Laplace-Weibull Mixtures for Modeling Price Changes

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B. Mandelbrot and E. Fama in the sixties, and W. Ziemba in the seventies, suggested stable laws for modeling stock returns and commodity prices. Geometric stable distributions, with Laplace distribution playing the role of a "normal" law, have been found to give better fit to such data. We study the "stability" properties of Laplace and a mixture of Laplace and Weibull and discuss the statistical inference for such mixture models. Application of the mixture distribution to modeling price changes in real estate prices in France is given.
(*Stable Distribution*; E-M *Algorithm*)

Introduction and Summary

The problem of modeling price changes is of prime importance and usefulness not only to an individual but to the society at large. Probability distributions have been enhanced for such purpose both since they enable objectivity and also because they lend themselves to statistical inference procedures, e.g., testing of certain characteristics of the underlying phenomenon, efficient market hypothesis, prediction over a short range, etc. Starting with normal distributions, the interplay between simple probability models and empirical evidence or lack of it thereof, have come a long way to evolve into the current interest in geometric stable distributions. In this quest, we propose below a mixture distribution incorporating members from both geometric stable and alternative stable distributions. As will be discussed below, this choice stems from both sound theoretical justifications and empirical evidence. We exhibit how the rather complicated associated statistical inference procedures, both estimation and testing, can be handled in practice. We conclude with an example on price changes for one-bedroom apartments in Paris in the period 1984–1989 to demonstrate a reasonably good fit of our suggested model to this real life data. Certainly we do not claim that the real world phenomenon is dictated by our model. Rather, we hope that our model is but merely one step further in that pursuit compared to the existing parametric probability models.

The genesis of modeling price changes or asset returns may be traced back to the original theory of speculation by Bachelier (1990) based on the normal distribution. However, detailed study of empirical distributions by Mandelbrot (1962, 1967) and Fama (1965) revealed excess kurtosis in the data, thereby leading them to abandon the normal theory and propose, with theoretical considerations, the stable Paretian distributions. These distributions are leptokurtic, possess domains of attraction and are richer in parameters (four in number), and are, hence, fairly flexible. The normal distribution is a member of this family. Further, such distributions generated interest since the stability of each stable distribution yielded the property that its "index of stability" (shape parameter) remains the same regardless of the scale (sampling interval) adopted—a characteristic felt to be desirable and expected to be possessed by the relevant data.

Though some empirical evidence was found for such models, further recent theoretical empirical studies exploring the "stability" of the stable Paretian distribution revealed that (i) the rate of convergence to the stable law may be too slow to be useful in practice (Du Mouchel 1973) and (ii) the characteristic exponent does not, as it should, remain constant with the change in the sampling period. This contradicted directly the basic property of stable distributions and sparked further research. DuMouchel (1973) and Bonness et al., (1974)

0025-1909/93/3908/1029$01.25

suggested mixture distributions as alternative fat-tailed, or distributions to explain the observed excess kurtosis. These, however, were not based on probabilistic schemes, i.e., they were not justified to evolve as limiting distributions (say, via the central limit theorems) as is the usual desirable physical interpretation for most models proposed for the real world context. For further discussion and extended review, the reader is referred to Mittnik and Rachev (1992).

These led us to look at two basic properties that a probability distribution should possess if it is to model the price changes. First, it should possess the "stability" property allowing for possibly different behavior in different segments of the data; i.e., it is able to model processes that may, with some small probability, change in each segment. In our context of price changes, such segmentation may arise from, say, "breaks" caused by unexpected major changes (e.g., catastrophes, wars, etc.), which could with some small probability, occur in any segment, causing a crash in the market. This distribution should also incorporate possible distortions within each period between two successive breaks, caused by minor changes (e.g., spread of rumours, mergers, filing of bankruptcy, etc.). The first necessitates a model capable of incorporating random breaks, while the second calls for its capability to incorporate "outliers." The random breaks can be related to the notion underlying Clark's (1973) subordinated stochastic process model wherein the investment horizon is a random variable. This gives rise to the geometric stable distributions for which the study of domain of attraction, which was pioneered by Robbins (1948), Gnednenko and Fahim (1969). Explicit representation for its characteristic function was obtained by Klebanov et al. (1984). Next, the outliers can be handled by expanding our model to mixture families. Further, note that, as already mentioned above, mixture distributions have also been found (DuMouchel 1973, Bonness et al. 1974) to capture the excess kurtosis and fat-tailed characteristics of the empirical data sets on common stock prices remarkably well. These motivated us to propose contaminated geometric stable distributions and in particular the Laplace-Weibull mixture. The second important property of a probability model, we believe, is its robustness. Since our models are mere approximations to the real world, it is imperative that inference procedures should not be too sensitive to the exact distributional assumptions; but rather, should hold for some "close" models as well. This statistical concept is translated to the probabilistic requirement that our distribution possess a "domain of attraction." We show below that this is indeed the case for our model. Further, we are able to give estimates of such "closeness." Finally, since successive price charges do appear to be non i.i.d. random variables, a legitimate concern is the i.i.d. approximation. This could possibly necessitate a plot of the time series to look at the autocorrelation function, i.e., a correlalogram analysis. However, this i.i.d. approximation has been, so far, universally adopted in practice based on empirical evidence; but it lacks theoretical justification. Here again, it is very reassuring to observe that recent results (e.g., Bolthausen 1982, Gudynas 1985, Rackauskas 1990, Rachev and Ruschendorf 1991) establish that geometric stable distributions arise by virtue of the central limit theorem for dependent non i.i.d. random variables, such as martingale differences which are applicable to our log differences for the price changes. This precludes the necessity of any correlalogram analysis. Section 2 below gives further insights on and justifications for the specific choices of the mixture components. Section 3 deals with the statistical inference procedures for our model and §4 illustrates it with a real data set.

