standard deviation of the demand uncertain factor σ, and hence the comparative static analysis corresponding to γ and σ are similar. To conserve space, we only discuss the comparative static analysis with respect to the retailer risk-aversion.

[11] The triple marginalization effect also takes place for other specifications of the uncertainty in the inverse demand function. For instance, it is easy to see that if the stochasticity in the demand function arises from a multiplicative shock to the intercept (that is, if $\hat{p} = \hat{a}\epsilon - \hat{B}\hat{x}$), or from the following additive shock $\hat{p} = \hat{a} - \hat{B}\hat{x} + \epsilon e$, where e is the vector of ones, then the effect of stochastic demand and risk aversion on the equilibrium is also triple marginalization, except that the new (third) margin arises from a decrease in the demand function intercept a (that is, from a decrease in the price consumers are willing to pay for the products if the supply of every product is zero) rather than an increase in the manufacturing cost.

[12] Note that the assumption that products are perfectly differentiated is not particularly strong in the context of this section because we are interested in studying the effect of risk aversion, in isolation from the effect of product and retailer differentiation.

marginalization leads to lower aggregate supply, higher expected retail prices, and lower expected utility, but leaves supply chain efficiency unaltered. In a different context (a single risk-averse newsboy), Eeckhoudt et al. [1995] also give conditions under which risk aversion leads to lower order quantities.

Corollary 4.5. Let the assumptions in Propositions 3.6 and 3.12 hold, then:

1. the expected retail price in the supply chain with risk averse retailers is higher than in the equivalent supply chain with risk neutral retailers,
2. the efficiency of the decentralized chain with risk averse retailers is equal to that in the chain with risk neutral retailers,

if, in addition, products are perfectly differentiated (H is diagonal), then:

3. the aggregate supply in the chain with risk-averse retailers is lower than in the equivalent chain with risk neutral retailers,
4. the manufacturer profit, expected retailer utility, and expected aggregate supply chain utility is smaller than in the equivalent chain with risk neutral retailers.

Corollary 4.5 shows that in the *symmetric* supply chain the efficiency does not depend on the degree of retailer risk aversion. The explanation for this is that in the symmetric supply chain all the retailers distribute the same quantity of each product (both in the decentralized and centralized chains), and as a result the relative impact of risk aversion on the supply chain utility is the same in the decentralized and centralized settings. In Section D, however, we show that for the case with asymmetric retailers the efficiency depends on retailer risk aversion.

5. The asymmetric supply chain

We now focus on the general (asymmetric) supply chain. We will use a numerical approach to analyze the asymmetric supply chain, with two purposes: (i) to study how the asymmetry of the supply chain affects the equilibrium, and (ii) to check which of the insights obtained for the symmetric supply chain hold also for asymmetric chains.

Section 5.1 gives a summary of our main findings from the analysis of asymmetric supply chains. Section 5.2 describes the numerical approach we use to study the asymmetric supply chains. Section 5.3 gives the details of our analysis.

5.1. Summary of insights.

Our first insight is that the efficiency of the decentralized supply chain may drop rapidly when manufacturers are asymmetric (they have different costs), or when retailers are asymmetric (they have different risk aversion). The reason for this is that while in the centralized supply chain only the cheapest manufacturer produces and only the least risk averse retailer distributes, in the decentralized chain even the most expensive manufacturers may produce and the most risk-averse retailers may distribute because of competition, and this leads to a sharp decrease in efficiency.

Our second insight is that, not surprisingly, the asymmetry of product assortments at different retailers leads to a decrease in the degree of competition among retailers, and thus it has a similar effect on the equilibrium quantities as a reduction in the number of retailers.

Our third insight is to show which of the insights from the symmetric chain hold also for the asymmetric chains we consider, and which do not. We find that the following results obtained for the symmetric chain also hold for the asymmetric chains we consider: (i) in asymmetric chains with one manufacturer (retailer), the efficiency increases with the number of retailers (manufacturers), and (ii) the efficiency is a unimodal function of the degree of retailer differentiation. We find, on the other hand, that some of the results obtained for the symmetric chain do not hold in general for asymmetric chains: (i) efficiency does depend on retailer risk aversion provided the retailers are asymmetric, and (ii) efficiency may depend on the degree of product differentiation because of the interaction between product differentiation and the asymmetry of either asymmetric manufactures or asymmetric retailers.

5.2. Computing the manufacturer retailer equilibrium

We first show that, for the general (asymmetric) supply chain, there exists a unique retailers equilibrium under the following assumption, which is equivalent to Assumption 3.7 in the context of the asymmetric supply chain.

Assumption 5.1. The matrix $\hat{B}+\hat{H}$ is positive definite, where

$$\hat{B}=\begin{pmatrix} H_1 & G_{12} & \dots & G_{1N} \\ G_{21} & H_2 & \dots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \dots & H_N \end{pmatrix} \quad \text{and} \quad \hat{H}=\begin{pmatrix} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_N \end{pmatrix}.$$

Proposition 5.2. Let Assumption 5.1 hold, then there exists a unique retailers equilibrium, which is the unique solution to the following linear complementarity problem (LCP):

$$0 \le \hat{q}+(\hat{B}+\hat{H})\hat{x} \perp \hat{x} \ge 0, \tag{9}$$

where $\hat{q}=(v/\eta_1-a_1, v/\eta_2-a_2,\dots,v/\eta_N-a_N)$.

From Proposition 5.2, we know that the jth retailer equilibrium order quantity x_j can be expressed as an implicit function of wholesale price vector v. This formally justifies our formulation of the ith manufacturer decision as problem (3)–(4). For computational purposes, however, it is more practical to rewrite the ith manufacturer decision problem as the following mathematical program with equilibrium constraints (MPEC):

$$\max_{y_i,\hat{x},v} \quad v^T y_i - c_i^T y_i, \tag{10}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} y_k = \sum_{j=1}^{N} x_j, \tag{11}$$

$$0 \leq \hat{q} + (\hat{B} + \hat{H})\hat{x} \perp \hat{x} \geq 0, \tag{12}$$
$$y_i \geq 0. \tag{13}$$

Note that the objective function of problem (10)–(13) is the ith manufacturer profit, the first constraint (11) represents the wholesale market clearing conditions, and the second constraint (12) is the linear complementarity problem characterizing the retailers equilibrium. As explained in Appendix A, the term mathematical program with equilibrium constraints refers to the fact that we have imposed the LCP characterizing the retailers equilibrium as a constraint to the ith manufacturer problem.

A manufacturer retailer equilibrium is then a set of retailer order quantities and manufacturer production quantities that solve the MPECs representing the decisions of all M manufacturers. Equivalently, the manufacturer retailer equilibrium is the solution to the equilibrium problem with equilibrium constraints (EPEC) that is defined by grouping the MPEC problems of all manufacturers. To find the equilibrium, we use the numerical solver PATH (Ferris and Munson [1998]) to find a point that simultaneously satisfies the optimality conditions of the MPECs corresponding to the M manufacturers. This numerical approach is similar to that proposed by Hu [2002] and used, for instance, by Ehrenmann [2004], Leyffer and Munson [2005], Hu and Ralph [2007], DeMiguel and Xu [2009].

Note that although we have shown that under mild assumptions the retailers equilibrium is unique, we have not shown that the manufacturer retailer equilibrium is unique for the asymmetric chain. This is not surprising as it is unusual to be able to show uniqueness of equilibrium for asymmetric multiple leader multiple follower games; see DeMiguel and Xu [2009], and the references therein. In our numerical experiments, however, we have always found a single equilibrium, which suggests that the equilibrium is unique for the examples we consider.

5.3. Discussion

5.3.1. Asymmetric manufacturing costs. We consider a supply chain with four manufacturers, two retailers, and two products. We assume that there is no retailer differentiation. We use the following supply chain parameters[13]:

$$a_j = a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \forall j, \quad \text{and} \quad H_j = G_{jk} = H = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \quad \forall j, k \neq j. \tag{14}$$

We assume that the retailers are identical with $\eta_1 = \eta_2 = 0.9$, but the manufacturers are asymmetric; three of the manufacturers have identical unit production cost $c = (0.1, 0.1)$, but the unit production cost of the fourth manufacturer is $c_4 = C_4(1,1)$, where $C_4 \in \mathbb{R}$ ranges between 0 and 0.5.

Our main insight is that the efficiency of the decentralized supply chain may drop sharply with the asymmetry of the manufacturers because while in the centralized supply chain only the

[13] In this example, products are substitutable, however we found similar observations in the case of complementary products.

cheapest manufacturers produce, in the decentralized chain all manufacturers may produce. This is illustrated in the left plot in Figure 2, which depicts the supply chain efficiency as a function of the fourth manufacturer's cost C_4. The plot shows that the efficiency attains a local maximum for the symmetric case, when all four manufacturers have the same cost equal to 0.1, and has a global maximum equal to 1 for high values of C_4. The explanation for this is simple: when the fourth manufacturer's production cost is 0.1 for both products, all four manufacturers are identical, and hence every manufacturer produces the same quantity, both in the centralized and decentralized chains.[14] When the fourth manufacturer's cost is smaller than 0.1, in the centralized chain only the cheapest fourth manufacturer produces, but in the decentralized setting all four manufacturers still produce, and therefore the supply chain efficiency drops rapidly. When the fourth manufacturer's cost is larger than 0.1, in the centralized chain only the first three manufacturers produce, but not the fourth, which is the most expensive, while in the decentralized setting all four manufacturers still produce. Therefore, the efficiency also drops but less rapidly. When C_4 approaches 0.3, in the decentralized chain the fourth manufacturer produces less as its costs become higher, and as a result the supply chain efficiency increases. For C_4 beyond 0.3, the fourth manufacturer's cost becomes prohibitively expensive, and hence only the three symmetric manufacturers produce. Consequently, the efficiency of the supply chain becomes equal to one, because we know from Proposition 3.13 that the efficiency of a symmetric supply chain with three manufacturers and two nondifferentiated retailers is one.[15]

5.3.2. Asymmetric retailer risk aversion. We now consider the case with four retailers and two manufacturers. We also assume that there is no retailer differentiation. We consider supply chain parameters as in (14). We assume that the two manufacturers are symmetric with $c_1 = c_2 = (0.1, 0.1)$, but the retailers are asymmetric: three of them have identical parameter $\eta = 0.9$, but the fourth retailer has a different parameter η_4.

