JOURNAL OF MULTIVARIATE ANALYSIS 45, 9–24 (1993)

# Comparison between the Locally Most Mean Power Unbiased and Rao's Tests in the Multiparameter Case

RAHUL MUKERJEE

*Indian Institute of Management, Post Box No. 16757, Calcutta 700 027, India*

AND

ASHIS SEN GUPTA

*Indian Statistical Institute, 203, B.T. Road, Calcutta 700 035, India*

This paper makes an asymptotic comparison, up to the third order, between the locally most mean power unbiased (LMMPU) and Rao's tests in the multiparameter case. The two tests are seen to have identical power up to second order. It is also seen that Rao's test, which is much simpler than the LMMPU test, is almost as good as the latter, in terms of third-order local average power, for small but reasonable test size, provided the statistical curvature of the model is not too large. 

## 1. INTRODUCTION

The problem of higher order asymptotic comparison of tests has received considerable attention over the last two decades; see Chandra and Joshi [4], Amari [1], Mukerjee [10] for references and Ghosh [6] for an excellent review. Recently, Mukerjee [11, 12] established, in a multiparameter setup, the optimality of Rao's test, in terms of maximization of third-order average power under contiguous alternatives, within a very large class of tests that includes in particular the likelihood ratio and Wald's tests. This result was established by considering locally unbiased (up to $o(n^{-1})$) versions of the tests. It may, therefore, be of interest to

Received June 5, 1991; revised September 1992.

AMS 1980 subject classification: 62F05.

Key words and phrases: average power, contiguous alternatives, local unbiasedness, statistical curvature.



0047-259X/93 $5.00

study the performance of Rao's test vis-a-vis the locally most mean power unbiased (LMMPU) test proposed by Sengupta and Vermeire [15] as a natural generalization of the locally most powerful unbiased (LMPU) test to a multiparameter setting. This has been attempted in the present work.

Earlier, in the one-parameter case, Mukerjee and Chandra [13] compared the LMPU and Rao's tests considering "square-root" versions of both the tests. Although the LMMPU test arises as a natural generalization of the LMPU test, the former, unlike the latter, does not admit a "square root" version in the general multiparameter case and, as such, the techniques in Mukerjee and Chandra [13] are not applicable in the present context. Essentially for similar reasons, the LMMPU test does not belong to the class considered in Mukerjee [12]. Therefore, for the comparison attempted here, new techniques, combining in a sense those in Mukerjee [11, 12] and Peers [14] (see also Hayakawa [8]) have to be used. Although our techniques differ from those in Mukerjee and Chandra [13], our final results, indeed, extend theirs to a multiparameter setup. It may be remarked that our approach also differs from the differential geometric one due to Amari [1] and Kumon and Amari [9] since we assume neither curved exponentiality of the model nor sphericity of the power function.

## 2. Notation and Preliminaries

With reference to a sequence $\{X_j\}$, $j \geqslant 1$, of i.i.d. possibly vector-valued random variables with a common density $f(x, \theta)$, where $\theta = (\theta_1, ..., \theta_p)' \in \mathscr{R}^p$, or an open subset thereof, we consider the problem of testing $H_0: \theta = \theta_0$ against $\theta \neq \theta_0$. We shall be considering contiguous alternatives of the form $\theta(n) = \theta_0 + n^{-1/2}\delta$, where $n$ is the sample size and $\delta = (\delta_1, ..., \delta_p)'$ is free from $n$. All formal expansions will be over a set $\mathscr{A}_n$ (see Chandra and Ghosh [3]) with $P_{\theta(n)}(\mathscr{A}_n) = 1 + o(n^{-1})$, uniformly over compact subsets of $\delta$. The per-observation information matrix at $\theta_0$, say $\mathscr{I}$, will be supposed to be positive definite. Then without loss of generality (if necessary, by a reparametrization—see Mukerjee [11]) it may be assumed that $\mathscr{I} = I$, the $p \times p$ identity matrix. The notational system of this paper, similar to that in Mukerjee [11, 12], is presented below for ease in reference.