2. A Contaminated Geometric Stable Distribution—Motivations and Justifications

Let X_i denote the log difference in the prices of an asset at the times $t_0 + i$ and $t_0 + (i - 1)$. Assume that X_i, $i = 1, 2, \ldots, M$, are i.i.d. random variables. Based on stability considerations discussed above, we require that

$$X_1 \stackrel{d}{=} a_m(X_1 \circ X_2 \circ \cdots \circ X_m) + b_m, \qquad (2.1)$$

where $\stackrel{d}{=}$ denotes equality in distribution, a_m and b_m are normalizing factors, $\circ$ stands for the compounding scheme—summation, multiplication, max, min, etc., and M is a deterministic or random integer. However, in each period we expect with probability $p \in (0, 1)$, the occurrence of an event which dramatically alters the characteristics of the underlying process as will be reflected through $\{X_i\}$. Denote by $T(p)$ the period in which such an event is expected to occur and thus $M \equiv T(p)$. Alternatively, $T(p)$ represents the random moment at which the crash in the price changes occurs. $T(p)$ is assumed to be independent of $\{X_i\}$. Prices, and thus price changes, up to M and after M will reflect the influence of the crash, and hence, may possibly be governed by a different probabilistic process. Thus, the stability properties of X_i are only preserved up to period $T(p)$, the moment of the crash. $T(p)$ is assumed to follow a geometric distribution, i.e.,

$$P[T(p) = k] = (1-p)^{k-1}p, \quad k = 1, 2, \ldots. \tag{2.2}$$

The choice of the geometric distribution stems naturally from physical considerations that the underlying phenomenon governing the price changes is not affected drastically till the occurrence of a disaster, or until its "waiting time" is over. Further, analytically, it can be shown (Melamed 1988) that the solution of (1.1) leads to a family of probability distributions for M, which are either simple versions of the geometric distribution in (2.2), or are constants.

DEFINITION. The random variable Y, with distribution function (d.f.) G, is said to be geometrically stable, if there exists a sequence of independent, identically distributed (i.i.d.) random variables $X^{(1)}, X^{(2)}, \ldots$, a geometric r.v. $T(p)$, $P\{T(p) = k\} = pq^{k-1}$, $k = 1, 2, \ldots$, $q = 1 - p$, $0 < p < 1$, independent of X's, and constants $a = a(p) > 0$ and $b = b(p) \in R$ such that

$$a(p) \sum_{i=1}^{T(p)} (X^{(i)} + b(p)) \xrightarrow{d} Y, \quad \text{as} \quad p \to 0. \tag{2.3}$$

PROPOSITION. (MITTNIK AND RACHEV 1991). *A nondegenerate d.f. G is geometric stable with index $\alpha \in (0, 2]$ if and only if its characteristic function (ch.f.) f_g has the form*

$$f_g(\theta) = \frac{1}{1 - \ln \phi(\theta)}, \quad \theta \in \mathbb{R}, \tag{2.4}$$

where $\phi(\theta)$ is a ch.f. of some α-stable distribution, i.e., there exist $\beta \in [-1, 1]$, $c \geq 0$, and $\delta \in \mathbb{R}$ such that

$$f_g(\theta) = \left(1 + c^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2}\right) + i\delta t\right)^{-1},$$
$$\text{if} \quad \alpha \neq 1, \quad \text{or} \tag{2.5}$$

$$f_g(\theta) = \left(1 + c|t|\left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(t) \ln |t|\right) + i\delta t\right)^{-1/2}$$
$$\text{if} \quad \alpha = 1. \tag{2.6}$$

RESULT 2.1. In terms of Robbins's mixtures (a terminology accepted by the Soviet authors) the representation (2.4) can be rewritten as

$$f_g(\theta) = \int_0^\infty \phi(\theta)^z e^{-z}\, dz, \quad \theta \in \mathbb{R},$$

where ϕ is a ch.f. of an α-stable law; that is, the distribution of α-geometric stable random variables is an exponential scaled mixture of α-stable laws. □

According to (2.4) and (2.5), in the geometric stable case the role of the normal distribution (stable with parameter $\alpha = 2$) is played by the Laplace distribution (geometric stable with parameter $\alpha = 2$) with density

$$g(t) = (2\lambda)^{-1} \exp\{-\lambda|t|\}, \quad t \in \mathbb{R}. \tag{2.7}$$

On the other hand, the Weibull distribution is "double stable." It arises as the limit distribution of two compounding probabilistic schemes in (2.1)—the geometric random summation and minimum scheme for i.i.d. r.v.'s, i.e.,

RESULT 2.2. (MITTNIK AND RACHEV 1991, PROPOSITION 4.6). If $X_1, X_2, \cdots$ are i.i.d. r.v.'s with Weibull distribution with parameters λ, then

$$n^{1/\alpha} \min_{1 \leq i \leq n} X_i \stackrel{d}{=} X_1, \quad n = 1, 2, \ldots;$$

$$p \sum_{i=1}^{T(p)} X_i^\alpha \stackrel{d}{=} X_i^\alpha \stackrel{d}{=} X_i^\alpha, \quad 0 < p < 1. \tag{2.8}$$

Further, empirical studies including those by Mittnik and Rachev (1992) over a variety of probabilistic compounding schemes (which generalize and include the Paretian stable summation scheme of Mandelbrot) with data on daily stock changes, S & P 500 index, revealed that the double Weibull distribution dominates all other alternative stable distributions. Thus, the double exponential and double Weibull distributions are both strong candidates for modeling price change data. Since we would like to incorporate outliers, which could exhibit possibly multimodal distributions, theoretically a mixture distribution is a reasonable choice. We also recall that, empirically also, as observed by Bonness et al.