Our main insight is that the supply chain efficiency may drop sharply as a result of retailer asymmetry because while in the centralized supply chain only the least risk averse retailers order, in the decentralized chain all retailers may order. This is illustrated in the right plot in Figure 2, which depicts the efficiency as a function of η_4. The plot shows that the efficiency has a local maximum for the symmetric case, where all four retailers are identical, and has a global maximum equal to one for low values of η_4, when the fourth retailer has very high risk aversion. This makes sense: when $\eta_4 = 0.9$, all four retailers have the same risk aversion, and hence, their order quantities are identical, both in the the centralized or decentralized chains. When η_4 increases from 0.9, in the centralized chain, only the fourth retailer (the least risk averse) orders a positive quantity, but in the decentralized setting all four retailers order, and therefore the efficiency drops rapidly. If η_4

[14] Note that although every manufacturer produces the same quantity, this quantity is different in the decentralized and centralized chains.

[15] We also studied the effect of varying C_4 on the consumer surplus, and we found that the consumer surplus decreases as C_4 increases because consumers benefit from lower production costs at the manufacturer level.

decreases from 0.9, in the centralized chain, the first three retailers order, but not the fourth, which is the most risk averse, while in the decentralized setting all four retailers still order. Therefore, the efficiency also drops but less rapidly. When η_4 decreases towards 0.65, in the decentralized chain the fourth retailer orders less as it becomes more risk averse and as a result the supply chain efficiency increases. We observe that for η_4 below 0.65, the fourth retailer is extremely risk averse and chooses an order quantity equal to zero. As a result, there are in effect only three symmetric nondifferentiated retailers and two symmetric manufacturers, and we know from Proposition 3.13 that the efficiency is then one.[16]

Remark 5.3. Our analysis shows that, while in symmetric chains the supply chain efficiency is independent of the retailer risk aversion, in an asymmetric chain the efficiency may depend on the degree of retailer risk aversion. To see why, note first that in the symmetric supply chain, all retailers distribute the same quantity of each product (both in the decentralized and centralized chains), and hence the relative effect of risk aversion on the utility is the same in both settings. In the asymmetric chain, on the other hand, while in the centralized setting only the least risk averse retailer distributes, in the decentralized setting *all* retailers may distribute (because of competition), and hence the efficiency depends on the difference between the risk aversions of the different retailers.

5.3.3. Asymmetric retailer product assortments. We now consider the case with two retailers, three manufacturers, and two products. We also assume that there is no retailer differentiation. We consider supply chain parameters as in (14). We assume that the retailers have identical risk aversion parameter $\eta = 0.9$, and the three manufacturers have identical unit production cost $c = (0.1, 0.1)$, but we compare the equilibria for the cases when the two retailers carry asymmetric product assortments (the first retailer carries only the first product and the second retailer carries only the second product), and the case when the retailers carry symmetric assortments (both retailers may carry both products).

Figure 3 shows that introducing asymmetric product assortment leads to a decrease in supply quantities and an increase in retail prices, while leaving wholesale prices unaffected. Thus, the retailers' expected utility grows, and the manufacturers' profits and the consumer surplus decrease. The explanation for this is that in the case with asymmetric assortments, the degree of competition among the two retailers is smaller because they sell different products. As a result, the retailers choose to order smaller quantities, which results in higher retail prices, while they still pay the same wholesale prices as in the case with symmetric assortment.[17] Consequently, the expected retailer

[16] We also studied the effect of varying η_4 on consumer surplus, and we found that the consumer surplus increases as η_4 increases because consumers benefit from a a reduced degree of risk aversion among retailers.

[17] Following our discussion in Section 4.1.2, the reason why the wholesale price does not change with the introduction of asymmetric assortment is that the retailers are price takers with respect to the wholesale market price, and hence the wholesale market price depends only on the intensity of competition among the manufacturers and not among retailers.

utility is higher in the case with asymmetric assortment, while the manufacturers profits are lower. Since the quantities are lower and the retail prices higher, consumer surplus decreases. Essentially, introducing asymmetric assortment has the same effect as reducing the number of retailers: they both decrease the degree of competition between retailers.

Finally, the introduction of asymmetric assortments leads (in the case with three manufacturers and two retailers) to a decrease in the supply chain efficiency. This is not surprising as we know from Section 4 that with three symmetric manufacturers and two symmetric retailers, there is a precise balance of competition between manufacturers and retailers that leads to efficiency one. Introducing asymmetric assortment, breaks this balance by reducing the degree of competition among retailers, and hence efficiency drops.

5.3.4. Robustness checks for the symmetric chain results. In this section, we check which of the insights obtained for the symmetric chain hold also for the asymmetric supply chains we consider.

5.3.4.1. Number of manufacturers and retailers. We consider two cases: one manufacturer and an increasing number of asymmetric retailers, and one retailer and an increasing number of asymmetric manufacturers. From our computations we observe that, similar to the symmetric chain, for the case with a single manufacturer the efficiency is growing on the number of retailers, and for the case with a single retailer the efficiency is growing on the number of manufacturers.

We now describe our experiments in detail. For the case with one manufacturer and multiple *asymmetric* retailers, we consider cases with numbers of retailers ranging between one and eight, with two products, we use the demand function given in equation (14), and we set the manufacturing cost equal to $c = (0.1, 0.1)$. For each case of number of retailers from one to eight, we randomly generate ten scenarios corresponding to different asymmetric risk aversion parameters.[18] We observe that the average efficiency across the ten scenarios grows with the number of asymmetric retailers. We performed a similar experiment with one retailer and an increasing number of asymmetric manufacturers, and we obtained a similar insight.

5.3.4.2. Retailer differentiation. In this section we show that the result obtained for the symmetric chain with one product that the efficiency is unimodal with respect to the retailer differentiation and has a maximum equal to one (Proposition 4.2), is quite robust to the presence of multiple products and asymmetric manufacturers and retailers. Specifically, we find that for the symmetric chain with two products, the efficiency is a unimodal function of retailer differentiation and it achieves a maximum of one. We also find that for the asymmetric chain, the efficiency is also a unimodal function of retailer differentiation, although its maximum may be smaller than one. In the remainder of this section we give the details of our experiments.

[18] Specifically, for each scenario we randomly draw the risk aversion parameter of each retailer from a uniform distribution on $[0.8, 1]$.

We first consider a symmetric chain with two products, four manufacturers, and two differentiated retailers with variable degree of retailer differentiation. We use the following supply chain parameters[19]:

$$a_j = a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \forall j, \quad \eta_j = 0.9 \quad \forall j, \quad c_i = c = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, \quad \forall i,$$

$$H_j = H = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \quad \forall j, \quad \text{and} \quad G_{jk} = G = \delta H, \quad \forall j, k \neq j,$$

where δ ranges from zero to one. Note that when $\delta = 1$, retailers are not differentiated, and hence we term $\nu = 1 - \delta$ the degree of retailer differentiation. We find that the result we proved in the case of one product holds for the case with two products with symmetric retailers and manufacturers.[20]

We now consider an asymmetric chain. Specifically, we consider the same chain as described above but with randomly generated values for the risk aversion parameter of each retailer and cost coefficient of each manufacturer.[21] We find that while the average efficiency is still concave and attains a maximum for a certain value of the degree of retailer differentiation, the maximum efficiency may be lower than one in the asymmetric case. The presence of asymmetries thus makes it impossible to align total profits of the equilibrium with profits of the centralized solution by adjusting the differentiation among the retailers. The reason for this is that manufacturers have asymmetric costs, and hence in the centralized chain only the cheapest manufacturer produces, whereas in the decentralized chain both manufactures may produce independently of the degree of retailer differentiation, leading to inefficiency in profits. Also, retailers are asymmetric, and hence in the centralized chain the least risk averse retailer will distribute more relative to the decentralized chain independently of the degree of retailer differentiation, as long as the retailers are not completely differentiated. As a result of this difference in the centralized and decentralized optimal solutions, the efficiency is lower than one even as its maximum with respect to retailer differentiation.

5.3.4.3. Product differentiation. In this section we show that the result for the symmetric chain that efficiency does not depend on product differentiation does *not* hold in general for asymmetric chains. Specifically, our numerical experiments show that the efficiency may depend on product differentiation when manufacturers and/or retailers are asymmetric. The reason for this is that while in the symmetric chain all manufacturers produce (and all retailers distribute) each product in the same quantity, this is not the case in general in the asymmetric chain. Then,

[19] In this example, products are substitutable, however we found similar observations in the case of complementary products.

[20] We have tried several supply chains with different parameters and the result holds for all the supply chains we have tried.

[21] We draw the risk aversion parameter of each retailer from a uniform distribution on $[0.8, 1]$ and we draw the unit variable cost for each product and for each manufacturer from a uniform distribution on $[0, 0.2]$.

the interaction between product differentiation and the asymmetric production and distribution schedules results in the efficiency changing with product differentiation.

To show this result, we consider a supply chain with two manufacturers, two retailers, two products, and the following demand function:

$$a_j = a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \forall j, \qquad H_j = H = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}, \quad \forall j, \quad \text{and} \quad G_{jk} = G = H, \quad \forall j, k \neq j,$$

where δ ranges from zero to one. Note that when $\delta = 0$ the products are completely differentiated, and hence $1 - \delta$ represents the degree of retailer differentiation. We consider two cases:

(a) two asymmetric manufacturers ($c_1 = (0.1, 0.3)$ and $c_2 = (0.3, 0.1)$; that is, the first manufacturer can produce the first product cheaper, whereas the second manufacturer can produce the second product cheaper) and two symmetric retailers ($\eta_1 = \eta_2 = 0.9$), and

(b) two symmetric manufacturers, $c_1 = c_2 = (0.1, 0.3)$, and two asymmetric retailers, $\eta_1 = 1$ and $\eta_2 = 0.9$.

