Optimal tests for the mean direction of a von Mises distribution

Ashis SenGupta
Department of Statistics
University of California - Riverside, CA 93106, USA &
Applied Statistics Division, Indian Statistical Institute - Kolkata
203 B.T. Road, WB 700 108, India
and
Ho-Jan Chang
Department of Applied Mathematics, National Chung-Hsing University
Taichung, Taiwan, Republic of China

Address to which Proofs to be sent:

Professor Ashis SenGupta, Applied Statistics Division, Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata, WB 700 108, INDIA

Short Running Title:

Tests for mean direction

e-mail address of author to be contacted:

ashis@isical.ac.in

Abstract The problem of constructing optimal unconditional tests for a specified value of the mean direction parameter μ of a von Mises or circular normal distribution $CN(\mu, \kappa)$ against two-sided alternatives is considered. Motivated by the curved exponential family nature of the distribution and the associated curvature when κ is known, locally most powerful unbiased test is argued to be a good choice. The test statistic is seen to admit of a major simplification. Exact cut-off points are given. It is also shown that small sample sizes suffice to make this test perform well even for non-local alternatives. The asymptotic distributions of this test statistic are presented. When κ is unknown, reduction by invariance or similarity fails. We derive an asymptotically locally optimal test and another equivalent but simpler test for this case. Two real life examples are discussed.

Keywords and phrases: C_α - test, Curved exponential family, Directional data, Locally most powerful unbiased test, Statistical curvature.

AMS 1985 Subject Classification: 62 F12.

1 Introduction

We consider here the problem of constructing and implementing both exact and asymptotic optimal tests for the mean direction parameter μ of a von Mises or circular normal population $CN(\mu, \kappa)$ in the absence as well as in the presence of the nuisance parameter κ respectively. Let $\theta_1, \ldots, \theta_n$ be a random sample from $CN(\mu, \kappa)$, $0 \leq \mu < 2\pi, \kappa > 0$. We are interested in testing $H_0 : \mu = \mu_0$ against two-sided alternatives $H_1 : \mu \neq \mu_0$.

The non-regular exponential family (REF) nature of the CN distribution when κ is known, has constrained the development of exact optimal tests to only conditional ones. When κ is unknown, the CN distribution becomes a member of the REF. However, since κ is not a scale parameter, here again one is constrained to look at conditional tests. Unconditional similar or invariant tests are not available.

Mardia (1972, p. 138) notes that there is no uniformly most powerful (UMP) test for testing H_0 against the even the usual one-sided composite alternatives. Also, Mardia (1972, pp. 138-140) proposes the unconditional likelihood ratio test and the conditional unbiased test using Fisher's ancillary principle (see, Fisher, 1959, Section 4.4; Kendall and Stuart, 1967, pp. 217-218). The LRT is not known to be optimal for small sample size, while the other test is a conditional test.

First in section 2 we present the notion of statistical curvature (Efron, 1975) for a curved exponential family (CEF) and note its role in the development of optimal tests in a CEF. This motivates us to present in section 3 unconditional, yet simple and optimal, tests for the above testing problems by exploiting the approach of locally most powerful (LMP) tests. LMP tests have been found convenient to derive as well as to implement in various complex testing problems, particularly in the context of univariate (Durairajan and Kale, 1979; SenGupta and Pal, 1993a) and multivariate (Sengupta and Pal, 1993b) mixture models, of multivariate inference (Gokhale and SenGupta, 1986; SenGupta, 1987), of univariate and bivariate reliability models (SenGupta, 1994; SenGupta and Pal, 2000) of directional data, e.g. in testing for isotropy (SenGupta and Pal, 2001), for mean direction (SenGupta, 1991; SenGupta and Jammalamadaka, 1991, 2003), for outliers (SenGupta and Laha, 2001), for change-points (SenGupta and Laha, 2004), for independence (Arnold and SenGupta, 2003), for symmetry (SenGupta and Rattihalli, 2004), etc. For generalizations and further discussion on LMP tests see e.g.

SenGupta and Vermeire (1986), SenGupta (1991), Mukerjee and SenGupta (1993).

We will consider here the two-sided testing problem for both the cases, κ known and unknown. The one-sided testing problem with its associated geometry has been dealt with in details in SenGupta and Jammalamadaka (2003). In section 3.1 the case of κ known is considered. The exact LMP unbiased (LMPU) test is presented and it is demonstrated that a major simplification results in its form. The LMPU test here is both an unconditional as well as an unbiased test. Further, it possesses the important exact (valid for all sample sizes) optimality property of having maximum power, among all locally unbiased tests, for small departures from the null - the most difficult alternatives to detect in practice. Based on associated statistical curvature, we present a table exhibiting encouraging values (even trivial, i.e. 2, for $\kappa \geq 2.2$) of the minimum sample size which is expected to make this test perform well throughout the parameter space. We also derive both the null and the non-null asymptotic distributions of the LMPU test statistic. Some exact cut-off points are also tabulated to enhance the use of the LMPU test in practice. The geometry of the various tests employed facilitates the higher order power comparison of the tests.

In section 3.2 we consider the case when κ is unknown. We first derive the $C_\alpha-$ or the asymptotic LMP test of Neyman (1959). We next propose an asymptotically eqivalent optimal test which has an extremely elegant and simple form. Finally two examples corresponding to the two cases are given in section 4 to demonstrate the ease in implementation of the optimal tests proposed.

2 Definitions and Discussions

2.1 Curved Exponential Family

It will be useful here to present the notion of a curved exponential family (CEF). Let Y be a random variable taking on values in a nonempty open subset $\mathcal{O}$ of an Euclidean space and let $P(\widetilde{\boldsymbol{\Theta}}) = \{P_\eta \mid \eta \ \epsilon \ \boldsymbol{\Theta}\}$ be a class of probability measures on $\mathcal{O}$, where the parameter space $\widetilde{\boldsymbol{\Theta}}$ is a nonempty open subset of R^p and for $\eta_1 \neq \eta_2$ in $\boldsymbol{\Theta}$, $P_{\eta_1} \neq P_{\eta_2}$. Next let, $\widetilde{\boldsymbol{\Theta}} = \{\eta \ \epsilon \ \widetilde{\boldsymbol{\Theta}} \mid \eta = \psi(\beta), \beta \ \epsilon \ L\}$ be a "surface" in $\widetilde{\boldsymbol{\Theta}}$ parameterized by β, where L is a nonempty open subset of R^q with $q < p$ and $\psi(\cdot)$ is a known Borel bimeasurable bijection from L onto its image $\psi(L)$ in $\widetilde{\mathbf{\Theta}}$. We call the subfamily $P(\widetilde{\mathbf{\Theta}}) = \{P_\eta \mid \eta \ \epsilon \ \mathbf{\Theta}\}$ a "curved" family in $P(\widetilde{\mathbf{\Theta}})$. Further, let