(1974), mixture distributions can remarkably improve the fit to the data rather than a single component distribution. Finally, in comparison to stable distributions, DuMouchel (1973) had some success by working with a mixture of a stable Paretian distribution and a normal distribution. In this spirit, we recall that for our geometric summation scheme, the role of the normal distribution is played by the double exponential while that of the stable Paretian distribution may be played by the Weibull distribution. Thus, we believe that a Laplace-Weibull mixture is well motivated from both physical (empirical) evidence and probabilistic (theoretical) justifications as a reasonable choice to model price changes. (In fact, as alluded to by Mittnik and Rachev (1991) in their conclusions, §8, such mixtures may even be capable of modeling data not only up to the occurrence of the crash but possibly for data over both volatile and calm periods, the mixing proportion now not necessarily being quite small as is usually the case for modeling data with outliers.) That is, we take the contaminated geometric stable distribution to model the price changes as the contaminated Laplace distribution,

$$\Lambda_\pi(x) = \pi\Lambda(x) + (x) + (1-\pi)w(x), \quad 0 < \pi < 1, \tag{2.9}$$

where Λ is the Laplace (double exponential) d.f. with parameter $\lambda > 0$, W is a symmetric (double) Weibull distribution with parameters $\mu > 0$ and $\gamma > 1$, and all parameters are unknown. Note that $\Lambda_\pi(x)$ can resemble unimodal (as $\pi \to 1$), bimodal (as $\pi \to 0$) or multimodal (for different combinations of the parameters) distributions. This flexibility is quite appealing in its capability to model data of such general structure.

Suppose $X^{(1)}, \ldots, X^{(T)}$ represent the daily price changes until the "disastrous" geometric moment $T = T(p)$. Clearly $p \in (0, 1)$ represents the probability of a disastrous event in each day. Assume that $X^{(i)}$'s are i.i.d. with distribution Λ_π. Consider the total normalized change $\sqrt{p}\sum_{i=1}^T X^{(i)}$ and let us compare it with the Laplace distribution. Theorem 2.2(c) in Rachev and SenGupta (1991) provides us with a bound of the deviation between the distribution of $\sqrt{p}\sum_{i=1}^T X^{(i)}$ and $X = F_U$ in terms of the Khinchin metric

$$\chi(X, Y) = \sup_{x\geq 0}\left|\int_x^\infty (F_X(x) - F_Y(x))\,dx\right|.$$

RESULT 2.3. Let the means and the variances of $X^{(i)}$ and U match. Then,

$$\chi\left(\sqrt{p}\sum_{i=1}^T X^{(i)}, U\right) \leq (1-\pi)\left[\sqrt{p}\tilde{\chi} + qp^{r/2-1}\tilde{\kappa}_r\frac{1}{r(r-1)}\left(\frac{\lambda}{2}\right)^{r-1} \times (\Gamma(1+1/\beta))^\beta\right], \tag{2.10}$$

where

$$\tilde{\chi} := \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x (\Lambda(t) - W(t))\,dt\right|$$

and

$$\tilde{\kappa}_r := r\int_{-\infty}^{+\infty} |x|^{r-1}\,|\Lambda(x) - W(x)|\,dx.$$

One could use the Khinchin metric to obtain bounds similar to (2.10) for the mixture case.

RESULT 2.4. Let $F_{X^{(i)}} = \pi F_U + (1-\pi)F_V$ where U is geometric stable with index $\alpha \in (0, 1)$. Then

$$\chi\left(p^{1/\alpha}\sum_{i=1}^T X^{(i)}, U\right) \leq p^{-1+1/\alpha}(1-\pi)\chi(V, U). \tag{2.11}$$

This follows by the "ideality" of χ (see Rachev 1991, Chapter 14 for the properties of *ideal metrics*). Analogously one treats the cases $1 \leq \alpha < 2$ and $\alpha = 2$. See Rachev and SenGupta (1991) for details.

3. Statistical Inference for the Mixture Model

3.1. Estimation of the Parameters

The proposed mixture model of (2.1) is,

$$p(x; \pi, \lambda, \mu, \gamma) \equiv p(x; \underset{\sim}{\psi}) = \pi f_1(x; \lambda) + (1-\pi)f_2(x; \mu, \gamma)$$

where

$$f_1(x; \lambda) = (\lambda/2)\exp(-\lambda|x|), \lambda > 0, \quad \text{and}$$

$$f_2(x; \mu, \gamma) = (\gamma\mu/2)|x|^{\gamma-1}\exp(-\mu|x|^\gamma), \quad \gamma > 1, \mu > 0, 0 \leq \pi \leq 1. \tag{3.1}$$

Suppose a random sample $X_1, \ldots, X_n$ is observed from the above distribution. There does not exist any nontrivial sufficient statistic for $\underset{\sim}{\psi}$. Also, since $f_2(\cdot)$ is not a member of the exponential family, Hasselblad's (1969) approach of obtaining the maximum likelihood estimator cannot be used. However, the general E-M algorithm, as presented by Dempster, Laird and Rubin (1977) can be used. The resulting estimating equations at the $(m + 1)$st stage are

$$\pi_j^{(m+1)} = n^{-1} \sum_{i=1}^{n} \omega_{ij}(\underset{\sim}{\psi}^{(m)}),$$

$$j = 1, 2, \pi_1 \equiv \pi, \pi_2 \equiv 1 - \pi,$$

$$\omega_{ij}(\underset{\sim}{\psi}^{(m)}) = \pi_j^{(m)} f_j(x_j)/p(x; \psi^{(m)}), \qquad (3.2)$$

$$1/\lambda^{(m+1)} = (n\pi_1^{(m+1)})^{-1} \sum \omega_{ij}(\underset{\sim}{\psi}^{(m)})|x_i|, \qquad (3.3)$$

$$1/\gamma^{(m+1)} - \left[\sum \omega_{i2}(\psi^{(m)}) ln|x_i|\right] \cdot \left[\frac{|x_i|\gamma^{(m+1)}}{\sum \omega_{i2}(\Psi^{(m)})|x_i|\gamma^{(m+1)}} - (n\pi_2^{(m+1)})^{-1}\right] = 0, \qquad (3.4)$$

$$1/\mu^{(m+1)} = (n\pi_2^{(m+1)})^{-1} \sum_{1}^{n} \omega_{i2}(\underset{\sim}{\psi}^{(m)})|x_i|\gamma^{(m)}. \qquad (3.5)$$