For $1 \leqslant i$, $u \leqslant p$, let

$$H_{1i} = n^{-1/2} \sum_{j=1}^{n} \partial \log f(X_j, \theta_0)/\partial\theta_i,$$

$$V_{1i} = n^{-1/2} \sum_{j=1}^{n} \partial \log f(X_j, \theta(n))/\partial\theta_i,$$

$$H_{2iu}=n^{-1/2}\sum_{j=1}^{n}\{\partial^2\log f(X_j,\theta_0)/\partial\theta_i\,\partial\theta_u-l_{iu}^{(0)}\},$$

$$V_{2iu}=n^{-1/2}\sum_{j=1}^{n}\{\partial^2\log f(X_j,\theta(n))/\partial\theta_i\,\partial\theta_u-l_{iu}\},$$

where $l_{iu}^{(0)}=E_{\theta_0}\{\partial^2\log f(X_j,\theta_0)/\partial\theta_i\,\partial\theta_u\}$, $l_{iu}=E_{\theta(n)}\{\partial^2\log f(X_j,\theta(n))/\partial\theta_i\,\partial\theta_u\}$. As the per-observation information matrix at $\theta_0$ is assumed to equal $I$, by standard regularity conditions $l_{iu}^{(0)}=-1$ if $i=u$ and $=0$ if $i\neq u$. Let $H_1$ be a $p\times 1$ vector with $i$th element $H_{1i}$ $(1\leqslant i\leqslant p)$.

For $1\leqslant i, u, r, s, v\leqslant p$, let

$$\gamma_{i.ur}^{(1)}=E_{\theta_0}[\{\partial\log f(X_j,\theta_0)/\partial\theta_i\}\{\partial^2\log f(X_j,\theta_0)/\partial\theta_u\,\partial\theta_r\}],$$

$$\gamma_{iur}^{(2)}=E_{\theta_0}\{\partial^3\log f(X_j,\theta_0)/\partial\theta_i\,\partial\theta_u\,\partial\theta_r\},$$

$$\gamma_{iur}^{(3)}=E_{\theta_0}[\{\partial\log f(X_j,\theta_0)/\partial\theta_i\}$$
$$\times\{\partial\log f(X_j,\theta_0)/\partial\theta_u\}\{\partial\log f(X_j,\theta_0)/\partial\theta_r\}],$$

$$\gamma_{iu.rs}^{(4)}=E_{\theta_0}[\{\partial^2\log f(X_j,\theta_0)/\partial\theta_i\,\partial\theta_u\}\{\partial^2\log f(X_j,\theta_0)/\partial\theta_r\,\partial\theta_s\}],$$

$$\gamma_{s.iu}^{(5)}=E_{\theta_0}[\{\partial^2\log f(X_j,\theta_0)/\partial\theta_s^2+1\}$$
$$\times\{\partial\log f(X_j,\theta_0)/\partial\theta_i\}\{\partial\log f(X_j,\theta_0)/\partial\theta_u\}],$$

$$g_{iurv}^{(s)}=\gamma_{iur}^{(3)}\gamma_{v.ss}^{(1)},\qquad w_{iu}=(\gamma_{1.iu}^{(1)},...,\gamma_{p.iu}^{(1)})',$$

$$R=(w_{11},...,w_{pp}),\qquad \sigma_{iu}=\gamma_{ii.uu}^{(4)}-1.$$

It is assumed that all the expectations defined above exist. The well-known regularity condition $\gamma_{i.ur}^{(1)}+\gamma_{u.ir}^{(1)}+\gamma_{r.iu}^{(1)}+\gamma_{iur}^{(2)}+\gamma_{iur}^{(3)}=0$ $(1\leqslant i, u, r\leqslant p)$ will also be assumed to hold.