$$P(\widetilde{\mathbf{\Theta}}) = [f(t \mid \beta) = \exp\{< t, \eta(\beta) > -h(t) - \tau(\beta)\}, \beta \epsilon L] \tag{1}$$

where $< t, \eta(\beta) >= \ \sum_{i=1}^{p} t_i \eta_i(\beta)$, with range of $t_i, i = 1, \ldots, p$ independent of β. $P(\widetilde{\mathbf{\Theta}})$ given by (1) can be looked upon as a "reduced" dimensional exponential family with respect to a σ-finite measure ν where $\dim(\theta) = q < p = \dim(Q), Q$ being a minimal sufficient statistic.

2.2 Statistical Curvature

Consider the family in (1) with $q = 1$, i.e. the one-parameter CEF. Let $\sum_\beta = Cov_\beta(T)$. Denote the componentwise derivatives of $\eta(\beta)$ with respect to β by, $\dot{\eta}(\beta) \equiv (\partial/\partial\beta)\eta(\beta), \ddot{\eta}(\beta) \equiv (\partial^2/\partial\beta^2)\eta(\beta)$. Assume that these derivatives exist continuously in a neighborhood of a value of β where we wish to define the curvature. Let,

$$M_\beta \equiv \begin{bmatrix} \nu_{20}(\beta) & \nu_{11}(\beta) \\ \nu_{11}(\beta) & \nu_{02}(\beta) \end{bmatrix} \equiv \begin{bmatrix} \dot{\eta}(\beta)\prime \sum_\beta \dot{\eta}(\beta) & \dot{\eta}(\beta)\prime \sum_\beta \ddot{\eta}(\beta) \\ \ddot{\eta}(\beta)\prime \sum_\beta \dot{\eta}(\beta) & \ddot{\eta}(\beta)\prime \sum_\beta \ddot{\eta}(\beta) \end{bmatrix},$$

$$\gamma_\beta^2 = \mid M_\beta \mid / \nu_{20}^3(\beta).$$

Then, γ_β is the statistical curvature of P at β.

2.3 CEF and LMP test

In a CEF if an exact ancillary statistic exists, then for purposes of inferences regarding β, the principle of conditionality is often used. However, whether or not an exact ancillary statistic exists, even then it would be desirable to utilize Q to form an optimal and preferably an unconditional (for the sake of simplicity) test. If an UMP test does not exist then the LMP test can be an attractive choice, particularly if it utilizes all the components of Q . However, in a non-regular exponential family there are specific examples (e.g. Chernoff., 1951) which demonstrate that the choice of the LMP test can be disastrous.

Consider in general the LMP test for say, $H_0 : \beta = \beta_0$. Efron suggests that a value of $\gamma^2_{\beta_0} < \frac{1}{8}$ is not "large" and one can expect linear methods to work "well" in such a case. In repeated sampling situations, the curvature ${}_m\gamma^2_{\beta_0}$ based on m observations satisfies, ${}_m\gamma^2_{\beta_0} = {}_1\gamma^2_{\beta_0}/m$, and hence one can determine the sample size which reduces the statistical curvature below 1/8.

3 Tests for mean direction

The probability density function of $CN(\mu, \kappa)$ is given by:

$$f(\theta;\mu,\kappa) \;=\; [2\pi I_0(\kappa)]^{-1}\exp\{\kappa\cos(\theta-\mu)\},\;\; 0<\theta,\mu\leq 2\pi, 0\leq\kappa<\infty. \quad (2)$$

Suppose we have a random sample $\theta_1, ..., \theta_m$ from (2). Define $(C, S) = (\Sigma_{i=1}^m \cos\theta_i, \Sigma_{i=1}^m \sin\theta_i)$. Then the mean direction $\bar{\theta}, 0 \leq \bar{\theta} < 2\pi$, and the resultant $R, 0 \leq R \leq m$, are defined through,

$(C, S) = (R\cos\bar{\theta}, R\sin\bar{\theta});\quad R = \Sigma_{i=1}^m \cos(\theta_i - \bar{\theta}) = (C^2 + S^2)^{1/2}$.

Observe that in general the CN population is a REF, but with κ known it becomes a member of (1,2) CEF, i.e. a CEF having a 1-dimensional parameter μ with a 2-dimensional sufficient statistic (C,S) for it. Since μ can be treated as a location parameter, without loss of generality, we will take $\mu_0 = 0$. Unlike the linear case, note that for the circular case it makes sense to have the mean (direction) at an endpoint (e.g. 0) of the support of the random variable. Further it also makes sense to have a two-sided alternative $H_1 : \mu \neq 0$ against the null hypothesis $H_0 : \mu = 0$. Note that we interpret the two-sided alternative for the circular case, in contrast to the linear case, in terms of the circular proximity of the values under the alternative to that the null. This may thus be interpreted for a two-sided local alternative as μ lying either in the arc $(0, \delta_1]$ or in the arc $[2\pi - \delta_2, 2\pi)$, where $\delta_i > 0, i = 1, 2$, is some small angle.

3.1 Case 1. κ known.

Consider testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. Assume κ is known, say, equal to 1. Note that since κ is not a scale parameter, tables of cut-off points need to be supplied for different values of the continuous parameter $\kappa > 0$ to render the tests to be useful in practice. An unconditional LMP test may be quite useful here, provided its performance is satisfactory which may be initially judged through the statistical curvature associated with the CEF nature of the present CN distribution and this testing problem.

We consider the following tests : a test based on the maximum likelihood estimator (MLE), the likelihood ratio test (LRT) and the LMPU tests. The first one is ad-hoc in nature while the form of the LRT is already available from SenGupta and Jammalamadaka (2001, pp. 114-116). The LMPU merits special mention here, since unlike the other two tests, not only does it possess an exact optimality property, but the test statistic also is quite elegant. This results from the symmetry of the CN distribution and an exploitation of the reflection principle. Exact cut-off points of the LMPU test can be easily obtained either by numerical integration or through simulations. Two essential properties that ant reasonable test should possess are shown to hold for the LMPU test: the exact (all sample size) property of admissibility and the large sample property of consistency. The latter is particularly important in the context of a LMP test since it is known that such tests may in some cases (see e.g. Chernoff, 1951; Ferguson, 1967) indeed turn out to be inconsitent. Asymptotic normality of the LMPU test statistic under both the null and alternative hypotheses are also established. We present in Sec. 3.2 a result on the large-sample higher-order power comparison of these three tests.