The first panel in Figure 4 shows the efficiency as a function of the degree of product differentiation for the case with asymmetric manufacturers. We find that the efficiency decreases substantially with product differentiation.[22] To understand this, we note first that, while in the centralized chain each manufacturer produces only its cheapest product, in the decentralized chain this occurs only for small degrees of product differentiation. Specifically, for the case without product differentiation ($1 - \delta = 0$), the two products are identical, and each manufacturer in the decentralized chain only produces its cheapest product. Competition is intense because both products are identical, and this intense competition leads to a relatively high value of the efficiency 98.7%. As product differentiation grows, however, the intensity of competition among the manufacturers is smaller because the two products are becoming more differentiated, while each manufacturer continues to produce only its cheapest product. This reduced degree of competition intensifies the double marginalization effect, and hence the efficiency drops. When the degree of product differentiation is large ($1 - \delta \geq 0.6$), we observe from our results that the two products are so differentiated that each manufacturer in the decentralized chain decides to produce both products, cheap and expensive. This results in increased competition, which alleviates double marginalization, but it also results in manufacturers incurring much higher costs than in the centralized chain, and hence the efficiency drops sharply.

The second panel in Figure 4 shows the efficiency as a function of the degree of product differentiation for the case with asymmetric retailers. The panel shows that for this case the efficiency decreases with the degree of product differentiation. To understand this, we note that in the centralized setting only the least risk averse retailer distributes, and hence the most risk averse retailer is not affected by product differentiation because it does not distribute any of the two products regardless of the degree of product differentiation. However, in the decentralized setting, the most

[22] We have found in other cases that, depending on the number of manufacturers and retailers and the supply chain parameters chosen, the efficiency may increase or decrease with product differentiation.

risk averse retailer *is* affected by the degree of product differentiation: we observe from our results that when products are identical (without product differentiation), the most risk averse retailer only distributes the cheapest product, but for $1-\delta \geq 0.3$ the most risk-averse retailers starts distributing the expensive product too, and as a result the efficiency decreases with the degree of product differentiation.

6. Conclusions

We have shown that the results in Corbett and Karmarkar [2001] regarding efficiency in a symmetric chain hold also for the more general case with product and retailer differentiation, stochastic demand, and risk averse retailers. In addition, we show how these results can be generalized for the case when there is retailer differentiation. Specifically, we find that provided there is a sufficient number of manufacturers and retailers, efficiency one can always be achieved in the symmetric chain by changing the degree of retailer differentiation. In the asymmetric chain, we find that efficiency is also a unimodal function of retailer differentiation, although its maximum may be smaller than one. The main implication of this result is that, in a decentralized chain with a sufficient number of manufacturers and retailers, the efficiency may be raised by using, for instance, loyalty programs to manipulate the degree of retailer differentiation.

Retailer risk aversion leads to triple marginalization: the retailers require a strictly positive margin to undertake the risk associated with uncertain consumer demand, even when they are perfectly competitive. We also find that, while in symmetric chains the efficiency is independent of product differentiation and risk aversion, in an asymmetric chain the efficiency may depend on these characteristics. This is because of the interaction of these characteristics with the asymmetry of manufacturers and retailers.

Finally, manufacturer and retailer asymmetry lead to steep declines in efficiency. The reason for this is that while in the centralized supply chain only the cheapest manufacturers produce and only the least risk-averse retailers distribute, in the centralized chain all manufacturers produce and all retailers distribute. Moreover, asymmetric product assortments lead to a decrease in the degree of competition among the retailers, which can also affect the supply chain efficiency.

Table 1 Equilibrium quantities in decentralized and centralized supply chains

This table gives the equilibrium quantities in the decentralized and centralized supply chains for the case without retailer differentiation ($H=G$). The first column in the table lists the eight different equilibrium quantities reported: the aggregate supply, the wholesale price, the expected retail price, the aggregate manufacturer profit, the aggregate retailer expected utility, the aggregate supply chain expected utility, the supply chain efficiency, and the expected consumer surplus. The second column reports the equilibrium quantities for the general case with M manufacturers and N retailers (M,N), which are given by Proposition 3.12. Columns third to sixth report the equilibrium quantities for the following particular cases: one manufacturer and one retailer ($M=N=1$), an infinite number of manufacturers and one retailer ($M=\infty, N=1$), one manufacturer and an infinite number of retailers ($M=1, N=\infty$), and an infinite number of manufacturers and retailers ($M=N=\infty$); these quantities are trivially obtained from Proposition 3.12. Finally, the seventh column reports the optimal quantities for the centralized supply chain, which are given by Proposition 3.6.

	Decentralized supply chain					*Centralized*
	General (M,N)	$M=1$, $N=1$	$M=\infty$, $N=1$	$M=1$, $N=\infty$	$M=\infty$, $N=\infty$	General (M,N)
Aggregate supply	$\frac{MN}{(M+1)(N+1)}H^{-1}\left(a-\frac{c}{\eta}\right)$	$\frac{1}{4}H^{-1}\left(a-\frac{c}{\eta}\right)$	$\frac{1}{2}H^{-1}\left(a-\frac{c}{\eta}\right)$	$\frac{1}{2}H^{-1}\left(a-\frac{c}{\eta}\right)$	$H^{-1}\left(a-\frac{c}{\eta}\right)$	$\frac{1}{2}H^{-1}\left(a-\frac{c}{\eta}\right)$
Wholesale price	$\frac{\eta}{M+1}\left(a+\frac{Mc}{\eta}\right)$	$\frac{\eta}{2}\left(a+\frac{c}{\eta}\right)$	c	$\frac{\eta}{2}\left(a+\frac{c}{\eta}\right)$	c	—
Retail price (exp.)	$a-\frac{MN}{(M+1)(N+1)}\left(a-\frac{c}{\eta}\right)$	$\frac{3}{4}a+\frac{c}{4\eta}$	$\frac{1}{2}\left(a+\frac{c}{\eta}\right)$	$\frac{1}{2}\left(a+\frac{c}{\eta}\right)$	$\frac{c}{\eta}$	$\frac{1}{2}\left(a+\frac{c}{\eta}\right)$
Manufacturer profit	$\eta\frac{MN}{(M+1)^2(N+1)}\theta$	$\frac{\eta}{8}\theta$	0	$\frac{\eta}{4}\theta$	0	—
Retailer utility (exp.)	$\eta\frac{M^2N}{(M+1)^2(N+1)^2}\theta$	$\frac{\eta}{16}\theta$	$\frac{\eta}{4}\theta$	0	0	—
Aggregate utility (exp.)	$\eta\frac{MN(M+N+1)}{(M+1)^2(N+1)^2}\theta$	$\frac{3\eta}{16}\theta$	$\frac{\eta}{4}\theta$	$\frac{\eta}{4}\theta$	0	$\frac{\eta}{4}\theta$
Efficiency	$\frac{4MN(M+N+1)}{(M+1)^2(N+1)^2}$	$\frac{3}{4}$	1	1	0	1
Consumer surplus (exp.)	$\frac{1}{2}\frac{M^2N^2}{(M+1)^2(N+1)^2}\theta$	$\frac{1}{32}\theta$	$\frac{1}{8}\theta$	$\frac{1}{8}\theta$	$\frac{1}{2}\theta$	$\frac{1}{8}\theta$

Table 2 Monotonicity properties of the decentralized equilibrium quantities and supply chain efficiency

This table gives the monotonicity properties of the decentralized equilibrium quantities and the supply chain efficiency for the case without retailer differentiation. Panel A gives the properties for the equilibrium quantities, whereas Panel B gives the properties for efficiency. The first column in Panel A lists the different equilibrium quantities considered: the aggregate supply, the wholesale price, the expected retail price, the aggregate manufacturer profit, the aggregate retailer expected utility, and the expected consumer surplus. The second and third columns report how the equilibrium quantities vary with the number of retailers and manufacturers, respectively. For Panel B, we observe that the monotonicity properties of the efficiency with respect to the number of retailers N for fixed number of manufacturers M are identical to the monotonicity properties with respect to the number of manufacturers M for fixed number of retailers. The first column in Panel B lists different cases for fixed number of manufacturers M (equivalently fixed number of retailers N). The second column reports how the supply chain efficiency varies with the number of retailers (equivalently with the number of manufacturers). The third column reports the value of the number of retailers (equivalently number of manufacturers) for which the efficiency is maximum. Finally, the fourth column reports the maximum efficiency.

Panel A: Decentralized equilibrium quantities

Quantity	Evolution as a function of M	Evolution as a function of N
Aggregate supply	increasing	increasing
Wholesale price	decreasing	constant
Expected retail price	decreasing	decreasing
Manufacturer profit	decreasing	increasing
Expected retailer utility	increasing	decreasing
Expected consumer surplus	increasing	increasing

Panel B: Supply chain efficiency

Number of manufacturers M (Number of retailers N)	Evolution with N (Evolution with M)	Efficiency-maximizing N (Efficiency-maximizing M)	Maximum efficiency (Maximum efficiency)
1	increasing	∞	1
2	strictly concave	3	1
3	strictly concave	2	1
4, 5	strictly concave	2	<1
≥ 6	decreasing	1	<1

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Material for on-line appendix

Figure 1 Supply chain efficiency as a function of the degree of retailer differentiation

This figure depicts the efficiency of the decentralized chain as a function of the degree of retailer differentiation. We consider a case with a single product, at least two (differentiated) retailers, and $H = 1$. The vertical axis gives the supply chain efficiency. The horizontal axis gives the value of the degree of retailer differentiation $\nu = H - G = 1 - G$ between 0 and 1. The different lines give the efficiency for a variety of cases with different number of retailers (N) and manufacturers (M).