Observe that the dispersion matrix of $\partial\log f(X_j,\theta_0)/\partial\theta_i$, $\partial^2\log f(X_j,\theta_0)/\partial\theta_i^2$, $1\leqslant i\leqslant p$, at $\theta_0$, is

$$\Sigma_0=\begin{pmatrix} I & R\\ R' & \Sigma\end{pmatrix},$$

where the $p\times p$ matrix $\Sigma$ is given by $\Sigma=((\sigma_{iu}))$. Hence the matrix $\Sigma^*=\Sigma-R'R$ is non-negative definite and $1'\Sigma^*1\geqslant 0$, where 1 is a $p\times 1$ vector with all elements unity. For $p=1$, $1'\Sigma^*1$ reduces to the square of Efron's curvature at $\theta_0$ [5] and in this sense $1'\Sigma^*1$ provides a generalization of Efron's curvature to a multiparameter setup. This generalization is, however, different from the one considered in Mukerjee [12].

For $\lambda\geqslant 0$ and positive integral $v$, let $k_{v,\lambda}(\cdot)$, $K_{v,\lambda}(\cdot)$ and $\eta(v,\lambda,\cdot)$ represent respectively the pdf, the cdf, and the characteristic function of a possibly non-central chi-square variate with $v$ degrees of freedom (d.f.) and non-centrality parameter $\lambda$; $\bar{K}_{\nu,\lambda}(\cdot)=1-K_{\nu,\lambda}(\cdot)$, $\Delta k_{\nu,\lambda}(\cdot)=k_{\nu+2,\lambda}(\cdot)-k_{\nu,\lambda}(\cdot)$, $\Delta K_{\nu,\lambda}(\cdot)=K_{\nu+2,\lambda}(\cdot)-K_{\nu,\lambda}(\cdot)$, $\Delta\bar{K}_{\nu,\lambda}(\cdot)=\bar{K}_{\nu+2,\lambda}(\cdot)-\bar{K}_{\nu,\lambda}(\cdot)$, $\Delta\eta(\nu,\lambda,\cdot)=\eta(\nu+2,\lambda,\cdot)-\eta(\nu,\lambda,\cdot)$. Let $z^2$ denote the upper $\alpha$-point of a central chi-square variate with pdf.

The average of any function $\chi(\delta)$ of $\delta=(\delta_1,...,\delta_p)'$ along the sphere $\delta'\delta=\lambda$ ($\geqslant 0$) will be defined as

$$\bar{\chi}(\lambda)=\chi(0) \qquad \text{if} \quad \lambda=0, \tag{2.1a}$$

$$=\left(\int\cdots\int_{\delta'\delta=\lambda}\chi(\delta)\,d\delta\right)\Big/\left(\int\cdots\int_{\delta'\delta=\lambda}d\delta\right) \qquad \text{if} \quad \lambda>0, \tag{2.1b}$$

provided the integral in the numerator of (2.1b) exists (cf. [11]). It may be clarified that while Mukerjee [11, 12] defines average power in the sense of (2.1) the definition in Sengupta and Vermeire [15] who average over regions of the form $\delta'\delta\leqslant\lambda$, is slightly different. However, one can easily prove a suitably modified version of Lemma 1 in Sengupta and Vermeire [15] to show that the exact optimality of their LMMPU test remains valid even when average power is defined in the sense of (2.1) above. Because of this reason and also because this work is primarily in continuation of Mukerjee [11, 12], we continue to define average power along the line of (2.1) in deriving our results. However, even if average power is defined as in Sengupta and Vermeire [15], then essentially repeating the present derivation it should not be hard to obtain similar results—such details are omitted here to save space.