3.1.1 Test Based on the MLE.

It is easy to see that the maximum likelihood estimator of μ is given by, $\hat{\mu} = \tan^{-1}(S/C)$. Motivated by the geometry of the problem and based on the description of the alternative as given above, we propose the test given by, $\omega : \hat{\mu} \in arc(\eta_1, 2\pi - \eta_2), \eta_i > 0, i = 1, 2$. For $\mu_0 = 0$, this may be written in the simpler form: $\omega : \hat{\mu} > K_1$ or $< K_2, 0 < K_1 < K_2 < 2\pi$. However in terms of general $\mu_0 (\neq 0)$, the critical region proposed takes the form: $\omega : \hat{\mu} \in arc(K_1, K_2)$ where K_1 may indeed be larger than K_2. The constants used above are to be determined from the size and (local) unbiasedness conditions given in (3) below. The determination of these constants however turns out to be non-trivial and we do not proceed with it further.

3.1.2 Locally Most Powerful Unbiased (LMPU) Test.

The concept of LMPU test may be attributed to Pearson and Neyman (1936, 1938) - see also Ferguson (1972, pp 237-238). The aim is to find a test φ_0

out of all α-level unbiased tests which maximizes the curvature (the slope for one-sided tests) of the power function at the null value. Consider the class $\mathcal{C}$ of tests such that any test $\varphi \in \mathcal{C}$ satisfies

$$\mu_\varphi(\mu)\,|_{\mu=0} = \alpha \qquad \text{and} \qquad \mu'_\varphi(\mu)\,|_{\mu=0} = 0 \tag{3}$$

The test $\varphi_0 \in \mathcal{C}$ is an LMPU test if it maximizes the value of the second derivative of $\mu_\varphi(\mu)$ at $\mu = 0$, that is

$$\mu''_{\varphi_0}(\mu)\,|_{\mu=0} > \mu''_\varphi(\mu)\,|_{\mu=0} \ .$$

We now motivate LMPU test for our problem. The LRT here is not known to have any exact optimality property. Note that also no small sample unconditional optimal test for H_0 is yet available. Mardia (1972, pp 138 - 141) and Mardia and Jupp (2000) present conditional tests based on the "Fisher Ancillary Principle". An interesting point to note though is that this conditioning does not introduce any additional dimension in the induced "parameter" (including the known value of the conditioning variable) space since the conditioning variable, the resultant R, is amalgamated in the new known concentration parameter. Fisher (1993, pp 93 - 94) suggests tests based on large samples and bootstrap technique. These are quite interesting approaches. However, small sample optimality properties of these tests are yet to be established.

The score test discussed by Mardia and Jupp (2000) also needs to be mentioned here. Though it also is based on derivative of the log likelihood function, only the first derivative is used in contrast to both the first and second derivative needed for the LMPU test. Further, unlike the LMPU test, this test is not known to possess any exact optimality property and hence will not be discussed further in this paper.

In practice we will usually be interested in testing against 'local' alternatives, i.e. alternatives close to the null, which however are also more difficult to detect. In the absence of a UMP test, a LMP test is then a natural choice. Since our underlying distribution is a member of the CEF, the suitability of this LMP test may be further evaluated through the associated statistical curvature. We show that here the LMP approach not only yields a simple and elegant test statistic but further that the test is expected to work "well" , as revealed by γ_0^2, for small sample size, e.g. even as small as 15 when $\kappa = 1$.

Inspite of these favorable poperties of a LMPU test, it cannot be unequivocally advocated. This test need not be even admissible or consistent. We establish below that the LMPU test for our testing problem does not suffer from any of these drawbacks.

Let [x] denote the greatest integer in x, $\bar{C} = C/m, \bar{S} = S/m\bar{R} = R/m$ and $A \equiv A(\kappa) = I_0(\kappa)/I_1(\kappa)$. We then have the following

Theorem 1. The LMPU test for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$, is given by

$$\begin{aligned} \omega : \quad & \{(\bar{\theta}, r) : \bar{\theta} \ \in (c_{1r}(K), c_{2r}(K))U(2\pi - c_{2r}(K), 2\pi - c_{1r}(K))| \\ & (0 \leq \bar{\theta} < 2\pi, 0 < r \leq n)\} \end{aligned} \tag{4}$$

where

$$\begin{aligned} c_{ir}(K) &= arc\cos a_{ir}(K), i = 1, 2. \ \ a_{1r}(K) > a_{2r}(K), \\ a_{1r}(K) &= [-\bar{r} + \{\bar{r}^2 - 4(m\kappa\bar{r}^2)(K - m\kappa\bar{r}^2)\}^{1/2}]/[2(m\kappa\bar{r}^2)], \\ a_{2r}(K) &= [-\bar{r} - \{\bar{r}^2 - 4(m\kappa\bar{r}^2)(K - m\kappa\bar{r}^2)\}^{1/2}]/[2(m\kappa\bar{r}^2)]. \end{aligned}$$

Using the size condition, K is determined from the equivalent critical region

$$\omega : \qquad -\bar{C} + m\kappa\bar{S}^2 > K. \tag{5}$$

(ii) This test is admissible. (iii) A sample size $m = [8\{1/A^2 - 1/A\kappa - 1\}] + 1$, which monotonically decreases with κ, suffices to reduce the statistical curvature below Efron's critical value.

Proof: A LMPU test can be found by using the generalized Neyman-Pearson Lemma (see, Lehmann, 1986, pp. 96-101). According to this lemma, for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$, a LMPU test will have critical region given by

$$\frac{\partial^2}{\partial\mu^2} L(\theta^*, \ \mu)|_{\mu=0} + (\frac{\partial}{\partial\mu} L(\theta^*, \mu)|_{\mu=0})^2 > k_1 + k_2(\frac{\partial}{\partial\mu} L(\theta^*, \ \mu)|_{\mu=0}) \tag{6}$$

where $\theta^* = (\theta_1, ..., \theta_m)'$, $L(\theta^*, \ \mu)$ is the log likelihood function of $(\theta_1, \ \ldots, \theta_m)$ and $k_1, \ k_2$ are determined by (3). Then the critical function of this LMPU test is given by

$$
\begin{aligned}
\varphi_0(\theta^*,\ \mu) &= 1 \qquad \text{if} \qquad (-\kappa\sum_{i=1}^{m}\cos\theta_i) + (\kappa\sum_{i=1}^{m}\sin\theta_i)^2 > k_1 + k_2(\kappa\sum_{i=1}^{m}\sin\theta_i) \\
&= 0 \qquad \text{otherwise} \qquad\qquad (7)
\end{aligned}
$$

Define $U = \kappa\sum_{i=1}^{m}\sin\theta_i$ and $V = -\kappa\sum_{i=1}^{m}\cos\theta_i + (\kappa\sum_{i=1}^{m}\sin\theta_i)^2$.