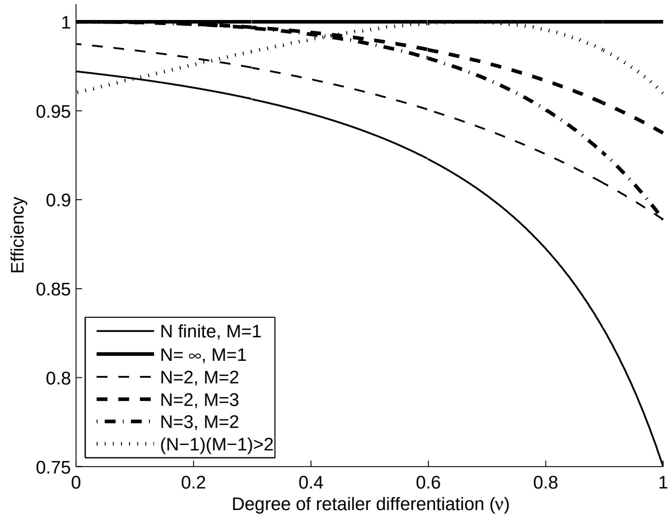
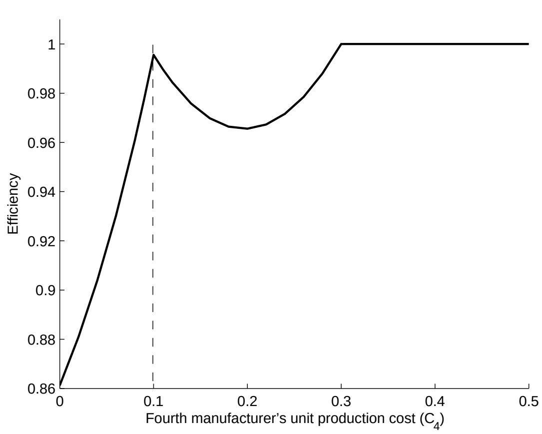


Figure 2 Efficiency with asymmetric manufacturers and retailers

The plot on the left depicts the supply chain efficiency for a supply chain with two products, two symmetric nondifferentiated retailers, and four asymmetric manufacturers—the first three manufacturers' production cost is 0.1 for every product, whereas the fourth manufacturer's production cost for every product varies from 0 to 0.5. The horizontal axis gives the cost of the fourth manufacturer and the vertical axis gives the efficiency. A vertical dashed line indicates the cost of the three symmetric manufacturers. The plot on the right depicts the supply chain efficiency for a supply chain with two products, two symmetric manufacturers, and four asymmetric retailers—the first three retailers' parameter η is 0.9, whereas the fourth retailer's parameter η_4 varies between 0.5 and 1. The horizontal axis gives the fourth retailer's parameter η_4 and the vertical axis gives the efficiency. A vertical dashed line indicates the parameter of the three symmetric retailers.



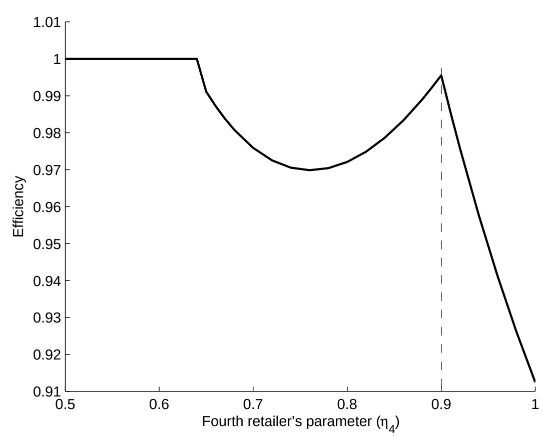


Figure 3 Effect of asymmetric product assortments

The figure depicts the aggregate supply quantity for each product, the manufacturer profit, the expected retailer utility, the expected supply chain utility, the expected retail price, the wholesale price, and the efficiency for a supply chain with two products, three symmetric manufacturers, two nondifferentiated retailers for the cases with asymmetric and symmetric product assortments—in the case with asymmetric assortment the first retailer carries product 1 only, and the second retailer carries product 2 only, while in the case with symmetric assortment both retailers may carry both products.

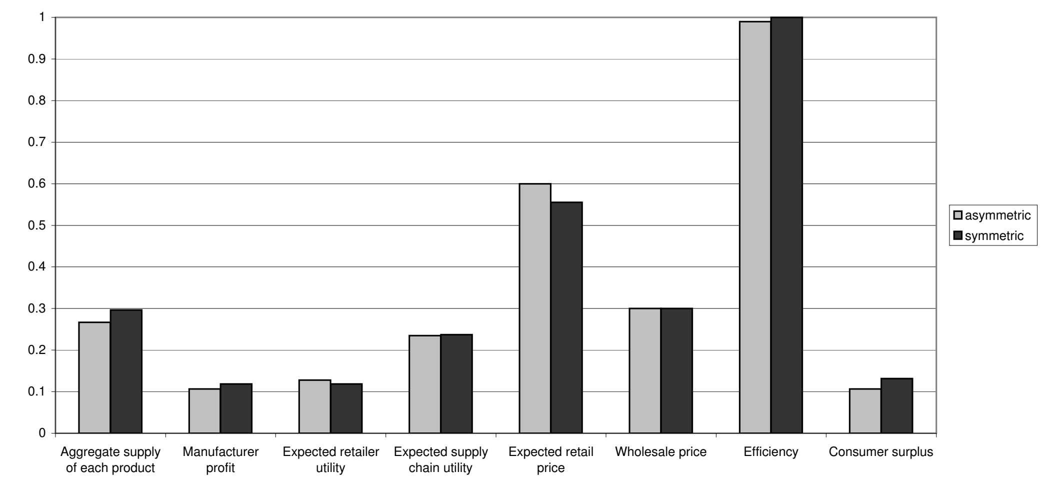
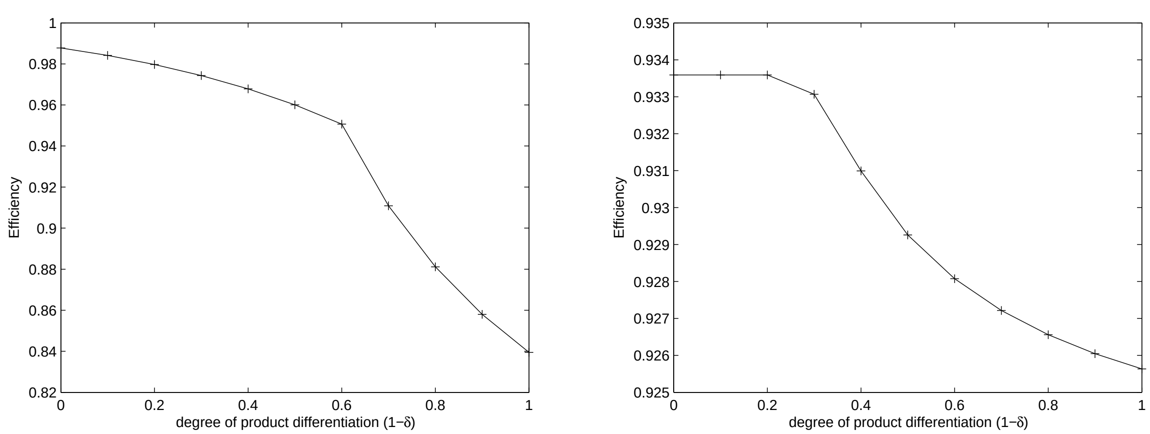


Figure 4 Effect of product differentiation with asymmetric retailers or manufacturers

The plot on the left depicts the supply chain efficiency for an asymmetric supply chain with two products, two asymmetric manufacturers, and two non differentiated symmetric retailers. The plot on the right depicts the supply chain efficiency for an asymmetric supply chain with two products, two symmetric manufacturers, and two non differentiated asymmetric retailers. In both plots, the horizontal axis gives the degree of product differentiation and the vertical axis gives the supply chain efficiency. The value $1-\delta=0$ corresponds to non differentiated products.



Appendix A: A premier on complementarity problems

The *complementarity problem* (CP) consists in finding a vector $x \in \mathbb{R}^n$ that satisfies the following system of equalities and inequalities:

$$\begin{aligned} x^T F(x) &= 0, \\ x &\geq 0, \\ F(x) &\geq 0, \end{aligned}$$

where $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function. Note that the fact that both x and $F(x)$ must be nonnegative, and their scalar product $x^T F(x)$ must be zero, implies that for each component $j = 1, 2, \ldots, n$, we must have that either x_j or $(F(x))_j$ must be zero. The CP is often denoted as

$$0 \leq x \perp F(x) \geq 0.$$

A *linear complementarity problem* (LCP) is a CP where $F(.)$ is a linear function: $F(x) = Mx + q$, where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The LCP can thus be denoted as

$$0 \leq x \perp Mx + q \geq 0. \tag{15}$$

It is well-known that under certain conditions linear and quadratic programs can be formulated as LCPs; see Cottle et al. [2009]. Moreover, the first-order conditions for Nash games can often be rewritten as LCPs. For instance, we show that the retailers equilibrium conditions can be rewritten as an LCP.

The following proposition gives a sufficient condition for the existence of a unique solution to the LCP.

Lemma A.1. If M is positive definite, then the LCP has a unique solution.

Proof. It is easy to show that if the matrix M is positive definite, then x is a solution to the LCP if an only if it is a minimizer to the following quadratic program:

$$\begin{aligned} \min_x \quad & x^T(Mx + q), \\ \text{s.t.} \quad & Mx + q \geq 0, \\ & x \geq 0. \end{aligned}$$

Moreover, the positive definiteness of M implies that the feasible region of this quadratic program is nonempty. Furthermore, given that the feasible region is nonempty, the positive definiteness of M implies that there exists a unique solution to the quadratic program, and hence to the LCP. □

A *mathematical program with equilibrium constraints* (MPEC) is an optimization problem that includes as part of its constraints a complementarity problem; see Luo et al. [1996] for an in-depth treatment of MPECs. An MPEC can be written as follows:

$$\begin{aligned} \min_x \quad & r(x) \\ \text{subject to} \quad & c_i(x) \geq 0 \\ & c_e(x) = 0 \\ & 0 \leq x \perp F(x) \geq 0. \end{aligned}$$

MPECs can be used to model Stackelberg games, where a leader makes a decision anticipating the equilibrium reached by a set of followers, which compete with each other. We use the MPEC to model the decision of each manufacturer, who anticipates the equilibrium reached by the retailers, who are the followers.

Finally, an *equilibrium problem with equilibrium constraints* (EPEC) is a mathematical program whose solution is the equilibrium point that simultaneously solves a set of MPECs. For instance, an EPEC can be used to represent the manufacturer-retailer equilibrium in our paper, as each manufacturer decision problem can be written as an MPEC, and their simultaneous equilibrium is an EPEC. Equilibrium problems with equilibrium constraints can be solved using numerical methods as the one proposed by Hu [2002].