## 3. Results

As in Mukerjee [11, 12], we consider a locally unbiased version of Rao's test given by the critical region,

$$Z_n^{(1)}=(H_1+n^{-1/2}b_1+n^{-1}b_2)'(H_1+n^{-1/2}b_1+n^{-1}b_2)$$
$$>z^2+n^{-1/2}b_{10}+n^{-1}b_{20}, \tag{3.1}$$

where the scalars $b_{10}$, $b_{20}$ and the elements of the $p\times 1$ vectors $b_1$, $b_2$ are constants, free from $n$, to be so determined that the test has size $\alpha+o(n^{-1})$ and is locally unbiased up to $o(n^{-1})$. Next observe that the LMMPU test [15] is given by a critical region of the form

$$\sum_{i=1}^{p}(H_{1i}+\beta_{in})^2+n^{-1/2}\sum_{i=1}^{p}H_{2ii}>q_n, \tag{3.2}$$

where $q_n$, $\beta_{1n}$, ..., $\beta_{pn}$ are chosen subject to the conditions of size and local unbiasedness. Following Mukerjee and Chandra [13] (see also Ghosh, Sinha, and Joshi [7]), we take

$$q_n = z^2 + n^{-1/2}a_{10} + n^{-1}a_{20} + o(n^{-1}),$$
$$\beta_{in} = n^{-1/2}a_{1i} + n^{-1}a_{2i} + o(n^{-1}), \qquad 1 \leqslant i \leqslant p, \tag{3.3}$$

where $a_{1i}$, $a_{2i}$ $(0 \leqslant i \leqslant p)$ are constants, free from $n$, to be so chosen that the test has size $\alpha + o(n^{-1})$ and is locally unbiased up to $o(n^{-1})$; as seen later, these constants exist uniquely from these specifications. By (3.3), the critical region (3.2) can be expressed as

$$Z_n^{(2)} = \sum_{i=1}^{p} (H_{1i} + n^{-1/2}a_{1i} + n^{-1}a_{2i})^2 + n^{-1/2} \sum_{i=1}^{p} H_{2ii}$$
$$> z^2 + n^{-1/2}a_{10} + n^{-1}a_{20} + o(n^{-1}). \tag{3.4}$$

We assume that the joint distribution of $(H_{11}, ..., H_{1p}, H_{211}, ..., H_{2pp})'$, under $\theta(n)$, admits a valid multivariate Edgeworth expansion up to $o(n^{-1})$ in the $L_1$-sense [2]. Then, following the last part of Section 4, one may justify the calculations for the proof of our main result:

THEOREM 1. *Let $b_{10}, b_{20}, b_1, b_2$ in (3.1) and $a_{1i}, a_{2i}$ $(0 \leqslant i \leqslant p)$ in (3.4) be chosen subject to the conditions of size and local unbiasedness up to $o(n^{-1})$. Then the (local) power functions of Rao's and LMMPU tests, under contiguous alternatives $\theta(n) = \theta_0 + n^{-1/2}\delta$, are respectively given by*

$$P^{(1)}(\delta) = P_0(\delta) + n^{-1/2}P_1(\delta) + n^{-1}P_2^{(1)}(\delta) + o(n^{-1}),$$
$$P^{(2)}(\delta) = P_0(\delta) + n^{-1/2}P_1(\delta) + n^{-1}P_2^{(2)}(\delta) + o(n^{-1}),$$

*where $P_0(\delta)$, $P_1(\delta)$, $P_2^{(1)}(\delta)$, $P_2^{(2)}(\delta)$ are free from $n$,*

$$P_0(\delta) = \bar{K}_{p,\lambda}(z^2),$$

$$P_1(\delta) = \sum\sum_{i,u,s=1}^{p}\sum \gamma_{ius}^{(3)} \delta_i\delta_u\delta_s (\tfrac{1}{6} \Delta^3\bar{K}_{p,\lambda}(z^2) + \tfrac{1}{2} \Delta^2\bar{K}_{p,\lambda}(z^2))$$
$$+ \tfrac{1}{2} \sum_{i,u=1}^{p}\sum \gamma_{iiu}^{(3)}\delta_u [\{1 - (p+2)^{-1} z^2\}$$
$$\times \Delta\bar{K}_{p,\lambda}(z^2) + \Delta^2\bar{K}_{p,\lambda}(z^2)]$$
$$+ \tfrac{1}{2} \sum\sum_{i,u,s=1}^{p}\sum (\gamma_{ius}^{(3)} + \gamma_{i,us}^{(1)})\, \delta_i\delta_u\delta_s\, \Delta\bar{K}_{p,\lambda}(z^2),$$