Exploiting the facts that the von Mises density is symmetric, that $(U,\ V)$ and $(U,\ -V)$ are equal in distributions since V is an even function under H_0 and the reflection principle (Ferguson, 1967), it follows (Chang, 1991) that $k_2 = 0$. Thus the test reduces to the simple form given in (4).

?? According to Lemma 1 above,?? the critical region of the LMPU test becomes

$$\omega: \quad -C + \kappa S^2 > k_1, \qquad \omega: \quad \bar{T} \equiv -\bar{C} + m\kappa\bar{S}^2 > K$$

where K is determined by the given level of significance.

Note that since

$$\bar{C} = \bar{R}\cos\bar{\theta}, \quad \bar{S} = \bar{R}\sin\bar{\theta}, \bar{R} = R/m; \bar{C}^2 + \bar{S}^2 = \bar{R}^2.$$

The above critical region then can be rewritten as,

$$\omega: \qquad m\kappa\bar{r}^2\cos^2\theta + \bar{r}\cos\theta - m\kappa\bar{r}^2 + K < 0, ...(XX)$$

which, due to the convexity of the function, can be equivalently written as,

$$a_{1r}(K) < \cos\bar{\theta} < a_{2r}(K), ...(YY)$$

where $0 \leq a_{1r}(K), a_{2r}(K) \leq \pi/2$ are the two roots of the quadratic expression on the LHS of (XX) equated to zero. Now, from the nature of the cosine curve it follows that (YY) yields that $\bar{\theta}$ for each $\bar{r}$ should lie in the union of two disjoint arcs as presented in the theorem.

(ii) Admissibility of the test follows due to the uniqueness of the LMPU test - a consequence of the non-randomized nature of the critical region corresponding to a continuous test statistic.

(iii) Finally, consider the associated statistical curvature. At $\mu = 0$, Var(sin θ) $= A(\kappa)/\kappa$; Var(cos θ) $= 1 - A(\kappa)/\kappa - A^2(\kappa)$ and Cov(cos θ, sin $\theta) = 0$. Further, $\dot{\eta}(0) = (0, \kappa), \ddot{\eta}(0) = (-\kappa, 0)$. Simplification yields $\gamma_0^2(\kappa) = 1/A^2 - 1/A\kappa - 1$. It then follows that $(\partial/\partial\kappa)\gamma_0(\kappa) < 0$, i.e. $\gamma_0(\kappa) \downarrow \kappa$. Also demanding ${}_m\gamma_0^2(\kappa)$ ¡ 1/8, the critical value suggested by Efron, yields the value of m as stated in the theorem.

Remarks: 1. Note that the LMPU test is a function of both the components C and S or equvalently $\bar{\theta}$ and R of the sufficient statistic and is an unconditional test.

2. Table 2 provides the desired minimum values of m, as per Efron's criterion, for various values of κ which should make the LMP test work well. In particular with $\kappa = 1$, to achieve ${}_m\gamma_0^2(1)$ ¡ 1/8, the critical value, it suffices to have $m > 14.266$, i.e. m =15. Such a sample size should be easily available, implying thereby that the LMPU test will work "well" in practice. Note that the required sample size m which suffices becomes trivial (i.e. 2) for $\kappa > 2.2$.

3.1.3 Exact cut-off points

Small sample cut-off points may be obtained through numerical integration or by simulation. For the former approach, the following representation of the tail probability of $\bar{T}$ under H_0 is convenient.

$$\alpha = P(\bar{T} > t_\alpha) = \int_0^m P(-\bar{r}\cos\bar{\theta} - m\kappa\bar{r}^2\cos^2\bar{\theta} + m\kappa\bar{r}^2 > t_\alpha | R = r) f_R(r)dr$$

$= \int_0^m P(\bar{\theta} \in \mathcal{S}_r | R = r) f_R(r)dr$ say.

Similar representation for the power as the above for the size of the test shows that the power of the LMPU test is a weighted average of the powers of the conditional (for each r) test. We can then use the facts that the conditional density of $\bar{\theta}|(R = r)$ is $CN(\mu, \kappa r)$ and $f_R(r)$, the marginal density of R is available from, e.g. equation (4.5.4) in Mardia and Jupp (2000), p. 69. Gaussian quadrature and iterative techniques (e.g. see SenGupta and Jammalamadaka, 2001, for the one-sided LMP test) may then be employed to get the cut-off points.

Alternatively, one can generate the distribution of $\bar{T}$ by simulating observations from the CN distribution, e.g. by the algorithm of Best and Fisher (1979). Cut-off points for $\alpha = 0.01,\ 0.025,\ 0.05,\ 0.10, \kappa = 0.1,\ 0.5,\ 1.0,\ 1.5,\ 2.0$, and sample sizes $m = 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 20,\ 30,\ 50,\ 100$ obtained by simulation are given in Table 1.

The cut-off points for large samples are easily obtained for all levels of significance α and all κ values by virtue of theorem 2 below.

3.1.4 Asymptotic distribution of $\bar{T}$

We now study both the null and non-null asymptotic distributions of $\bar{T}$. These are shown to be normal.

Let us denote the raw trigonometric moments as

$$E(\cos p\theta) = \alpha_p, \quad E(\sin p\theta) = \beta_p$$

and the central trigonometric moments as,

$$E(\cos(p(\theta - \mu))) = \alpha_p^*, \quad E(\sin(p(\theta - \mu))) = \beta_p^* = 0.$$

Also for brevity we will write α_p^* as B_p.