Note that, as is usually done in solving the likelihood equations for the parameters of a Weibull distribution, we have written (3.4) in terms of $\gamma^{(m+1)}$ as the only unknown parameter at the $(m + 1)$st stage. An iterative method is then employed to solve for $\gamma^{(m+1)}$. Once $\gamma^{(m+1)}$ is obtained, $\mu^{(m+1)}$ can be obtained easily from (3.5). Hence at each stage, we need to iteratively solve for $\gamma^{(\cdot)}$ and $\mu^{(\cdot)}$. This procedure will be as easy or as difficult as one would face in obtaining the MLE's of the parameters of a Weibull distribution. Further, one can study the convergence of this algorithm using the general theory provided by Wu (1980). Of course, one would expect to face computational difficulties for values of γ which are known to create problems in obtaining the MLE's of a Weibull distribution. As in the general case, statistical properties, e.g., consistency, efficiency, etc. of these estimators are unknown. We plan to study this aspect in the future.

3.2. Tests for No Mixture

In model (3.1), the Laplace density $f_1(x; \lambda)$ is to be understood as the main underlying density with the Weibull density $f_2(x; \mu, \gamma)$ a possible contaminant. We want to test H_0: No mixture (i.e., $p \equiv f_1$) against H_1: Mixture distribution for X. The test needs to be derived depending on whether one of the crucial parameters $\pi \in [0, 1]$ and $\gamma \geq 1$ can be considered as a nuisance parameter or if both need to be represented in the parametric formulation of H_0. This leads to interesting and diverse situations. Also we consider exhaustively the four possible cases for λ and μ; (i) λ and μ both known, (ii) $\mu = \mu_0$ known, λ unknown, (iii) $\lambda = \lambda_0$ unknown, μ unknown, and (iv) λ and μ both unknown. A variety of tests are presented below.

Case 1. $\gamma = \gamma_0 > 1$ known. For all the cases (i)–(iv), H_0 and H_1 above reduce to H_{01}: $\pi = 1$ and H_1: $\pi < 1$ respectively.

(i) Let $\mu = \mu_0$ and $\lambda = \lambda_0$ be known. Considering appropriate one-sided derivative (see, e.g., Durirajan and Kale 1983) the locally most powerful (LMP) test given by,

$$\omega: \sum_{i=1}^{n} \left.\frac{\partial ln p(x_i, \psi)}{\partial \pi}\right|_{\pi=1} > c_0$$

reduces to

$$\omega: T^* = \sum_{i=1}^{n} \{(\gamma_0 - 1)ln|x_i| - \mu_0|x_i|^{\gamma_0} + \lambda_0|x_i|\}/\sqrt{n} > C. \qquad (3.6)$$

Note that T^* is easy to compute and is asymptotically normally distributed under both H_0 and H_1.

(ii) Though λ is a nuisance parameter, unfortunately here and also for (iii) below, no reduction is available through similarity or invariance. There does not even exist any nontrivial sufficient statistic. Nevertheless, it can be shown that all the five conditions for the validity of Neyman's (1959) C_α test hold, provided $\gamma \leq K < \infty$. Let

$$T^* = \left[\sum \frac{\partial ln p(x_i, \psi)}{\partial \pi} - a_1^0 \sum \frac{\partial ln p(x_i, \psi)}{\partial \lambda}\right]\Bigg|_{\pi=1}. \qquad (3.7)$$

Unfortunately, since λ and π are not orthogonal, a_1^0, the regression coefficient of the first term on the second does not vanish.

Define $Z_n^* = \hat{T}^*/\hat{\sigma}_0(T^*)$, where $\sigma_0(T^*)$ is the standard deviation of T^* computed under H_0, and $\hat{T}^*$ and

$\hat{\sigma}_0(T^*)$ are computed by replacing the unknown parameter λ in T^* and σ_0 by its any root-n consistent estimator, e.g., the maximum likelihood estimator, under H_0. Then, the test becomes,

$$\omega: Z^* > C_1. \tag{3.8}$$

For any sequence $\pi^* = \{\pi_n\}$ such that $\pi_n\sqrt{n} \to \tau$, the asymptotic value of the power of the test is given by

$$1 - (1/\sqrt{2\pi}) \int_{-\infty}^{\tau_\alpha} \exp\{-(t - \sigma_0\tau)^2/2\}\, dt.$$

Among all tests Z_n^{**} for H_0: $\pi = 1$ with asymptotic level of significance α, for whatever sequence of alternatives $\pi_n > 0$ with $\pi_n \to \pi_0 = 1$, and whatever fixed $\lambda > 0$,

$$\underline{\lim}[\text{Power}\{Z_n^*(\pi_n, \lambda)\} - \text{Power}\{Z_n^{**}(\pi_n, \lambda)\}] \geq 0.$$

The test Z_n^* is in this sense an asymptotically locally most powerful test.

(iii) Since the nuisance parameter μ is present only under H_{11}, the Neyman's C_α test fails to be applicable here—the requirement of a root-n consistent estimator of μ under H_{01} is meaningless. Consider however T^* from (i) and rewrite it as $T^*(\mu_0)$. Let $T^*(\mu)$ be standardized such that $S(\mu) = \{T^*(\mu) - E(T^*(\mu)]/[\text{Var}(T^*(\mu))]^{1/2}$ has asymptotically a standard normal distribution under H_{01}. We rejected H_{01} for large values of $S(\mu_0)$. Assume that $\mu \in [L, U]$. Then $S(\mu)$ is continuous on $[L, U]$ with a continuous derivative (except possibly for a finite number of jumps in the derivative) and forms a Gaussian process. Hence, from Davis (1977, 1987) we reject H_0 for large values of

$$M = \sup\{S(\mu): L \leq \mu \leq U\}. \tag{3.9}$$

To obtain the cut-off points, we use the bound

$$P\{\sup S(\mu) > c: L \leq \mu \leq U\}$$
$$\leq \phi(-c) + \exp\left(-\frac{1}{2}c^2\right) \int_L^U \{-\rho_{11}(\mu)\}^{1/2}\, d\mu/2\pi$$