*and* $P_2^{(1)}(\delta)$, $P_2^{(2)}(\delta)$ *are such that for* $\lambda \geqslant 0$,

$$\bar{P}_2^{(2)}(\lambda) - \bar{P}_2^{(1)}(\lambda) = \lambda \psi_p(z^2) + O(\lambda^2),$$

*with* $\lambda = \delta'\delta$, *and* $\psi_p(z^2) = \frac{1}{4} p^{-1} k_{p,0}(z^2)(1'\Sigma^* 1)$.

It may be observed from Theorem 1 that up to second order Rao's and LMMPU tests have (point-by-point) identical powers. This seems to be non-trivial in a multiparameter setting and may be contrasted with the findings in Mukerjee [11]. Also, $\psi_p(z^2)$ as in Theorem 1, although non-negative, is always small, provided $z^2$ is large; that is, the test size is small, especially when the "generalized statistical curvature" $1'\Sigma^*1$ is not too large. Thus Rao's test, which is much simpler than the LMMPU test, is beaten by the latter only in terms of third-order (local) average power and that, too, usually by a narrow margin for small but reasonable test size (see the example below). For $p = 1$, it can be seen that Theorem 1 is in agreement with the results in Mukerjee and Chandra [13].

EXAMPLE. As in Mukerjee [11], consider the sequence $X_j = (X_{j1}, ..., X_{jp})'$, $j \geqslant 1$, of i.i.d. random variables with a common $p$-variate pdf over $\mathscr{R}^p$ given by

$$f(x, \theta) = \prod_{i=1}^{p} [\theta_i^{-1} (2\pi)^{-1/2} \exp\{-\tfrac{1}{2}\theta_i^{-2}(x_i - \theta_i)^2\}],$$

where $\theta = (\theta_1, ..., \theta_p)' > 0$. Suppose the interest lies in testing $H_0: \theta = \theta_0$ against $\theta \neq \theta_0$, where $\theta_0$ is a $p \times 1$ vector with each element equal to $\sqrt{3}$. Then the per observation information matrix at $\theta_0$ equals $I$ and it can be seen that $\Sigma^* = \frac{2}{27} I$, so that $1'\Sigma^*1 = 2p/27$ and $\psi_p(z^2) = \frac{1}{54} k_{p,0}(z^2)$. The table below shows that $\psi_p(z^2)$ is, indeed, small for $\alpha = 0.05$ and $\alpha = 0.01$, so that Rao's test is almost as good as the LMMPU test in terms of third-order average local power.

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\psi_p(z^2)$ ($\alpha = 0.05$) | 0.00055 | 0.00046 | 0.00041 | 0.00038 | 0.00036 | 0.00034 |
| $\psi_p(z^2)$ ($\alpha = 0.01$) | 0.00010 | 0.00009 | 0.00009 | 0.00008 | 0.00008 | 0.00008 |

## 4. Proof of Theorem 1

Let $T_n$ be a $p \times 1$ vector with $i$th element $H_{1i} + n^{-1/2} a_{1i} + n^{-1} a_{2i}$ $(1 \leqslant i \leqslant p)$. Then by (3.4), $Z_n^{(2)} = T_n' T_n + n^{-1/2} \sum_{i=1}^{p} H_{2ii}$ and under contiguous alternatives $\theta(n)=\theta_0+n^{-1/2}\delta$, the power function of the LMMPU test is given by

$$P^{(2)}(\delta)=P_{\theta(n)}(Z_n^{(2)}>z^2+n^{-1/2}a_{10}+n^{-1}a_{20})+o(n^{-1}). \tag{4.1}$$