Appendix B: Auxiliary results

This appendix contains the statements and proofs of some auxiliary results used in the paper.

LEMMA B.1. Let the matrix $\hat{T} \in \mathbb{R}^{NP\times NP}$ be

$$\hat{T} = \begin{pmatrix} 2T & T & \dots & T \\ T & 2T & \dots & T \\ \vdots & \vdots & \ddots & \vdots \\ T & T & \dots & 2T \end{pmatrix},$$

where $T \in \mathbb{R}^{P\times P}$. The matrix $\hat{T}$ defined is positive definite if and only if matrix T is positive definite.

Proof: A matrix is positive definite if and only if its symmetric part is positive definite. Hence without loss of generality, we assume that $\hat{T}$ and T are symmetric. Assume that $\hat{T}$ is positive definite. By definition $\hat{x}^T\hat{T}\hat{x} > 0 \quad \forall \hat{x} \neq 0 \in \mathbb{R}^{NP}$. Let $x \neq 0 \in \mathbb{R}^P$, and let $\hat{x} = (x, 0, \dots, 0) \in \mathbb{R}^{NP}$.

$$\hat{x}^T\hat{T}\hat{x} = 2x^T T x > 0,$$

where the last strict inequality follows from the fact that $\hat{T}$ is positive definite. Therefore, T is positive definite.

For the reverse, assume that T is positive definite; then all its eigenvalues are strictly positive. Any eigenvalue λ of $\hat{T}$ associated with eigenvector $\hat{x} \neq 0 \in \mathbb{R}^{NP}$ such that $\hat{x} = (x_1, \dots, x_N)$ satisfies

$$2Tx_i + \sum_{j\neq i} Tx_j = \lambda x_i, \quad i = 1, \dots, N.$$

Summing these equalities over $i = 1, \dots, N$ yields $(N+1)T\sum_i x_i = \lambda \sum_i x_i$. Thus if $\sum_i x_i \neq 0$, then $\lambda\frac{1}{N+1}$ is an eigenvalue of T, which implies $\lambda > 0$. Otherwise, consider i such that $x_i \neq 0$ (there must exist at least such i since $\hat{x} \neq 0$). We have $\sum_{j\neq i} x_j = -x_i$ and thus $2Tx_i + -Tx_i = \lambda x_i$ which imply that λ is an eigenvalue of T associated with eigenvector x_i. Thus $\lambda > 0$. It follows that $\hat{T}$ is positive definite. □

The following proposition states a useful consequence of Assumption 3.2.

PROPOSITION B.2. Let Assumption 3.2 hold, then the matrix D is positive definite.

Let $\hat{x} = (x, x, \ldots, x)$. Then $\hat{x}^T \hat{B} \hat{x} = N x^T (H + (N-1)G)x$. Moreover, by Assumption 3.2 we know that $\hat{B}$ is positive definite. Thus for any $x \neq 0$, we have that $x^T (H + (N-1)G)x > 0$. □

The following proposition states some useful implications of Assumption 3.7.

Proposition B.3. Let Assumptions 3.1 and 3.7 hold, then

1. the matrix H is positive definite,
2. the matrix $B = (2H + (N-1)G)/(N+1)$ is positive definite.

Part 1. Let $\hat{x} = (x, 0, \ldots, 0)$. Then $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = 2x^T H x$. Because by Assumption 3.7, $\hat{B} + \hat{H}$ is positive definite, for any $x \neq 0$, we must have $2x^T H x > 0$.

Part 2. Let $\hat{x} = (x, x, \ldots, x)$. Then, $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = N x^T (2H + (N-1)G)x$. Because by Assumption 3.7 we know that $\hat{B} + \hat{H}$ is positive definite, this implies that for any $x \neq 0$, we have that $x^T (2H + (N-1)G)x > 0$. □

Proposition B.4. Let Assumption 5.1 hold, then the matrices H_1, H_2, ..., H_N are positive definite.

Let $\hat{x} = (0, \ldots, 0, x, 0, \ldots, 0)$. Then $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = 2x^T H_i x$. Because by Assumption 5.1, $\hat{B} + \hat{H}$ is positive definite, for any $x \neq 0$, we must have $2x^T H_i x > 0$, which implies that H_i is positive definite. □

Appendix C: Proof for the results in the paper

Proof of Theorem 3.4

Part 1 and 2. The decision problem in a centralized supply chain with a risk averse central planner is

$$\max_{\hat{x} \geq 0} \eta \hat{x}^T \left(\hat{a} - \hat{B} \hat{x} \right) - \hat{c}^T \hat{x}, \tag{16}$$

where $\hat{c} = (c, c, \ldots, c)$. By Assumption 3.2, matrix $\hat{B}$ is positive definite, and therefore there exists a unique solution to the central planner decision problem, which is the solution to the following LCP:

$$0 \leq -\hat{a} + 2\hat{B}\hat{x} + \frac{\hat{c}}{\eta} \perp \hat{x} \geq 0. \tag{17}$$

Because the assumptions of Proposition B.2 in Appendix B hold, the matrix D is positive definite and thus invertible. Moreover, because Assumption 3.3 holds, we have that all the components of the vector $D^{-1}(a - c/\eta)$ are strictly positive. Then it is easy to see that the unique solution to the central planner problem is given by $\hat{x} = (x, x, \ldots, x)$, where

$$x = \frac{1}{2N} D^{-1} \left(a - \frac{c}{\eta} \right).$$

Part 3. The expected retail price at any retailer is given by the linear inverse demand function $E[p] = a - NDx$. Using the result in Part 1 to substitute x yields the result.

Part 4. The central planner expected utility can be rewritten as $CTU = \eta N x^T (a - NDx - c)$. Using the result in Part 1 to substitute x yields

$$CTU = \eta N \frac{1}{2N} \left(a - \frac{c}{\eta} \right)^T D^{-T} \left(a - c - ND \frac{1}{2N} D^{-1} \left(a - \frac{c}{\eta} \right) \right).$$

The result is obtained by straightforward simplification.

Part 5. From [Vives 1999, Exercise 6.9], the consumer utility that corresponds to the linear demand considered is

$$U(\hat{x}) = \hat{a}^T \hat{x} - \frac{1}{2}\hat{x}^T \hat{B}\hat{x}$$

therefore, the expected consumer surplus is

$$CS = U(\hat{x}) - \hat{x}^T E[\hat{p}] = \hat{a}^T \hat{x} - \frac{1}{2}\hat{x}^T \hat{B}\hat{x} - \hat{x}^T(\hat{a} - \hat{B}\hat{x}) = \frac{1}{2}\hat{x}^T \hat{B}\hat{x}$$

In the symmetric supply chain the expected consumer surplus simplifies to $N^2 x^T D x/2$. The result follows by replacing x with its expression given in Part 1. □

Proof of Proposition 3.6

Part 1. We have assumed that H is positive definite. Therefore the matrix $\hat{B}$ is positive semidefinite. As a result, the central planner decision problem in (16) is convex and a supply schedule is a global maximizer to problem (16) if and only if it satisfies the first-order optimality conditions in (17). For the case without retailer differentiation we have that $H = G$. Moreover, by Assumption 3.1 the supply chain is symmetric, and thus the LCP in (17) is equivalent to the following set of LCPs:

$$0 \le -a + 2H\sum_{k=1}^{N} x_k + \frac{c}{\eta} \perp x_j \ge 0 \quad \forall j. \tag{18}$$

Note that Assumption 3.3 implies that $H^{-1}(a - c/\eta) > 0$, and therefore it is easy to see that the supply schedule

$$x_j = x = \frac{1}{2N}H^{-1}\left(a - \frac{c}{\eta}\right) \quad \forall j,$$

satisfies (18) and hence is a global maximizer of the central planner problem. Moreover, it is easy to see that there are other asymmetric supply schedules that will lead to identical total supply and identical central planner expected utility.

Parts 2, 3, and 4. These parts follow trivially by using the result in Part 1 to substitute x in the expressions given in Parts 3, 4, and 5 of Theorem 3.4.

Proof of Proposition 3.8

Part 1. For the symmetric supply chain, the jth retailer decision given in equation (2) can be rewritten as:

$$\max_{x_j \ge 0} \quad -x_j^T v + \eta x_j^T (a - Hx_j - G\sum_{l \ne j} x_l). \tag{19}$$

Note that because Assumption 3.7 holds, by Proposition B.3 in Appendix B we have that the matrix H is positive definite, and thus for any wholesale price (v) and for any order quantities from the other retailers (x_l for $l \ne j$), there exists a unique maximizer to problem (19). Moreover, from Lemma A.1 we know that the unique maximizer coincides with the unique solution to the following LCP:

$$0 \le \frac{v}{\eta} - (a - 2Hx_j - G\sum_{l \ne j} x_l) \perp x_j \ge 0.$$

In other words, for any v and x_l for $l \ne j$, there exists a unique jth retailer best response.

Hence, for given v, the retailer order vector $\hat{x} = (x_1, x_2, \ldots, x_N)$ is a retailers equilibrium if and only if it solves the following LCP, which is obtained by concatenating the LCPs characterizing the best response of the N retailers:

$$0 \leq \hat{q} + (\hat{B} + \hat{H})\hat{x} \perp \hat{x} \geq 0, \tag{20}$$

where $\hat{q} = (v/\eta - a, v/\eta - a, \ldots, v/\eta - a)$. Note that by Assumption 3.7, $\hat{B} + \hat{H}$ is positive definite and thus the LCP in (20) has a unique solution (see Appendix A), which is the unique retailers equilibrium.

Part 2. Because the retailers equilibrium is unique and the game is symmetric with respect to all retailers, the equilibrium must be symmetric. Indeed, if the equilibrium was not symmetric, because the game is symmetric with respect to all retailers, it would be possible to permute the strategies among the retailers and obtain a different equilibrium, hereby contradicting the uniqueness of the equilibrium.