where ϕ denotes the cumulative normal distribution function, $\rho_{11}(\mu) = [\partial^2\rho(\phi, \mu)/\partial\phi^2]_{\phi=\mu}$, $\rho(\phi, \mu) = \text{corr}\{S(\phi), S(\mu)\}$. An estimate of the significance probability is given by

$$\Phi(-M) + V\exp\left(-\frac{1}{2}M^2\right)\Big/(8\pi)^{1/2},$$

where M denotes the maximum of $S(\mu)$ and V the total variation,

$$V = \int_L^U |T(\mu)|\, d\mu$$
$$= |S(\mu_1) - S(L)|$$
$$+ \sum_{i=1}^{n-1} |S(\mu_{i+1}) - S(\mu_i)| + |S(U) - S(\mu_n)|,$$

$T(\mu) \equiv \partial S(\mu)/\partial\mu$, with $\mu_1, \ldots, \mu_n$ $(n < \infty)$ being the successive turning points of $S(\mu)$.

(iv) No statistical test is known for this situation. We propose to combine the approaches of (ii) and (iii) above. Consider from (ii), the C_α-test statistic $\hat{T}^* \equiv \hat{T}^*(\gamma_0, \mu_0, \hat{\lambda})$. Let $S^*(\mu) = \hat{T}^*(\gamma_0, \mu, \hat{\lambda})$ we can now proceed exactly as in (iii) above with $S(\mu)$ replaced by $S^*(\mu)$.

Case 2. $\pi \in (0, 1)$ known. H_0 and H_1 reduce to H_{02}: $\gamma = 1$ and H_{12}: $\gamma > 1$ provided $\lambda = \mu$. Thus knowledge of λ or/and μ needs to be incorporated in the hypotheses.

(i) Here the common value of $\lambda = \mu$ is known, say equal to 1. Then H_{02} reduces to H_{02}^1: $\gamma = 1$.

$$\omega: \sum_{i=1}^n \frac{\partial \ln p(x_i, \Psi)}{\partial\gamma}\bigg|_{\gamma=1} > c_0 \quad \text{i.e.,}$$

$$\omega: T_1 \equiv \sum_{i=1}^n \ln|x_i|(1 - |x_i|) > C. \tag{3.10}$$

T is simple in form and it is easy to simulate its null distribution to get the percentage points. Further, $T/\sqrt{n}$ is asymptotically normally distributed under both H_0 and H_1.

(ii) $\mu = \mu_0$ known, λ unknown and (iii) $\lambda = \lambda_0$ known, μ unknown. H_0 now becomes the multiparameter hypothesis H_{02}^2 (H_{02}^3): $\gamma = 1$, $\lambda = \mu_0$ ($\mu = \lambda_0$). An optimal multiparameter test can be obtained by using the locally most mean powerful unbiased (LMMPU) test derived by SenGupta and Vermeire (1986). The test is given by

$$\omega: T_2 \equiv \sum_{t=1}^2 \dot{L}_{tt}(x, \psi_0) - cL(x, \psi_0)$$
$$- \sum_{t=1}^2 c_t\dot{L}_t(x, \psi_0) \geq 0, \tag{3.11}$$

where $L \equiv \prod_{i=1}^{n} p(x_i, \psi)$, $\psi = (\gamma, \lambda)$, $\psi_0 = (1, \mu_0)$ for case (ii) and $(1, \lambda_0)$ for case (iii) and c, c_1 and c_2 are to be determined from,

(1) $\int_\omega L(x, \psi_0) = \alpha$ and
(2) $\int_\omega \dot{L}_t(x, \psi_0) = 0$, $t = 1, 2$.

Note that $\sqrt{n}\, T_2$ can be explicitly computed and, further, is asymptotically normally distributed under both H_0 and H_1. This test is locally optimal for all sample sizes in the sense that among all locally unbiased tests it maximizes at ψ_0, the mean curvature of the power hypersurface. Further, this LMMPU test φ possesses the property that for any other locally unbiased test φ^*, there exists an $r_0 > 0$ (depending on φ^*) such that

$$\int_{S_r} \beta_\varphi(\psi)\, d\psi > \int_{S_r} \beta_{\varphi^*}(\psi)\, d\psi,$$

$$r < r_0,\ S_r = \{\psi\colon |\psi - \psi_0| < r\};$$

where β_ξ is the power function of the test ξ. One could also use other multiparameter tests for testing simultaneously the hypotheses $\gamma = 1$ and $\mu = \lambda$ ($\lambda = \mu_0$); e.g., a step-down test could be used. However, the exact level of significance then is not known; any optimality property of such a test is also not known.

(iv) Both λ and μ are unknown. H_0 now becomes H_{02}^4: $\gamma = 1$, $\mu = \lambda$, which is a composite hypothesis; and, hence, the previous optimal LMP test can no longer be used. Standard multiparameter large sample tests (e.g., the likelihood ratio, Wald's or Rao's tests, which are all asymptotically equivalent), can be explored. However, the test statistics need not have closed forms. For example, for the likelihood ratio test, even the maximum likelihood estimator of p is not available in a closed form.

Case 3. Both π and γ are unknown, but $\pi \in [L, U] \subset (0, 1)$. The testing problem is still identifiable and H_0 reduces to H_{03}: $\gamma = 1$, provided $\lambda = \mu$. This is exactly the same situation as in Case 2 above with π being an additional nuisance parameter. Thus, the same approaches as for Case 2 (i)–(iv) can be used here with the modification described in Case 1 (iii) now applied to π. For example, T_1 and T_2 of (3.9) and (3.10), which now become functions of the unknown π, should be respectively replaced by

$$\tilde{S}_1 = \sup\{S(\pi)\colon L \le \pi \le U\} \quad \text{and}$$
$$\tilde{S}_2 = \sup\{S_2(\pi)\colon L \le \pi \le U\}$$

where $S_1(\pi)$ and $S_2(\pi)$ are standardized versions of $T_1(\pi)$ and $T_2(\pi)$ respectively.