With $\xi=(-1)^{1/2}\,t$, the approximate characteristic function of $Z_n^{(2)}$, under $\theta(n)$, is given by

$$\begin{aligned} &E_{\theta(n)}\{\exp(\xi Z_n^{(2)})\}\\ &\quad=E_{\theta(n)}\{\exp(\xi T_n'T_n)\}+n^{-1/2}\xi\sum_{s=1}^{p}E_{\theta(n)}\{H_{2ss}\exp(\xi T_n'T_n)\}\\ &\quad\quad+\tfrac{1}{2}n^{-1}\xi^2\sum_{s,u=1}^{p}\sum E_{\theta(n)}\{H_{2ss}H_{2uu}\exp(\xi T_n'T_n)\}+o(n^{-1}). \end{aligned} \tag{4.2}$$

Now, $H_{1i}=\delta_i+V_{1i}+o(1)$, $H_{2ii}=\delta' w_{ii}+V_{2ii}+o(1)$ $(1\leqslant i\leqslant p)$, so that for $s\neq u$, under $\theta(n)$, up to the first order of approximation the distribution of $(H_1', H_{2ss}, H_{2uu})'$ is $(p+2)$-variate normal with mean vector $\mu_{su}=(\delta', \delta' w_{ss}, \delta' w_{uu})'$ and dispersion matrix

$$\sum_{su}=\begin{pmatrix} I & w_{ss} & w_{uu}\\ w_{ss}' & \sigma_{ss} & \sigma_{su}\\ w_{uu}' & \sigma_{us} & \sigma_{uu}\end{pmatrix}.$$

Hence for $s\neq u$,

$$\begin{aligned} E_{\theta(n)}\{H_{2ss}H_{2uu}\exp(\xi T_n'T_n)\}&=E_{\theta(n)}\{H_{2ss}H_{2uu}\exp(\xi H_1'H_1)\}+o(1)\\ &=E\{Y_{ss}Y_{uu}\exp(\xi Y'Y)\}+o(1), \end{aligned} \tag{4.3}$$

where $Y=(Y_1, ..., Y_p)'$ and the distribution of $(Y', Y_{ss}, Y_{uu})'$ is $(p+2)$-variate normal with mean vector $\mu_{su}$ and dispersion matrix $\sum_{su}$ as stated above. Evaluating the expectation in the right-hand side of (4.3) by conditioning on $Y$, it can be seen after some simplification that for $s\neq u$,

$$\begin{aligned} E_{\theta(n)}\{H_{2ss}H_{2uu}\exp(\xi T_n'T_n)\}&=\eta(p,\lambda,\xi)\,\sigma_{su}+\eta(p+4,\lambda,\xi)(w_{ss}'\delta)(w_{uu}'\delta)\\ &\quad+\Delta\eta(p,\lambda,\xi)(w_{ss}'w_{uu})+o(1), \end{aligned} \tag{4.4}$$

where $\lambda=\delta'\delta$. Similarly, an expression, up to $o(1)$, for the left-hand side of (4.4) can be obtained for $s=u$. Hence

$$\sum_{s,u=1}^{p}\sum E_{\theta(n)}\{H_{2ss}H_{2uu}\exp(\xi T_n'T_n)\}=B(p,\lambda,\xi)+o(1), \tag{4.5}$$

where

$$B(p, \lambda, \xi) = \eta(p, \lambda, \xi)\left(\sum\sum_{s,u=1}^{p} \sigma_{su}\right) + \eta(p+4, \lambda, \xi)\left(\sum_{s=1}^{p} w'_{ss}\delta\right)^2$$
$$+ \Delta\eta(p, \lambda, \xi)\left(\sum_{s=1}^{p} w_{ss}\right)'\left(\sum_{s=1}^{p} w_{ss}\right). \tag{4.6}$$