The asymptotic distribution of the LMPU test statistic is given by

Theorem 2. As $m \to \infty$,

$$\sqrt{2m}\{\bar{T} + A(\kappa)\cos\mu - \kappa m(A(\kappa)\sin\mu)^2\} \xrightarrow{\mathcal{L}} N(0,\ \sigma^2),$$

where

$$\begin{aligned}\sigma^2 \;&=\; \{1 + B_2(\kappa)\cos 2\mu - 2(A(\kappa)\cos\mu)^2\} - 4A(\kappa)\kappa m \sin\mu\{B_2(\kappa) - A^2\}\sin 2\mu \\ &\quad +4(A(\kappa)\kappa m\sin\mu)^2\{1 - B_2(\kappa)\cos 2\mu - 2(A(\kappa)\sin\mu)^2\}.\end{aligned}$$

Corollary. Under H_0, as $m \to \infty$,
$\sqrt{2m}\{\bar{T} + A(\kappa)\} \xrightarrow{\mathcal{L}} N(0, \{1 + B_2(\kappa) - 2A^2(\kappa)\})$
and $\bar{C}$ and $\bar{S}$ are asymptotically independent.
Proof: We first need to prove the following

Lemma 2: For a $CN(\mu, \kappa)$ population, the trigonometric moments are given by,

$$\begin{gathered}\alpha_p^* = I_p(\kappa)/I_0(\kappa) = B_p(\kappa), \quad \beta_p^* = 0, \\ E(\cos(p(\theta - \mu))\sin(q(\theta - \mu))) = 0, p, q \geq 1,\end{gathered} \tag{8}$$

where $I_p(\kappa) = \int_0^2 \pi \cos p\theta \exp(\kappa \cos p\theta) d\theta$ is the modified p-th order Bessel function of the first kind. Also, in terms of the standard notations (see e.g., (A.11) of Mardia and Jupp, 2000, p. 350)

$$B_1(\kappa) = I_1(\kappa)/I_0(\kappa) = A_2(\kappa) \equiv A(\kappa), A_p(\kappa) = I_{p/2}(\kappa)/I_{p/2-1}(\kappa).$$

Proof: The results follow on noting the symmetry of the CN distribution, the even and odd nature of the cosine and sine functions respectively, and the definition of $I_p(\kappa)$.

Now,

$$\begin{array}{l} E\cos\theta = \alpha_1 = A(\kappa)\cos\mu, E\sin\theta = \beta_1 = A(\kappa)\sin\mu \\ Var(\cos\theta) = \frac{1}{2}\{1 + \alpha_2 - 2\alpha_1^2\} = \frac{1}{2}\{1 + B_2\cos 2\mu - 2(A\cos\mu)^2\} \quad \equiv \quad \sigma_{11} \\ Var(\sin\theta) = \frac{1}{2}\{1 - \alpha_2 - 2\beta_1^2\} = \frac{1}{2}\{1 + B_2\cos 2\mu - 2(A\sin\mu)^2\} \quad \equiv \quad \sigma_{22} \\ Cov(\cos\theta, \sin\theta) = \frac{1}{2}\{\beta_2 - 2\alpha_1\beta_1\} = \frac{1}{2}\{B_2 - A^2\}\sin 2\mu \quad \equiv \quad \sigma_{12}. \end{array}$$

Then by the Central Limit Theorem and on using lemma 2,

$$\sqrt{m}(\bar{C} - \alpha_1, \bar{S} - \beta_1) \xrightarrow{\mathcal{L}} N_2(\underline{0},\ \Sigma) \tag{9}$$

where the covariance matrix $\Sigma = ((\sigma_{ij}))$ has elements defined above.

To derive the asymptotic distribution of the test statistic, we invoke the δ- method given by

Lemma 3. [Rao (1973), p 387.] Let $(\sqrt{m}(T_{1m} - \eta_1),\ \ldots,\ \sqrt{m}(T_{km} - \eta_k))$ have asymptotic k-variate normal distribution with mean zero and covariance matrix $\Sigma = ((\sigma_{ij}))$ with $\sigma_{ij} = Cov(T_i,\ T_j)$, $i = 1,\ \ldots,\ k$ and $j = 1,\ \ldots,\ k$. Furthermore, let g be a function of k variables which is totally differentiable. Then, $\sqrt{m}[g(T_{1m},\ \ldots,\ T_{km}) - g(\eta_1,\ \ldots,\ \eta_k)]$ has the asymptotic normal distribution with mean zero and variance $\sum_i \sum_j \sigma_{ij} \frac{\partial}{\partial\theta_i} g \frac{\partial}{\partial\theta_j} g$.

Define $g(\bar{C}, \bar{S}) = -\bar{C} + \kappa m \bar{S}^2$. Then, $g(\alpha_1,\ \beta_1) = -\alpha_1 + \kappa m \beta_1^2$.

By lemma 3 we are able to conclude that

$$\sqrt{m}\{-\bar{C} + \kappa m \bar{S}^2 - (-\alpha_1 + \kappa m \beta_1^2)\} \xrightarrow{\mathcal{L}} N(0,\ \sigma^2),$$

where

$$\sigma^2 \quad = \quad \sigma_{11}\left(\frac{\partial}{\partial\alpha_1} g(\alpha_1,\ \beta_1)\right)^2 + 2\sigma_{12}\left(\frac{\partial}{\partial\alpha_1} g(\alpha_1,\ \beta_1)\right)\left(\frac{\partial}{\partial\beta_1} g(\alpha_1,\ \beta_1)\right)$$

$$
\begin{aligned}
& +\sigma_{22}\left(\frac{\partial}{\partial\beta_1}g(\alpha_1,\ \beta_1)\right)^2 \\
= \quad & \sigma_{11} - 4\kappa m\beta_1\sigma_{12} + 4(\kappa m\beta_1)^2\sigma_{22}
\end{aligned}
$$

On simplifictions, the expressions in the theorem now result.

The corollary follows on putting $\mu = 0$.

3.1.5 Consistency of the LMPU test.

Let us indicate with the superfixes 0 and 1 the relevant quantities under $H_0 : \mu = 0$ and $H_1 : \mu = \mu(\neq 0)$ respectively. For example,

$$\alpha_1^0 = A(\kappa), \beta_1^0 = 0; \alpha_1^1 = A(\kappa)\cos\mu, \beta_1^1 = A(\kappa)\sin\mu.$$

Then we have the following **Theorem 3.** The LMPU test given above is consistent. Proof: By theorem 2 it follows that for large samples $K = (\sigma_0\tau_\alpha)/(m^{1/2}) - \alpha_1^0$, where τ_α is the upper 100 $\alpha\%$ point of the standard normal distribution. Then for large samples,

$$P[\bar{T} > K|H_1] = P[Z > (\sigma_0/\sigma_1)\tau_\alpha\ \ -m^{1/2}\{(m\kappa(\beta_1^1)2) + A(\kappa)(1-\cos\mu)\}] \longrightarrow ????$$

Remarks: Sometimes moments of $\sin\theta$ and $\cos\theta$ (instead of trigonometric moments of θ) are needed. These may be obtained by repeated differentiation under the integration sign as many times as needed and exploiting the properties used above for obtaining the trigonometric moments. Alternatively, on noting that the CN distribution is a member of the exponential family, these may also be obtained by an useful generalization of Stein's identity to the univariate multiparameter exponential families noted by Arnold et al (2001).