Case 4. Both π and γ are unknown, $\pi \in [0, 1]$.

(i)(a). Suppose $\gamma > 1$ but γ is otherwise unknown. This greatly simplifies the situation. Further, let H_0 specify H_0: $p = f_1(\lambda_0)$, λ_0 given, i.e., $H_0 \equiv H_{04}^1$: $\pi = 1$, $\lambda = \lambda_0$. This reduces to a case of "strongly identifiable" mixtures and is much easier to handle, e.g., one may use the likelihood ratio test to test the single hypothesis H_{04}^1. However, even under this special case, the classical distribution theory of the log likelihood ratio statistic does not hold. We no longer have asymptotically a χ^2 distribution. Rather, as demonstrated by Ghosh and Sen (1985), the likelihood ratio statistic is distributed as a certain functional $W^2 I_{\{W>0\}}$, where $W = \sup\{T(\mu, \gamma)\}$ and $T(\cdot)$ is a Gaussian process with zero mean and covariance kernel depending on the true value λ_0 under H_0. If $\mu = \mu_0$ is known, we simply replace $T(\mu, \gamma)$ by $T(\mu_0, \gamma)$.

(i)(b). Suppose $\lambda = \mu$ but the common value is unknown. This results in greatly simplifying the situation. H_0 reduces to

$$H_0 \equiv \bar{H}_{04} = \bar{H}_{04}^{1(1)} \cup \bar{H}_{04}^{1(2)} \cup (\bar{H}_{04}^{1(1)} \cap \bar{H}_{04}^{1(2)})$$

where we have $\bar{H}_{04}^{(1)}$: $\pi = 1$ and $\bar{H}_{04}^{1(2)}$: $\gamma = 1$. Note that $\bar{H}_{04}^{1(1)}$ and $\bar{H}_{04}^{1(2)}$ can be tested by suitably modifying the methods already discussed above, since the value of λ (and hence $\mu = \lambda$) can be estimated under $\bar{H}_{04}^{1(1)}$ and $\bar{H}_{04}^{1(2)}$ easily; while the nuisance parameter $\lambda(\pi)$ may be dealt with as in Case 1(iv). However, we also now need to test $\bar{H}_{04}^{1(1)} \cap \bar{H}_{04}^{1(2)}$, i.e., to test the multiparameter composite hypotheses $\pi = 1$ and $\gamma = 1$ with the nuisance parameter $\lambda = \mu$. This calls for a multiparameter generalization of Neyman's C_α test and one such generalization due to Büehler and Puri (1966) can be exploited here.

(ii) and (iii). Both these situations require modifications of $\bar{H}_{04}^{1(2)}$ and can be dealt with by the same approach. For (ii) ((iii)) we have $H_{04}^{2(2)}$ ($H_{04}^{3(2)}$): $\gamma = 1$, $\lambda = \mu_0$ ($\mu = \lambda_0$). Setting π to be known, the LMMPU test for this hypothesis can be derived. This can then be used to construct $T^*(\pi)$ from which the test can be obtained as in Case 1(iii).

(iv). All parameters π, γ, μ and λ are unknown and no constraint (assumption) is imposed on the usual parameter space. We now face the problem of nonidentifiability. H_0 is represented by $H_0 \equiv H_{04}^4$: $H_{04}^{4(1)} \cup H_{04}^{1(2)} \cup (H_{04}^{4(1)} \cap H_{04}^{4(2)})$ where we have $H_{04}^{4(1)}$: $\pi = 1$ and $H_{04}^{4(2)}$: $\gamma = 1$ and $\mu = \lambda$. Since under $H_{04}^{(1)}$, both γ and μ appear as nuisance parameters only under the alternative, we need a multiparameter generalization of the results of Davies (1977, 1987) to construct a statistic from, say $S(\gamma, \mu)$ as a generalization of $S(\mu)$ of Case 1(iii) above. Note that $H_{04}^{4(1)}$ specifies the parameter value on the boundary and hence (see Chernoff 1954) classical asymptotic theory for likelihood ratio statistic will not work. Further, complications also exist with the likelihood ratio test for $H_{04}^{4(2)}$. In even a simpler case withnormalmixtures,Hartigan(1985)demonstratedthat $L_n \to \infty$ in probability as $n \to \infty$, where L_n was the corresponding likelihood ratio statistic. Note that H_{04}^2 may be viewed as an "Intersection-Union" test, in contrast to the usual Union-Intersection test in multivariate analysis. Whereas some results on optimal properties, e.g., Bahadur optimality, of usual Union-Intersection tests or pooled tests are known, we are unaware of any optimality property of the Intersection-Union test.

We propose here a simple ad hoc test based on the pivotal parametric product (P^3),

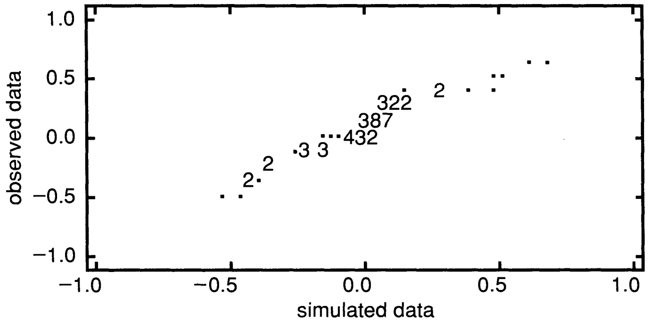
$$P^3 = (\pi - 1)\{(\gamma - 1)^2 + (\mu - \lambda)^2\}.$$

Observe that H_{04}^4 holds if and only if $P^3 = 0$ and under the alternative H_1 we have $P^3 < 0$. We propose rejecting H_0 ($\equiv H_{04}^4$) for small values of $\hat{P}^3$ where $\hat{P}^3$ is obtained from P^3 by replacing the parameters by their consistent estimators (in case efficient estimators are not easily available). The cut-off points need to be obtained by simulation. In many cases, tests based on P^3's are L-optimal (e.g., see SenGupta 1991). We have, however, not been yet able to establish this property for the test in the Laplace-Weibull mixture here.