Part 3. Because the assumptions of Proposition B.3 in Appendix B hold, we know that the matrix $2H + (N-1)G$ is positive definite and thus the LCP in (6) has a unique solution. Moreover, it is easy to see that the vector $\hat{x} = (x, x, \ldots, x)$, where x is the unique solution to the LCP in (6), solves the LCP in (20) and thus is the unique retailers equilibrium. □

Proof of Proposition 3.10
Given that $B^{-1}\Big(a - v/\eta\Big) > 0$, it is easy to see that x given by (7) is the unique solution to the LCP in (6). Then, the result follows from Proposition 3.8. □

Statement of Theorem C.1

THEOREM C.1. Let Assumptions 3.1, 3.7, and 3.9 hold, then

1. there exists a unique manufacturer retailer equilibrium,
2. at equilibrium every manufacturer supplies the same quantity y, every retailer orders the same quantity x, and the total supply is

$$\text{Total supply} = My = Nx = \frac{MN}{(M+1)(N+1)} B^{-1}\Big(a - \frac{c}{\eta}\Big),$$

3. the wholesale market price is

$$v = \frac{\eta}{M+1}\Big(a + \frac{Mc}{\eta}\Big),$$

4. the expected retail price at every retailer is

$$E[p] = a - \frac{MN}{(M+1)(N+1)} DB^{-1}\Big(a - \frac{c}{\eta}\Big),$$

5. the decentralized manufacturers profit is

$$DMP = \eta\frac{MN}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T B^{-1}\Big(a - \frac{c}{\eta}\Big),$$

6. the decentralized retailers utility is

$$DRU = \eta\frac{M^2N}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T\Big(I - \frac{N}{N+1}B^{-T}D\Big)B^{-1}\Big(a - \frac{c}{\eta}\Big),$$

7. the decentralized supply chain total expected utility is

$$DTU = \eta \frac{MN}{(M+1)(N+1)} \Big(a - \frac{c}{\eta}\Big)^T \Big(I - \frac{MN}{(M+1)(N+1)} B^{-T} D\Big) B^{-1} \Big(a - \frac{c}{\eta}\Big).$$

8. the decentralized expected consumer surplus is

$$CS = \frac{1}{2} \frac{M^2 N^2}{(M+1)^2 (N+1)^2} \Big(a - \frac{c}{\eta}\Big)^T B^{-T} D B^{-1} \Big(a - \frac{c}{\eta}\Big).$$

Proof of Theorems 3.11 and C.1

Part 1. Under Assumption 3.9, we have from Proposition 3.10 that the ith manufacturer decision problem can be rewritten as

$$\max_{y_i \geq 0,\ v} \quad v^T y_i - c^T y_i \tag{21}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} y_k = \frac{N}{N+1} B^{-1} \Big(a - \frac{v}{\eta}\Big). \tag{22}$$

From the equality constraint we know that

$$v = \eta \left[a - \frac{N+1}{N} B \sum_{k=1}^{M} y_k \right], \tag{23}$$

and hence the ith manufacturer decision is

$$\max_{y_i \geq 0} \ \eta \left[a - \frac{N+1}{N} B \sum_{k=1}^{M} y_k \right]^T y_i - c^T y_i.$$

Because the assumptions in Proposition B.3 hold, we have that B is positive definite and thus the ith manufacturer decision has a unique maximizer that is given by the first-order optimality conditions:

$$0 \leq -\eta \left[a - \frac{N+1}{N} B \sum_{k=1}^{M} y_k \right] + \eta \frac{N+1}{N} B y_i + c \perp y_i \geq 0. \tag{24}$$

Concatenating the optimality conditions for all M manufacturers we have that the supply vector $\hat{y} = (y_1, y_2, \ldots, y_M)$ is an equilibrium if and only if it is the solution to the following LCP:

$$0 \leq \hat{r} + \hat{F} \hat{y} \perp \hat{y} \geq 0, \tag{25}$$

where $\hat{r} = (\frac{c}{\eta} - a, \frac{c}{\eta} - a, \ldots, \frac{c}{\eta} - a)$, and

$$\hat{F} = \frac{N+1}{N} \begin{pmatrix} 2B & B & \ldots & B \\ B & 2B & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & 2B \end{pmatrix}.$$

Moreover, since B is positive definite, Lemma B.1 implies that $\hat{F}$ is positive definite. Thus there exists a unique manufacturer retailer equilibrium.

Part 2. Because the manufacturers equilibrium is unique and the game is symmetric with respect to all manufacturers, the equilibrium must be symmetric. Indeed, if the equilibrium was not symmetric, because the game is symmetric with respect to all manufacturers, it would be possible to permute the strategies among the manufacturers and obtain a different equilibrium, hereby contradicting the uniqueness of the equilibrium. Therefore at the equilibrium $\hat{y} = (y, y, \ldots, y)$. Constraint (22) thus implies

$$My = \frac{N}{N+1} B^{-1}\left(a - \frac{v}{\eta}\right).$$

Therefore, under Assumption 3.9, $My > 0$ and hence $y > 0$. It follows that the nonnegativity constraint in problem (21) is redundant and can be ignored. Therefore y is obtained by solving the first order optimality conditions of the manufacturers decision problem

$$-\eta\left(a - \frac{N+1}{N} MBy\right) + \eta \frac{N+1}{N} By + c = 0.$$

It follows that

$$y = \frac{N}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right) \tag{26}$$

and thus $\hat{y} = (y, y, \ldots, y)$ is the unique solution to the LCP in (25). By adding the supply quantities of all manufacturers we get that

$$\text{Total supply} = My = \frac{MN}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right). \tag{27}$$

Moreover, we also know by Proposition 3.10 that at the unique equilibrium all retailers order the same quantity x. Because the wholesale market clearing conditions must hold at the unique equilibrium, we must then have that

$$Nx = \frac{MN}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right). \tag{28}$$

Part 3. Substituting (26) into (23), we have that

$$v = \eta\left[a - \frac{N+1}{N} B \frac{MN}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right)\right].$$

Simplifying we have

$$v = \eta\left[a - \frac{M}{M+1}\left(a - \frac{c}{\eta}\right)\right].$$

The result follows by simple algebra.

Part 4. The linear inverse demand function implies that at every retailer the expected retail price is $E[p] = a - (H + (N-1)G)x$. Using the result in Part 2, we have that

$$E[p] = a - (H + (N-1)G)\frac{M}{(M+1)(N+1)} B^{-1}\left(a - \frac{c}{\eta}\right).$$

The result follows because by definition $D = (H + (N-1)G)/N$.

Part 5. The decentralized manufacturers profit is

$$\begin{aligned}DMP &= M(v^T y - c^T y) = \frac{M}{M+1}(\eta a - c)^T \frac{N}{(M+1)(N+1)} B^{-1}\Big(a - \frac{c}{\eta}\Big)\\ &= \eta \frac{MN}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T B^{-1}\Big(a - \frac{c}{\eta}\Big)\end{aligned}$$

Part 6. The decentralized retailers utility is

$$\begin{aligned}DRU &= -Nv^T x + N\eta x^T(a - Hx - (N-1)Gx)\\ &= N\eta\Big(a - \frac{v}{\eta}\Big)^T x - N^2\eta x^T Dx\\ &= \eta\frac{MN}{M+1}\Big(a - \frac{c}{\eta}\Big)^T x - N^2\eta x^T Dx\\ &= \eta\frac{M^2N}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T B^{-1}\Big(a - \frac{c}{\eta}\Big) - \eta\frac{M^2N^2}{(M+1)^2(N+1)^2}\Big(a - \frac{c}{\eta}\Big)^T B^{-T}DB^{-1}\Big(a - \frac{c}{\eta}\Big)\\ &= \eta\frac{M^2N}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T\Big(I - \frac{N}{N+1}B^{-T}D\Big)B^{-1}\Big(a - \frac{c}{\eta}\Big)\end{aligned}$$

Part 7. The result follows by adding the expressions for the decentralized manufacturers profit and the decentralized retailers utility provided in the two previous parts.

Part 8. From [Vives 1999, Exercise 6.9], it is easy to show that for the linear demand considered and in the symmetric supply chain the expected consumer surplus is $N^2x^TDx/2$. The result follows by replacing x with its expression given in Part 2. □

Proof of Proposition 3.12

The result follows trivially from Theorem 3.11 because for the case without retailer differentiation we have that the three matrices B, D, and H are identical. □

Proof of Proposition 3.13

The expression of the supply chain efficiency given in (8) follows trivially from Propositions 3.6 and 3.12.

We now prove the monotonicity results shown in Table 2. The **total supply** is equal to

$$\frac{MN}{(M+1)(N+1)}H^{-1}\Big(a - \frac{c}{\eta}\Big).$$

Because $H^{-1}(a - c/\eta) > 0$, and since $M/(M+1) = 1 - 1/(M+1)$ is increasing in M and $N/(N+1) = 1 - 1/(N+1)$ is increasing in N, it follows that the total supply is strictly increasing in the number of manufacturers M and in the number of retailers N.

The **wholesale market price** is

$$v = \frac{\eta}{M+1}\Big(a + \frac{Mc}{\eta}\Big) = \frac{\eta a - c}{M+1} + c,$$

thus it is independent of the number of retailers N. Because we have assumed that $a - c/\eta > 0$, the wholesale price is strictly decreasing in the number of manufacturers M.

The **expected retail price** at every retailer is

$$E[p] = a - \frac{MN}{(M+1)(N+1)}\Big(a - \frac{c}{\eta}\Big).$$

Because $M/(M+1)$ is increasing in M, $N/(N+1)$ is increasing in N, and $a-c/\eta>0$, it follows that the expected retail price is strictly decreasing in the number of manufacturers M and in the number of retailers N.

The **manufacturers profit** is

$$DMP=\eta\frac{MN}{(M+1)^2(N+1)}\theta.$$

We know that $N/(N+1)$ is increasing in N, $\eta>0$, and by Assumption 3.7 and Proposition B.3 H is positive definite and thus $\theta>0$. Moreover, it is easy to derive

$$\frac{\partial}{\partial M}\frac{M}{(M+1)^2}=\frac{1-M}{(M+1)^3}\leq 0,$$

which is zero only for M=1, and hence the manufacturer profit is strictly decreasing in the number of manufacturers M and strictly increasing in the number of retailers N.

The **expected retailers utility** is

$$DRU=\eta\frac{M^2N}{(M+1)^2(N+1)^2}\theta.$$

Because $\eta>0$, $\theta>0$, $M^2/(M+1)^2$ is increasing in M and $N/(N+1)^2$ is decreasing in N, it follows that the expected retailer utility is strictly increasing in the number of manufacturers M and strictly decreasing in the number of retailers N.