3.1.6 Higher-order Power Comparison.

For the case of the two-sided alternatives, the test based on the MLE and the LR test discussed above and also the LMP test are unconditional tests. Except for the LMP test, which is optimal for all sample sizes in the sense of maximum local power, no small-sample property of the other two tests is known. However, using standard results, e.g., following Amari (1985), we get the following results on the deficiencies of the tests.

Lemma 4. The third order power losses of the LMP test compared to the MLE and the LR tests are respectively given by

$$\begin{aligned} {}_LL_M(t) &= [(1-1/(2\tau^2_{\alpha/2})-J(t))/J(t)]^2 \quad \text{and} \\ {}_LL_R(t) &= [(1-1/(2\tau^2_{\alpha/2})-J(t))/(1/2-J(t))]^2, \end{aligned} \tag{10}$$

where $\tau_{\alpha/2}$ is the upper $100\alpha/2\%$ point of the standard normal distribution $\phi(\cdot)$,

$\xi(t) = (t/2)[\phi(\tau_{\alpha/2}-t)-\phi(\tau_{\alpha/2}+t)]$ and $\mathrm{J(t)} = 1 - \mathrm{t}/[2\tau_{\alpha/2} \quad \tanh \mathrm{t}\tau_{\alpha/2}]$.

Proof: This result follows from Theorems 6.6, 6.7, and 6.8 of Amari.

3.2 Case 2. κ unknown

When κ is unknown, the principle of similarity or meaningful invariance does not lead to any reduction and hence no unconditional useful test is available. A conditional test may be derived, and even a conditional LMP test may be envisaged, i.e. a LMP test obtained from the conditional distribution free from the nuisance parameter κ. but as for the κ known case, this will also call for extensive tables corresponding to the conditioning continuous random variable, to be useful in practice. One may ofcourse use the usual LRT. However, then neither any simple test statistic results nor is any small sample optimal property of this LRT known.

3.2.1 The C_α Test.

Here, we show that a simple and elegant yet an unconditional asymptotically optimal test, e.g. Neyman's C_α test can be derived.

Theorem 3. The C_α-test for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ is given

by

$$\omega : \quad |Z_m| = |\sqrt{\hat{\kappa}} \sum_{i=1}^{m} \sin\theta_i/(mA(\hat{\kappa})^{1/2}| > \quad \tau_\alpha/2 \tag{11}$$

Proof: Let, $\phi = ln \quad f(\theta,\kappa)$. Then at $\mu = 0, \phi_\mu = \kappa\sin\theta, \phi_\kappa = \cos\theta - A(\kappa)$. Assume $\kappa < K_0 < \infty$. Then straightforward computations establish that all the conditions for ϕ_μ and ϕ_κ to be Crámer functions are satisfied.

The C_α- test is given as

$$\omega : \quad |Z_m^*| = |\sum_{i=1}^{m}\{\phi_\mu\{\theta, \hat{\kappa}\} - a_1^0 \phi_\kappa(\theta, \hat{\kappa})\}/\sqrt{m}\sigma_0(\hat{\kappa})| > \quad \tau_\alpha/2 \qquad (12)$$

where $\hat{\kappa}$ is any locally root m consistent estimator of κ under $H_0, \sigma_0(\hat{\kappa})$ is the standard deviation of $\phi_\mu(\theta, \kappa) - a_1^0\phi_\kappa(\theta, \kappa)$ under H_0 and evaluated at $\kappa = \hat{\kappa}, a_1^0$ is the partial regression coefficient of ϕ_μ on ϕ_κ and τ_α is the upper $100\alpha\%$ point of the standard normal distribution. One may, e.g., take $\hat{\kappa}$ as the MLE of κ under H_0, i.e., $\hat{\kappa} = Max\{0; A^{-1}(C/m), C > 0\}$. Further, a_1^0 is seen to be 0 by direct computation. Also, $E(\delta^2\ell/\delta\mu\delta\kappa) = 0$ under H_0, holds. Then, the numerator of Z_m^* reduces to $\kappa \sum \sin\theta_i$ and thus $\sigma_0^2(\kappa)$ reduces to, $\sigma_0^2(\kappa) = Var_{\mu=0}(\kappa \sin\theta) = \kappa A(\kappa)$. Thus (12) reduces to the simple form given in theorem 3.

For any sequence $\mu^* = \{\mu_n\}$ such that $\mu_n\sqrt{n} \to \gamma$, the asymptotic value of the power of the test is given by

$$1 - (1/\sqrt{2\pi}) \int_{-\tau_\alpha/2}^{\tau_\alpha/2} \exp\{-(t - \sigma_0(\kappa)\gamma)^2/2\}dt.$$

Among all tests, T_m^*, for $H_0 : \mu = 0$ with asymptotic level of significance α, whatever be the sequence of alternatives $\mu_m \neq 0$ with $\mu_m \to \mu_0 = 0$, and whatever be the fixed $\kappa > 0$,

$$\underline{\lim} \; [\text{ Power } \{T_m(\mu_m, \kappa)\} - \text{ Power } \{T_m^*(\mu_m, \kappa)\}] \quad \geq 0.$$

The test T_m is in this sense an asymptotically locally most powerful test.

3.2.2 A simple optimal test

(11) involves computation of $\hat{\kappa}$. This may be avoided to give an even simpler but nevertheless (asymptotically) equivalent test. Note that, $\sigma_0^2(\kappa) = \kappa^2 E_0(\sin^2\theta)$ and $\sum_{i=1}^m \sin^2\theta_i/m$ is a consistent estimator of $E_0(\sin^2\theta)$. Then (11) reduces to,

$$\omega : \quad |T_m| = |\sum_{i=1}^{m} \sin\theta_i/(\sum_{i=1}^{m}\sin^2\theta_i)^{1/2}| \quad > \quad \tau_\alpha/2. \qquad (13)$$

T_m is asymptoticaly equivalent to Z_m in the sense that it has, by Slutsky's theorem, the same limiting distribution as that of Z_m. In Example 2 we show that the numerical equivalence of these two test statistics can hold for even as small a sample size as $m = 15$.

4 Examples

Here we present two examples, one each corresponding to the κ known and unknown cases. These demonstrate the ease in the implementation of our LMPU tests proposed above.