Table 1 E-M Estimates of the Parameters of (3.1) for Certain Apartments in Paris

ψ	ϵ	$\hat{\psi}$	ϵ	$\hat{\psi}$
p	0.01	0.85	0.001	0.852
γ	0.01	5.00	0.001	5.070
λ	0.10	8.0	0.01	7.97
μ	0.10	45.0	0.01	45.39
	$n = 70$		$n = 173$	

Figure 1



4. Application

Here, we use (3.1) to model price changes for real estate data from Paris. The usefulness of Laplace and Weibull distributions to model commodity prices/price changes has been exemplified in §§1 and 2. Further, the mixture model seems to be an appropriate choice since the contamination may arise from the possible (small) change in the corresponding buyers/investors population due to immigration or emigration. Mixture distributions have been previously used in the context of housing prices by Quandt (1972) (see also Quandt and Ramsey 1978). The average price for one-bedroom apartments in Paris for each month of the period 1984–1989 were available. Define $x_i = ln(\xi_{2i}/\xi_{1i})$, where ξ_2 and ξ_1 are average prices for one-bedroom apartments in consecutive months, $i = 1, \ldots, 60$. The E-M algorithm of §3.2 was employed. In Table 1 we present the estimates of the parameters, number of iterations (n) needed for convergence and ϵ, the preassigned value such that the iteration terminates if the absolute difference between two successive estimates is smaller than this value. We used several different sets of starting values but in all the cases the algorithm converged to almost the same estimates. We next simulated 60 observations from (3.1) with the parameters being assigned the values of the corresponding estimators for $n = 173$. The Q—Q plot (Figure 1) exhibited almost a straight line and the quantile plots for the two data sets, observed and simulated, were fairly close.[1]

[1] The authors would like to thank the referee for several constructive remarks which have greatly improved the presentation of the paper. S. T. Rachev would like to thank M. Balinski, B. Cornet and V. Renard (Ecole Polytechnique) for very useful conversations about the distributions of commodity prices. A. SenGupta would like to acknowledge with thanks the support from NBHM, Department of Atomic Energy, Government of India and Indian Statistical Institute for this research. The data (prices of apartments in different districts of Paris) was kindly provided by M. Baudesson (Chambre des Notaires de Paris), whose assistance is greatly appreciated. The authors would also like to thank T. Kozubowski and A. Panorska for the computations of Table 1 and Figure 1.

This work was supported in part by NSF Grants DMS-89 02330 and DMS-9103452. S. T. Rachev's research was also sponsored by the Ministere de la Recherche et de la Technologie (France), with a "subvention pour chercheur etranger de haut niveau et de renomme internationale."

References

Bachelier, L., "Theorie de la Speculation," *Ann. Ecole Norm. Sup. Ser.* 3 17 21–86. English Transl. in: Coonter, P. H. (Ed.), (1967). *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press.

Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968.

Bones, A., A. Chen and S. Jatusipitak, "Investigations of Nonstationary Prices," *J. Business*, 47 (1974), 518–537.

Bolthausen, E., "Exact Convergence Rate in Some Martingale Central Limit Theorems," *Ann. Probab.*, 10 (1982), 672–688.

Büehler, W. J. and Puri, P. S., "On Optimal Asymptotic Tests of Composite Hypotheses with Several Constraints," *Z. Wahrscheinlichkeitstheorie.*, 5 (1966), 71–88.

Chernoff, H., "On the Distribution of the Likelihood Ratio, *Ann. Math. Statist.*, 25 (1954), 573–578.

Clark, P. K., "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," *Econometrica*, 41 (1973), 135–155.

Davies, R. B., "Hypothesis Testing When a Nuisance Parameter Is Present only under the Alternative," *Biometrika*, 64 (1977), 247–254.

——, "Hypothesis Testing When a Nuisance Parameter Is Present only under the Alternative, II," *Biometrika*, 74 (1987), 33–43.

Dempster, A. P., N. M. Laird and D. B. Rubin, "Maximum Likelihood from Incomplete Data via the EM Algorithm (with Discussion)," *J. Roy. Statist. Soc. Ser.* B, 39 (1977), 1–38.

DuMouchel, W., "Stable Distributions in Statistical Inference: 1. Symmetric Stable Distribution Compared to Other Symmetric Long-Tailed Distributions," *J. Amer. Statist. Assoc.*, 68 (1973), 469–477.

Durairajan, T. M. and B. K. Kale, "Locally Most Powerful Test for the Mixing Proportion," *Sankhya*, 41 (1979), 91–100.

Fama, E., "The Behavior of Stock Market Prices," *J. Business*, 38 (1965), 34–105.

Ghosh, J. K. and P. K. Sen, "On the Asymptotic Performance of the log Likelihood Ratio Statistic for the Mixture Model and Related Results," In *Proc. Berkeley Conf. in honor of Jerzy Neyman and Jack Kiefer, Vol.* II, L. M. Le Cam and R. A. Olshen (Eds.), Wadsworth, Belmont, CA, pp. 789–806.

Gnedenko, B. V. and H. Fahim, "On a Transfer Theorem," *Dok. Akad. Nauk SSSR*, 187 (1969), 15–17 (Russian).

Gudynas, P., "Approximation by Distributions of Sums of Conditionally Independent Random Variables," *Litovsk. Mat. Sbo.*, 24 (1985), 68–80,

Hasselblad, V., "Estimation of Finite Mixtures of Distributions from the Exponential Family," *J. Amer. Statist. Assoc.*, 64 (1969), 1459–1471.