The **expected consumer surplus** is

$$CS=\frac{1}{2}\frac{M^2N^2}{(M+1)^2(N+1)^2}\theta.$$

The result follows by noting that $\eta>0$, $\theta>0$, $M^2/(M+1)^2$ is increasing in M, and $N^2/(N+1)^2$ is increasing in N.

We now prove the **supply chain efficiency** properties.
From (8), the supply chain efficiency is given by

$$\text{efficiency}=\frac{4MN(M+N+1)}{(M+1)^2(N+1)^2}.$$

Note that the efficiency is given by a function that is symmetric with respect to M and N. Therefore it suffices to characterize the monotonicity properties of the efficiency as a function of N for fixed M, and we know that the same monotonicity properties hold for the efficiency as a function of M for N fixed.

We derive

$$\frac{\partial}{\partial N}\frac{N(M+N+1)}{(N+1)^2}=\frac{M+1-N(M-1)}{(N+1)^3},$$

which is positive if $M=1$ or if $N<(M+1)/(M-1)$, and is negative otherwise. It follows that:

- If $M=1$, the supply chain efficiency is strictly increasing in the number of retailers N. Moreover, if $M=1$ and $N=\infty$, then $DRU=0$ and $DMP=DTU=\eta\theta/4=CTU$ so the efficiency converges to one.

• If $M=2$, the supply chain efficiency is strictly increasing for $N<(M+1)/(M-1)=3$ and strictly decreasing otherwise, therefore it is strictly concave in N and reaches a maximum for $N=3$. We find that the efficiency equals 1 at the maximum.

• If $M=3$, the supply chain efficiency is strictly increasing for $N<(M+1)/(M-1)=2$ and strictly decreasing otherwise, therefore it is strictly concave in N and reaches a maximum for $N=2$. We find that the efficiency equals one at the maximum.

• If $M\geq 4$, the supply chain efficiency is strictly increasing for $N<(M+1)/(M-1)$ and strictly decreasing otherwise. Note that $(M+1)/(M-1)\in(1,2)$ is not an integer. Thus we must compare the value of the efficiency at $N=1$ and $N=2$. Some easy calculations lead to

$$\text{efficiency}(M,N=1)-\text{efficiency}(M,N=2)=\frac{M(M+2)}{(M+1)^2}-\frac{8M(3+M)}{9(M+1)^2}=\frac{M(M-6)}{9(M+1)^2}.$$

Therefore, if $M=4$ or $M=5$, efficiency$(M,N=1)<$ efficiency$(M,N=2)$ so the supply chain efficiency is strictly concave in N and reaches a maximum for $N=2$. If $M=4$ and $N=2$, we obtain that the efficiency equals $224/225<1$. If $M=5$ and $N=2$ (or $N=5$ and $M=2$), we obtain that the efficiency equals $80/81<1$.

• If $M\geq 6$, as seen above, the supply chain efficiency is strictly decreasing for $N>(M+1)/(M-1)>1$. If $M=6$, efficiency$(M,N=1)=$ efficiency$(M,N=2)$, so the supply chain efficiency is decreasing in N. If $M\geq 7$, efficiency$(M,N=1)>$ efficiency$(M,N=2)$, so the supply chain efficiency is strictly decreasing in N. Therefore if $M\geq 6$ the efficiency reaches its highest value is for $N=1$, where it equals $M(M+2)/(M+1)^2=((M+1)^2-1)/(M+1)^2<1$.

As discussed above, due to the symmetry of the efficiency as a function of M and N, the same results hold for the monotonicity properties of the efficiency as a function of M for fixed N.

Proof of Proposition 4.1

Because $G=H$ we know from the linear inverse demand function that $p=(a-HX)\epsilon$, where X is the total supply. Then we can rewrite this expression as $X=a-H^{-1}p/\epsilon$. This implies that the aggregate demand function depends only on the retail price p, and not on the number of retailers in the market. □

Proof of Proposition 4.2

In the case with one product ($P=1$) and $H=1$, we can simplify the expression of the efficiency, in order to study how it varies with respect to the value of G. We have

$$\text{Efficiency}=\frac{DTU}{CTU}=\frac{\eta\frac{MN}{(M+1)(N+1)}\left(a-\frac{c}{\eta}\right)\left(1-\frac{MN}{(M+1)(N+1)}\frac{D}{B}\right)\frac{1}{B}\left(a-\frac{c}{\eta}\right)}{\frac{\eta}{4}\left(a-\frac{c}{\eta}\right)\frac{1}{D}\left(a-\frac{c}{\eta}\right)}$$

After some straightforward calculations, we obtain

$$\text{Efficiency}=\left(\frac{4M}{M+1}\right)\left(\frac{1+(N-1)G}{2+(N-1)G}\right)\left(1-\frac{M}{M+1}\frac{1+(N-1)G}{2+(N-1)G}\right).$$

Let $r\equiv M/(M+1)$ and $\varphi(G)\equiv 1+(N-1)G$. Then the efficiency can be rewritten as

$$\text{Efficiency}=4r\frac{\varphi(G)}{1+\varphi(G)}\left(1-r\frac{\varphi(G)}{1+\varphi(G)}\right).$$

We have

$$\frac{d(\varphi(G)/(1+\varphi(G)))}{dG} = \frac{\varphi'(G)}{(1+\varphi(G))^2} = \frac{N-1}{(1+\varphi(G))^2},$$

therefore the partial derivative of the efficiency with respect to G is:

$$\frac{\partial \text{ Efficiency}}{\partial G} = 4r\frac{N-1}{(1+\varphi(G))^2}\Big(1-2r\frac{\varphi(G)}{1+\varphi(G)}\Big).$$

Since we assume $N \geq 2$, it is easy to derive

$$\frac{\partial \text{ Efficiency}}{\partial G} \geq 0 \quad \text{if and only if} \quad G(N-1)(M-1) \leq 2,$$

which is trivially true when $M=1$. Moreover, if $M \geq 2$,

$$\frac{\partial \text{ Efficiency}}{\partial G} \geq 0 \quad \text{if and only if} \quad G \leq \frac{2}{(N-1)(M-1)}$$

which implies that the efficiency is a unimodal function of G. It attains its maximum for $G = 2/((N-1)(M-1))$ (as long as this value is within the allowed range $[0,1]$ for G), where $\varphi(G)/(1+\varphi(G)) = 1/2r$, that is Efficiency $= (4r/2r)(1-r/2r) = 1$.

Part 1. If $M=1$ and $N<\infty$, then $r=1/2$ and the efficiency is increasing with G, so it is decreasing with ν. Therefore its maximum value is reached for $G=1$ (or $\nu=0$) and is equal to

$$\frac{2N}{N+1}\Big(1-\frac{N}{2(N+1)}\Big) = 1-\Big(1-\frac{N}{N+1}\Big)^2$$

which is strictly less than 1.

If $M=1$ and $N=\infty$, then $r=1/2$ and $\varphi(G)=\infty$ so Efficiency $= 4r(1-r) = 1$ $\forall\nu$.

If $N=M=2$, then $2/((N-1)(M-1)) = 2$ therefore the efficiency is an increasing function of G on $[0,H]=[0,1]$, so it is a decreasing function of ν on $[0,1]$. It thus attains its maximum for $G=1$ (or $\nu=0$) where it takes a value lower than 1 (the efficiency is actually 80/81).

Part 2. if $(N,M)=(2,3)$ or $(3,2)$, then $2/((N-1)(M-1)) = 1$ therefore the efficiency is an increasing function of G on $[0,1]$, so it is a decreasing function of ν on $[0,1]$, and it reaches a maximum for $G=1$ (or $\nu=0$) where the efficiency is equal to 1.

Part 3. If $(N-1)(M-1)>2$, then $2/((N-1)(M-1))<1$ therefore the efficiency is a strictly concave function of G on $[0,1]$, so it is a strictly concave function of ν on $[0,1]$, and it reaches its unique maximum equal to 1 for $G = 2/((N-1)(M-1))$. □

Proof of Proposition 4.3

This follows in a straightforward manner from the results in Propositions 3.12 and 3.6. □

Proof of Proposition 4.4

Part 1. The aggregate supply, wholesale price, and expected retail price in a supply chain with risk-averse retailers depend on the risk aversion only via the term c/η, therefore risk neutral retailers with a cost of $\tilde{c} = c/\eta$ would yield the same results.

Part 2. The manufacturer profit, expected retailer utility, and aggregate expected utility in a supply chain with risk-averse retailers depend on the risk aversion via the term c/η and because of a factor η. □

Proof of Corollary 4.5

We first obtain a detailed expression for $H^{-1}(a-c/\eta)$ and θ as a function of η to study how the expected retail price, efficiency, aggregate supply, manufacturer profit, expected retailer utility, and expected aggregate supply chain utility vary with η.

It is clear that each component $a^k - c^k/\eta$ of the vector $a-c/\eta$ is increasing with η. Moreover, if H is diagonal with diagonal coefficients h^k, from Proposition B.3 H is positive definite so $h^k > 0$ for all k. Then the kth component of vector $H^{-1}(a-c/\eta)$ is $(1/h^k)(a^k - c^k/\eta)$ and is thus an increasing function of η. By Assumption 3.3, $H^{-1}(a-c/\eta) > 0$ therefore $a - c/\eta > 0$. Hence

$$\theta = \sum_{k=1}^{P} \frac{1}{h^k}\left(a^k - \frac{c^k}{\eta}\right)^2$$

is an increasing function of η. We finally recall that risk neutral retailers correspond to $\eta = 1$, and a lower value of η represents a higher risk aversion.

Part 1. Based on Propositions 3.12 and 3.6, the expected decentralized (resp. centralized) retail price in the supply chain with risk-averse retailers are given by

$$E[p] = a - \frac{MN}{(M+1)(N+1)}\left(a - \frac{c}{\eta}\right) \quad \text{(resp.} \quad E[p] = \frac{1}{2}\left(a + \frac{c}{\eta}\right) \text{)},$$

which are clearly decreasing with η, hence both prices are increasing with the risk aversion.