Example 1. We first consider the κ known case. This example is the example 6.5 in Mardia (1972, pp 141-142) or the example 7.1 in Mardia and Jupp (2000, p 121) where a conditional test has been used. A sample of size 10 on the dip-directions of cross-beds of a section of river gave $\bar{\theta} = 278°$ and $\bar{R} = 0.35$. It is known that $\mu = 342°$ and $\kappa = 0.8$ for a neighboring section of the river. We are interested to know whether the claim that the mean direction for the section sampled is the same as that of the neighboring one.

To use our framework as in section 3, i.e to test $H_0 : \mu = 0$, we introduce the translation $\theta = \alpha - 342°$ (mod 2π), where θ is the transformed observation obtained from the original observation α. This gives

$\bar{C}_\theta = \cos 342° \bar{C}_\alpha + \sin 342° \bar{S}_\alpha$, $\bar{S}_\theta = \cos 342° \bar{S}_\alpha - \sin 342° \bar{C}_\alpha$.

These give $(\bar{C}_\theta, \bar{S}_\theta) = (.153442, -.314586)$.

Note that $\bar{R}_\theta = \bar{R}_\alpha = .35$, since the resultant direction is invariant under any location shift. These give $T = -.0742$, which falls below the critical value at the 5% level of significance as is easily seen from Table 1. Thus the claim is not refuted, which is also the conclusion arrived at earlier.

Example 2. We finally consider the κ unknown case. This example is the example 4.23 in Fisher (1993, p. 94) or example 7.2 in Mardia and Jupp (2000, p 125). Schmidt-Koenig(1963) give the vanishing angles of 15 pigeons from an experiment on their homing ability. Tests have been conducted for "the null hypothesis that their mean vanishing direction μ is in fact in the direction of their loft (149^0), against the alternative that they cannot navigate straight home."

Fisher has used an intuitive large sample test based on $\sin(\bar{\theta} - \mu_0)$, where we have $H_0 : \mu = \mu_0$. Mardia and Jupp use the inversion of the conditional confidence interval for μ based on the conditional distribution of $\bar{\theta}$ given R.

As in example 1 above, we introduce the transformation $\theta = \alpha - 149^0$ (mod 2π). We conduct the test by using both the statistics T_m and Z_m, for the sake of illustration and more importantly for evaluating the performance of the approximation.

Calculations give $C_\theta = 10.32098, S_\theta = 3.64687$, so that $\bar{R}_\theta = 0.72976$. Then under $H_0, \hat{\kappa} = A^{-1}(\bar{C}) = 1.95$ (from Fisher, 1993, Table A.3, p. 224). Thus from (11) we have,

$|Z_m| = |\sqrt{\hat{\kappa}} \sum_{i=1}^{m} \sin\theta_i / \sqrt{C}| = 1.5852 < 1.9604 = \tau_{.025}$.

Next from (13) we have, $|T_m| = 1.5877$.

Both these test statistics lead to the same conclusion at the 5% level of significance. There is not enough evidence to refute the ability of the pigeons to navigate to their home, which is also the conclusion arrived at earlier e.g., by Fisher, Mardia and Jupp, and others.

The remarkable closeness of the values of these two statistics is worth noting.

Acknowledgements

Part of the research of the first author was supported by ONR Grant No. N00014-93-1-0174. The first author is thankful to both University of California - Santa Barbara and Indian Statistical Institute - Kolkata for the Grants to support this research. Part of this paper constitutes a part of the Ph.D. Dissertation of the second author conducted at University of California - Santa Barbara and at Indian Statistical Institute - Kolkata under the guidance of the first author.

References

[1] Amari, S. (1985). *Differential geometric methods in statistics.* Springer-Verlag, New York.

[2] Arnold, B.C., Castillo, E. and Sarabia, J.M. (2001). A multivariate version of Stein's identity with applications to moment calculations and estimation of conditionally specified distributions. *Commun. Statist., Theory and Methods,* **30**, 2517-2542.

[3] Arnold, B.C. and SenGupta, A. (2003). Distributions with circular normal conditionals and tests for independence on the torus. Tech. Report, Department of Statistics, University of California - Riverside. Also submitted for publication.

[4] Ash, B.R. (1970). Real analysis and probaility. Academic Press, London

[5] Best, D.J. and Fisher, N.I. (1979). Efficient simulation of the von Mises distribution. **Appl. Statist., 24,** 152-157.

[6] Chang, H. (1991) *Some optimal tests in directional data.* Ph.D. thesis. University of California-Santa Barbara.

[7] Chernoff, H. (1951). A property of some Type A regions. *Ann. Math. Statist.,* **22,** 472-474.

[8] Durairajan, T.M. and Kale, B.K. (1979). Locally most powerful test for the mixing proprtion. *Sankhya, Ser. B*, **41**

[9] Efron, B. (1975). Defining the curvature of a statistical problem. *Ann. Statist.,* **3,** 1189-1242.

[10] Ferguson, T.S. (1967). *Mathematical statistics - a decision theoretic approach.* Academic Press, New York.

[11] Fisher, N.I. (1993). *Statistical analysis of circular data.* Cambridge University Press, Cambridge.

[12] Fisher, R.A. (1959) *Statistical methods and scientific inferences.* 2nd ed., Oliver and Boyd, Edinburgh.

[13] Gokhale, D.V. and SenGupta, A. (1986). Optimal tests for the correlation coefficient in a symmetric multivariate normal population. *J. Statist. Plann. Inf.,* **14,** 256-263.

[14] Kendall, M.G. and Stuart, A. (1967). *The advanced theory of statistics.* Vol. 2, 2nd ed., Griffin, London.

[15] Lehmann, E.L. (1986). *Testing statistical hypotheses.* 2nd ed., Wiley, New York.

[16] Mardia, K.V. (1972). *Statistics of Directional Data.* Academic Press, London.

[17] Mardia, K.V. and Jupp, P.E. (2000). *Directional statistics.* J. Wiley, New York.

[18] Mukerjee, R. and SenGupta, A. (1993). Comparison between the locally most mean power unbiased and Rao's test in the multiparameter case. *J. Mult. Anal.,* **45,** 9-24.

[19] Neyman, J. (1959). Optimal asymptotic tests of composite statistical hypotheses. In: Grenander U (ed.) *Probability and statistics.* J. Wiley, New York, 213-234.

[20] Neyman, J. and Pearson E.S. (1936). Contributions to the theory of testing statistical hypotheses. *Statist. Res. Mem.,* **1,** 1-37.