Kalashnikov, V. V. and S. Yu. Vsekhsvyatskii, "On the Connection of Renyi's Theorem and Renewal Theory," In *Stability Problems for Stochastic Models*, V. V. Kalashnikov and V. M. Zolotarev (Eds.), Lecture Notes in Math., Springer-Verlag, Berlin and New York, 1989, pp. 83–109.

Klebanov, L. B., G. M. Maniya and I. A. Melamed, "A Problem of Zolotarev and Analogs of Infinitely Divisible and State Distribution in a Scheme for Summing a Random Number of Random Variables," *Theory. Probab. Appl.*, 29 (1984), 791–794.

——, —— and ——, "ν_p-strictly Stable Laws and Estimation of Their Parameters," In *Stability Problems for Stochastic Models*, V. V. Kalashnikov, B. Penkov and V. M. Zolotarev (Eds.), Lecture Notes in Math., Springer-Verlag, Berlin and New York, 1985, pp. 23–31.

——, —— and S. T. Rachev, "On the Products of a Random Number of Random Variables in Connection with a Problem from Mathematical Economics," In *Stability Problems for Stochastic Models*, Lecture Notes in Math., Springer-Verlag, Berlin and New York, 1989, pp. 103–109.

Korolev, V. Yu., "Nonuniform Estimates of the Accuracy of Approximation of Distributions of the Sums of a Random Number of Independent Random Variables by Mixtures of Normal Laws," *Teor. Veroyatnost. i Primenen.*, 34, No. 3 (1989), 589–593.

Levy, P., *Calcul des Probabilities*, Gauthier-Villars, Paris, 1925.

Mandelbrot, B. B., "The Pareto-Lévy Law and the Distribution of Income," *Internat. Econom. Rev.*, 1 (1960), 79–106.

——, "Sur Certain Prix Spéculatifs: Faits Empiriques et Modele Base sur les Processes Stables Additifs de Paul Lévy," *Comptes Rendus*, 254 (1962), 3968–3970.

——, "The Variation of Some Other Speculative Prices," *J. Business*, 40 (1967), 393–413.

——, *The Fractal Geometry of Nature*, W. H. Freeman & Co., New York, 1977.

Melamed, J. A., "Limit Theorems in the Set-up of Summation of a Random Number of Independent Identically Distributed Random Variables, In *Stability Problems for Stochastic Models*, V. V. Kalashnikov and V. M. Zolotarev (Eds.), Springer-Verlag, Berlin, 1988, 194–228.

Mittnik, S. and S. T. Rachev, "Stable Distributions for Asset Returns," *Appl. Math. Lett.* 2, No. 3 (1989), 301–304.

—— and ——, "Alternative Multivariate Stable Distributions and Their Applications to Financial Modeling," In *Stable Processes and Related Topics*, S. Cambanis et al. (Eds.), Birkhaüser, Boston, 1991, 107–121.

—— and ——, "Alternative Multivariate Stable Distributions and Their Applications to Financial Modeling," In *Stable Processes and Related Topics*, S. Cambanis et al. (Eds.), Birkhäuser, Boston, 1991, 107–119.

—— and ——, "Modeling Asset Returns with Alternative Stable Distributions," *Econometric Rev.* (to appear).

Neyman, J., "Optimal Asymptotic Tests of Composite Statistical Hypotheses," In *Probability and Statistics* (*The Harald Cramér Volume*), Almquist and Wiksells, Uppsala, Sweden, 1959, 213–234.

Quandt, R. E., "A New Approach to Estimating Switching Regressions," *J. Amer. Statist. Assoc.*, 67 (1972), 306–310.

—— and J. B. Ramsey, "Estimating Mixtures of Normal Distributions and Switching Regressions," *J. Amer. Statist. Assoc.*, 73 (1978), 730–738.

Rachev, S. T., *Probability Metrics*, Wiley, Chichester-New York, 1991.

—— and A. SenGupta, "Geometric Stable Distributions and Laplace-Weibull Mixtures," *Statist. Decisions* (to appear).

—— and L. Rüshendorf, "On the Rate of Convergence in the CLT with Respect to the Kantorovich Metric" (submitted).

Rackauskas, A., "On the Convergence Rate in Martingale CLT in Hilbert Spaces," Preprint 90-031, University of Bielefeld, 1990.

Rényi, A., "A Characterization of Poisson Processes," MTA *Mat. Kut. Int. Kōzel.*, 1 (1956), 325–335 (Hungarian).

Robbins, H. "On the Asymptotic Distribution of the Sum of a Random Number of Random Variables," *Proc. Nat. Acad. Sci.* USA, 34 (1948), 162–163.

——, "The Asymptotic Distribution of the Sum of Random Number of Random Variables," *Bull. Amer. Math. Soc.*, 54 (1948b), 1151–1161.

SenGupta, A., "Optimality of Multivariate Tests with Applications to Mixtures and Directional Distributions," *Statist. Probab. Lett.*, Special issue on Optimality of Multivariate Procedures, A. SenGupta and S. R. Jammalamadaka (Eds.) (1991 to appear).

—— and L. Vermeire, "Locally Optimal Tests for Multiparameter Hypotheses," *J. Amer. Statist. Assoc.*, 81 (1986), 819–825.

Szász, D., "Limit Theorems for the Distributions of the Sums of a Random Number of Variables," *Ann. Statist.*, 43 (1972), 1902–1913.

Wu, C. F., "On the Convergence Properties of the EM Algorithm," *Ann. Statist.*, 11 (1983), 95–103.

Ziemba, W. T., "Choosing Investment Portfolios when the Returns have Stable Distributions," In *Mathematical Programming in Theory and Practice*, P. L. Hammer and G. Zoutendijk (Eds.), North-Holland, Amsterdam, 1974, pp. 443–482.

Zolotarev, V. M., "One Dimensional Distributions," *Amer. Math. Soc. Transl.* Ser. 2, Vol. 65, 1986.

Accepted by William T. Ziemba, former Departmental Editor; received December 1991. *This paper has been with the authors* 3 *months for revision.*