Part 2. Based on Proposition 4.3, the efficiency of the decentralized chain with risk-averse retailers is $4MN(M+N+1)/((M+1)^2(N+1)^2)$, which is independent of η.

Part 3. Based on Propositions 3.12 and 3.6, the decentralized (resp. centralized) aggregate supply in the chain with risk-averse retailers are given by

$$\frac{MN}{(M+1)(N+1)} H^{-1}\left(a - \frac{c}{\eta}\right) \quad \text{(resp.} \quad \frac{1}{2} H^{-1}\left(a - \frac{c}{\eta}\right) \text{)},$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion.

Part 4. Based on Propositions 3.12 and 3.6, the decentralized manufacturer profit and expected retailer utility are given by

$$\eta \frac{MN}{(M+1)^2(N+1)}\theta \quad \text{and} \quad \eta \frac{M^2 N}{(M+1)^2(N+1)^2}\theta,$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion. Moreover, the decentralized (resp. centralized) expected aggregate supply chain utility are given by

$$\eta \frac{MN(M+N+1)}{(M+1)^2(N+1)^2}\theta \quad \text{(resp.} \quad \frac{\eta}{4}\theta \text{)},$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion. $\square$

Proof of Proposition 5.2

The jth retailer decision can be rewritten as:

$$\max_{x_j \geq 0} \quad -x_j^T v + \eta_j x_j^T (a_j - H_j x_j - \sum_{l \neq j} G_{jl} x_l).$$

Note that because Assumption 5.1 hold, we know by Proposition B.4 in Appendix B, that the matrix H_j is positive definite and thus for any v and x_l for $l \neq j$ there exists a unique maximizer to the jth retailer decision, which is the unique solution to the following LCP:

$$0 \leq \frac{v}{\eta_j} - (a_j - 2H_j x_j - \sum_{l \neq j} G_{jl} x_l) \perp x_j \geq 0.$$

In other words, for any v and x_l for $l \neq j$, there exists a unique jth retailer best response.

Hence, for a given v, the order quantity vector $\hat{x} = (x_1, x_2, \ldots, x_N)$ is a retailers equilibrium if and only if it solves the LCP in (9), which is simply the concatenation of the LCPs characterizing the best response of the different players. Note also that by Assumption 5.1, $\hat{B} + \hat{H}$ is positive definite and thus the LCP in (9) has a unique solution. □

Proof of Proposition D.1

Under the revenue-sharing contracts described, the jth retailer objective is

$$\max_{x_j \geq 0} \quad -\phi c^T x_j - \phi \eta x_j^T G \sum_{l \neq j} x_l + \phi \eta x_j^T (a - H x_j - G \sum_{l \neq j} x_l),$$

where the first term is the wholesale cost, the second term is the externality payment, and the third term is the expected utility from sales in the wholesale market.

Because Assumption 3.7 holds, by Proposition B.3 we have that matrix H is positive definite, and hence there exists a unique maximizer to the jth retailer decision problem, which is the solution to the following LCP:

$$0 \leq -a + 2H x_j + 2G \sum_{l \neq j} x_l + \frac{c}{\eta} \perp x_j \geq 0. \tag{29}$$

Therefore $\hat{x}$ is a retailers equilibrium under the revenue-sharing contracts if and only if it solves the following LCP:

$$0 \leq -\hat{a} + 2\hat{B}\hat{x} + \frac{\hat{c}}{\eta} \perp \hat{x} \geq 0.$$

Because Assumption 3.3 holds, it is easy to see that the unique solution to this LCP is given by $\hat{x} = (x, x, \ldots, x)$, where

$$x = \frac{1}{2N} D^{-1} \Big(a - \frac{c}{\eta} \Big), \tag{30}$$

and the order quantity given by (30) coincides with that in the centralized chain.

Note that each manufacturer knows the contractual conditions and thus knows the expected equilibrium retail price, which is entirely determined by the retailers equilibrium and thus is independent of its own decisions. In particular, each manufacturer knows that the expected retail price is, using expression (30) and the inverse demand function,

$$E[p] = a - \Big(H + (N-1)G \Big) \frac{1}{2N} D^{-1} \Big(a - \frac{c}{\eta} \Big) = \frac{1}{2} \Big(a + \frac{c}{\eta} \Big).$$

We can then write the manufacturer decision problem as

$$\max_{y_i \geq 0} \quad (1-\phi)\eta y_i^T \frac{1}{2}(a + \frac{c}{\eta}) - (1-\phi)c^T y_i - (1-\phi)\eta \frac{M}{M+1} y_i^T D \sum_{k=1}^{M} y_k,$$

where the first term is the expected utility from the revenue shared by the retailers, the second term is the wholesale market revenues minus the manufacturing cost, and the third term is the externality payment. Note that we have not included the wholesale market clearing conditions in the manufacturer's decision problem because the wholesale market price is set by the revenue-sharing contracts. We are going to see, however, that the externality payments are such that the manufacturers choose production quantities that satisfy the retailers demand and clear the wholesale market.

Because the assumptions in Proposition B.2 in Appendix B hold, we know that matrix D is positive definite, therefore there exists a unique solution to the manufacturer best response problem, which is the solution to the following LCP:

$$0 \leq -\frac{1}{2}(a+\frac{c}{\eta})+\frac{c}{\eta}+\frac{M}{M+1}D\sum_{k\neq i} y_k + 2\frac{M}{M+1}Dy_i \perp y_i \geq 0,$$

which can be rewritten as

$$0 \leq -\frac{1}{2}\frac{M+1}{M}(a-\frac{c}{\eta})+D(2y_i+\sum_{k\neq i} y_k) \perp y_i \geq 0. \tag{31}$$

Hence a supply schedule $\hat{y}=(y_1,y_2,\ldots,y_M)$ is a manufacturers equilibrium if an only if it solves the following LCP, which is formed as the concatenation of the LCPs (31) for all M manufacturers:

$$0 \leq \hat{r}+\hat{S}\hat{y} \perp \hat{y} \geq 0,$$

where $\hat{r}=((M+1)/(2M))(c/\eta-a, c/\eta-a,\ldots,c/\eta-a)$, and and

$$\hat{S}=\begin{pmatrix} 2D & D & \ldots & D \\ D & 2D & \ldots & D \\ \vdots & \vdots & \ddots & \vdots \\ D & D & \ldots & 2D \end{pmatrix}. \tag{32}$$

Because the assumptions of Proposition B.2 in Appendix B hold, the matrix D is positive definite, and hence, by Lemma B.1 $\hat{S}$ is positive definite, and hence there exists a unique manufacturers equilibrium under the proposed revenue-sharing contracts.

Because the equilibrium is unique and the problem is symmetric, the equilibrium is symmetric. Therefore, the equilibrium is the solution to the following LCP:

$$0 \leq -\frac{1}{2}(a-\frac{c}{\eta})+MDy \perp y \geq 0.$$

Since Assumption 3.3 holds, it is easy to see that the solution is

$$y=\frac{1}{2M}D^{-1}\left(a-\frac{c}{\eta}\right)$$

and thus $\sum_{i=1}^{M} y_i = \frac{1}{2}D^{-1}\left(a-\frac{c}{\eta}\right)$; that is, the manufacturing production quantities are identical to those in the centralized supply chain and the wholesale market clears. □

Appendix D: Supply chain coordination

We show that there exists a continuum of theoretical revenue sharing contracts that coordinate the symmetric supply chain and allow for any arbitrary split of utility between manufacturers and retailers. Our contracts are similar to those studied by Cachon and Lariviere [2005], but we extend the analysis to the case with competing manufacturers. We must note, however, that the purpose of our studying the revenue-sharing contracts is of a theoretical nature. Specifically, our purpose is to improve our understanding of the effect of competition in a supply chain with competing manufacturers and retailers, but we do not advocate the practical implementation of these complex contracts.

Our model allows for both vertical competition (manufacturers versus retailers) as well as horizontal competition (manufacturers versus manufacturers, or retailers versus retailers). Consequently, the coordinating contracts mush achieve vertical as well as horizontal coordination. To achieve vertical coordination, the contracts stipulate that the retailers must *share* their revenue from retail sales with the manufacturers. Specifically, every retailer must keep a proportion ϕ of its revenue, where $\phi \in (0,1)$, and pass on to the manufacturer a proportion $(1-\phi)$ of its revenue. Moreover, both the manufacturers and the retailers agree to a wholesale market price $v = \phi c$, where c is the manufacturer cost. As a result, manufacturers and retailers share revenue, costs, and risk, and hence their incentives are perfectly aligned. [23] To achieve horizontal coordination, the contracts require that every retailer pays a quantity equal to the externality that it imposes on its fellow competing retailers as a result of its supplying a particular quantity to the retail market. Likewise, every manufacturer must make an externality payment that depends on the amount they produce for each of the products.[24]

Proposition D.1. Let the assumptions of Theorems 3.4 and 3.11 hold. Assume also that the manufacturers and retailers have accepted the following revenue-sharing contracts:

1. every retailer keeps a proportion ϕ of its revenue, where $\phi \in (0,1)$, and passes on to the manufacturer a proportion $(1-\phi)$ of its revenue,
2. the wholesale market price is $v = \phi c$, where c is the manufacturer cost,
3. the jth retailer has to make an externality payment equal to $\phi \eta x_j^T G(\sum_{l \neq j} x_l)$ for ordering a quantity x_j,
4. the ith manufacturer has to make an externality payment for supplying a quantity y_i equal to $(1-\phi)\eta(M/(M+1)) y_i^T D(\sum_{k=1}^M y_k)$.

Then there exists a unique decentralized equilibrium at which the retailer order quantities and manufacturer supply quantities are equal to those in the centralized supply chain given in Theorem 3.4.

[23] Note that in the decentralized supply chain the manufacturer profit is deterministic, and hence the manufacturer risk aversion does not play a role. With the revenue sharing contracts in place, the manufacturers receive a share of the uncertain retail revenue, and hence their risk aversion comes into play. For tractability we assume in this section (and in the context of the symmetric supply chain) that all manufacturers and retailers in the supply chain have the same risk-aversion parameter.

[24] The spirit of the proposed revenue-sharing contracts is that the externality payments of the manufacturers and retailers are collected by a third party, and that this third party will decide how to reallocate these externality payments.