Considering a multivariate Edgeworth expansion, up to $o(n^{-1/2})$, for the distribution of $(H_{11}, ..., H_{1p}, H_{2ss})'$, under $\theta(n)$ (cf. Peers [14]), one obtains in a similar fashion, but with much more algebra,

$$E_{\theta(n)}\{H_{2ss}\exp(\xi T_n' T_n)\} = \eta(p+2, \lambda, \xi)(w'_{ss}\delta) + n^{-1/2} C_s(p, \lambda, \xi) + o(n^{-1/2}), \tag{4.7}$$

where

$$C_s(p, \lambda, \xi) = \tfrac{1}{2}\eta(p+4, \lambda, \xi)\sum\sum_{u,r=1}^{p} \gamma^{(5)}_{s.ur}\delta_u\delta_r + \tfrac{1}{2}\Delta\eta(p, \lambda, \xi)\sum_{u=1}^{p} \gamma^{(5)}_{s.uu}$$
$$+ \tfrac{1}{6}\sum_{i=1}^{p} g^{(s)}_{iiii}\{3\Delta^2\eta(p, \lambda, \xi)$$
$$+ 6\delta_i^2\Delta\eta(p+4, \lambda, \xi) + \delta_i^4\eta(p+8, \lambda, \xi)\}$$
$$+ \tfrac{1}{6}\sum\sum_{i\neq u=1}^{p} (g^{(s)}_{iiiu} + 3g^{(s)}_{iiui})\{\delta_i^3\delta_u\eta(p+8, \lambda, \xi)$$
$$+ 3\delta_i\delta_u\Delta\eta(p+4, \lambda, \xi)\}$$
$$+ \tfrac{1}{2}\sum\sum_{i\neq u=1}^{p} g^{(s)}_{iiuu}\{\delta_i^2\delta_u^2\eta(p+8, \lambda, \xi)$$
$$+ (\delta_i^2 + \delta_u^2)\Delta\eta(p+4, \lambda, \xi) + \Delta^2\eta(p, \lambda, \xi)\}$$
$$+ \tfrac{1}{2}\sum\sum\sum_{i\neq u\neq r=1}^{p} (g^{(s)}_{iiur} + g^{(s)}_{iuri})\{\delta_i^2\delta_u\delta_r\eta(p+8, \lambda, \xi)$$
$$+ \delta_u\delta_r\Delta\eta(p+4, \lambda, \xi)\}$$
$$+ \tfrac{1}{6}\sum\sum\sum\sum_{i\neq u\neq r\neq v=1}^{p} g^{(s)}_{iurv}\delta_i\delta_u\delta_r\delta_v\eta(p+8, \lambda, \xi)$$
$$+ (w'_{ss}a_1)\Delta\eta(p, \lambda, \xi) + (w'_{ss}\delta)(a_1'\delta)\Delta\eta(p+2, \lambda, \xi)$$
$$+ \tfrac{1}{2}\sum\sum_{u,r=1}^{p} \delta_u\delta_r\{(w'_{ss}w_{ur})\Delta\eta(p, \lambda, \xi)$$
$$+ (w'_{ss}\delta)(w'_{ur}\delta)\eta(p+4, \lambda, \xi) + (\gamma^{(4)}_{ss.ur} - \rho_{ur})\eta(p, \lambda, \xi)\}$$
$$+ \tfrac{1}{6}\sum\sum\sum_{u,r,v=1}^{p} \delta_u\delta_r\delta_v\gamma^{(2)}_{urv}(w'_{ss}\delta)\eta(p+2, \lambda, \xi), \quad 1 \leqslant s \leqslant p, \tag{4.8}$$

with $a_i = (a_{i1}, ..., a_{ip})'$ $(i = 1, 2)$ and $\rho_{ur} = 1$ if $u = r$ and $= 0$ if $u \neq r$, using in particular the fact that the $g^{(s)}_{iurr}$ are invariant under permutation of the subscripts $i, u, r$.