[21] Neyman, J. and Pearson E.S. (1938). Contributions to the theory of testing statistical hypotheses. *Statist. Res. Mem.,* **2,** 25-27.

[22] Rao, C.R. (1973). *Linear statistical inference and its applications.* 2nd ed., J. Wiley, New York.

[23] SenGupta, A. (1998a). Optimal parametric statistical inference. In: SenGupta A (ed.) *Analysis of directional data.* Indian Statistical Institute, Calcutta, pp. 23-52.

[24] SenGupta, A. (1998b). *DDSTAP - Statistical package for the analysis of directional data.* Applied Statistics Division, Indian Statistical Institute, Calcutta.

[25] SenGupta, A. (1991). A review of optimality of multivariate tests. In: Jammalamadaka, S.R. and SenGupta, A. (eds.) *Special Issue-Multivariate Optimality and Related Topics. Statist. Prob. Lett.,* **12,** 527-535.

[26] SenGupta, A. and Jammalamadaka, S.R. (1991). On locally optimal tests for the mean direction of the Langevin distribution. In: Jammalamadaka, S.R. and SenGupta, A. (eds.) *Special Issue-Multivariate Optimality and Related Topics. Statist. Prob. Lett.,* **12,** 537-544.

[27] SenGupta, A. and Jammalamadaka, S.R. (2001). Geometry and optimal unconditional tests for the mean direction of the von Mises distribution. *Stoch. Model. Applns.* **4,** 16-37.

[28] SenGupta, A. and Laha, A.K. (2001). The slippage problem for the circular normal distribution. *Austr. New Zlnd J. Statist.,* **43,** 461-471.

[29] SenGupta, A. and Laha, A.K. (2004). Change-point tests for the mean direction in the presence of general nuisance parameters. Submitted for publication.

[30] SenGupta, A. and Pal, C. (1993a). Optimal tests in some applied mixture models with non-regularity problems. In: Basu, S.K. and Sinha, B.K. (eds.) *Probability and Statistics,* Narosa Publ., Calcutta, 151-164.

[31] SenGupta, A. and Pal, C. (1993b). Optimal tests for no contamination in symmetric multivariate normal mixtures. *Ann. Inst. Statist. Math.,* **45,** 137-146.

[32] SenGupta, A. and Pal, C. (2000). Optimal tests for no contamination in reliability models. *Lifetime Data Anal.,* **6,** 281-290.

[33] SenGupta, A. and Pal, C. (2001). On optimal tests for isotropy against the symmetric wrapped stable-circular uniform mixture family. *J. Appl. Statist.,* **28,** 129-143.

[34] SenGupta, A. and Rattihalli, R.N. (2004). Optimal inference for an asymmetric circular normal distribution. Tech. Report. Applied Statistics Unit, Indian Statistical Institute.

[35] SenGupta, A. and Vermeire, L. (1986). On locally optimal tests for multiparameter hypotheses. *J. Amer. Statist. Assocn.,* **81,** 819-825.

[36] Schmidt-Koenig, K. (1963). On the role of the loft, the distance and site of release in pigeon homing (the "cross-loft experiment"). *Biol. Bull.* **125,** 154-164.

[37] von Mises, R. (1918). Über die "Ganzzahligkeit" der Atomgewichte und Verwandte Fragen. *Physikal. Z.,* **19,** 490-500.

Table 1. Cut-off points for LMPU Test for μ.

$\kappa = 0.1$				
m	α			
	0.01	0.025	0.05	0.10
5	.25588	.24837	.18745	.10488
6	.35424	.31454	.18830	.16045
7	.38346	.33863	.24501	.19843
8	.44309	.41304	.32930	.23780
9	.48662	.41365	.33164	.24985
10	.50016	.43030	.35260	.27876
20	.50708	.43755	.39570	.33135
30	.51225	.45019	.42963	.36538
50	.64130	.52308	.47680	.38592
100	.71001	.58147	.47750	.45746

$\kappa = 0.5$				
m	α			
	0.01	0.025	0.05	0.10
5	.84478	.67084	.57803	.24942
6	.91490	.70716	.58964	.39720
7	1.08133	.76623	.65221	.49258
8	1.16989	.76941	.71303	.43613
9	1.18185	.89727	.72266	.49217
10	1.19025	.93554	.73670	.50227
20	1.40034	1.10769	.73890	.50576
30	1.40411	1.13666	.74545	.50657
50	1.55547	1.16322	.86932	.54465
100	2.04561	1.28866	1.12061	.70739

$\kappa = 1.0$				
m	α			
	0.01	0.025	0.05	0.10
5	1.12936	1.05149	.75287	.59208
6	1.82596	1.13438	.88205	.59342
7	1.94141	1.59096	1.03125	.61784
8	2.20843	1.61217	1.13123	.71449
9	2.26283	1.64510	1.20029	.76255
10	2.38881	2.01740	1.36466	.85740
20	2.41958	2.20682	1.51555	.93066
30	75708	2.21226	1.56393	.98148
50	2.94436	2.31443	1.67136	1.13649
100	4.00131	3.34759	2.33409	1.99327

$\kappa = 1.5$				
m	α			
	0.01	0.025	0.05	0.10
5	2.02244	1.69334	1.04481	.64752
6	2.30280	1.74715	1.12702	.69839
7	2.48052	1.81820	1.21107	.76943
8	2.48615	1.90590	1.56377	.78720
9	2.50477	2.02335	1.58742	.91436
10	2.58416	2.03765	1.60232	.98784
20	3.10915	2.24143	1.66584	1.14907
30	3.15341	2.40212	1.75804	1.26063
50	3.39589	2.56502	1.77667	1.33914
100	4.47562	3.00727	2.56775	1.42439

$\kappa = 2.0$				
m	α			
	0.01	0.025	0.05	0.10
5	2.62098	1.42209	1.31822	.83484
6	2.90299	2.19884	1.47681	.91614
7	2.71539	2.31785	1.61512	.93788
8	3.09756	2.57843	1.68101	.94641
9	3.09985	2.59223	1.78824	1.04548
10	3.40628	2.74985	1.83594	1.18627
20	3.55619	2.79965	1.85397	1.22673
30	3.69003	2.92983	2.17252	1.58584
50	3.90731	3.63646	2.35456	1.61081
100	5.85477	4.72540	3.57381	2.75516

Table 2. Minimum sample size m corresponding to γ_κ^2.

κ	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
m	399	99	43	24	15	10	7	5	4	3	3
κ	2.4	2.6	2.8	3.0							
m	2	2	2	1							