By (4.2), (4.5), and (4.7),

$$\begin{aligned} E_{\theta(n)}\{\exp(\xi Z_n^{(2)})\} &= E_{\theta(n)}\{\exp(\xi T_n' T_n)\} + \tfrac{1}{2} n^{-1/2} \Delta\eta(p, \lambda, \xi) \sum_{s=1}^{p} (w_{ss}'\delta) \\ &\quad + n^{-1}\left\{\xi \sum_{s=1}^{p} C_s(p, \lambda, \xi) + \tfrac{1}{2}\xi^2 B(p, \lambda, \xi)\right\} + o(n^{-1}). \end{aligned} \tag{4.9}$$

For $\lambda \geqslant 0$ and positive integral $\nu$, $n^{-1}\xi\eta(\nu, \lambda, \xi) = \{\exp(n^{-1}\xi) - 1\}\, \eta(\nu, \lambda, \xi) + o(n^{-1})$, $\frac{1}{2} n^{-1}\xi^2\eta(\nu, \lambda, \xi) = \{\exp(\frac{1}{2} n^{-1}\xi^2) - 1\}\, \eta(\nu, \lambda, \xi) + o(n^{-1})$. Note that $\exp(n^{-1}\xi)\, \eta(\nu, \lambda, \xi)$ and $\exp(\frac{1}{2} n^{-1}\xi^2)\, \eta(\nu, \lambda, \xi)$ are characteristic functions respectively of $U_{\nu,\lambda} + n^{-1}$ and $U_{\nu,\lambda} + n^{-1/2}\tau$, where $U_{\nu,\lambda}$ follows the (possibly non-central) chi-square distribution with $\nu$ d.f. and non-centrality parameter $\lambda$, $\tau$ follows the standard univariate normal distribution, and $U_{\nu,\lambda}$ and $\tau$ are independent. Since, with $U_{\nu,\lambda}$ and $\tau$ so defined,

$$\begin{aligned} &P(U_{\nu,\lambda} + n^{-1} > z^2 + n^{-1/2}a_{10} + n^{-1}a_{20}) - P(U_{\nu,\lambda} > z^2 + n^{-1/2}a_{10} + n^{-1}a_{20}) \\ &\quad = n^{-1}k_{\nu,\lambda}(z^2) + o(n^{-1}), \end{aligned} \tag{4.10a}$$

$$\begin{aligned} &P(U_{\nu,\lambda} + n^{-1/2}\tau > z^2 + n^{-1/2}a_{10} + n^{-1}a_{20}) - P(U_{\nu,\lambda} > z^2 + n^{-1/2}a_{10} + n^{-1}a_{20}) \\ &\quad = -\tfrac{1}{2} n^{-1}k'_{\nu,\lambda}(z^2) + o(n^{-1}), \end{aligned} \tag{4.10b}$$

where $k'_{\nu,\lambda}(z^2) = \{dk_{\nu,\lambda}(\omega)/d\omega\}_{\omega = z^2}$, it follows from (4.1), (4.6), (4.8), and (4.9) that

$$\begin{aligned} P^{(2)}(\delta) &= P_{\theta(n)}(T_n' T_n > z^2 + n^{-1/2}\, a_{10} + n^{-1}a_{20}) + \tfrac{1}{2} n^{-1/2} \sum_{s=1}^{p} (w_{ss}'\delta)\, \Delta\bar{K}_{p,\lambda}(z^2) \\ &\quad + n^{-1} \sum_{s=1}^{p} C_s^*(\delta) - \tfrac{1}{2} n^{-1} B^*(\delta) + o(n^{-1}), \end{aligned} \tag{4.11}$$

where $\lambda = \delta'\delta$,

$$\begin{aligned} C_s^*(\delta) &= \tfrac{1}{2} k_{p+4,\lambda}(z^2) \sum_{u,r=1}^{p}\sum \gamma_{s.ur}^{(5)}\, \delta_u \delta_r + \tfrac{1}{2}\Delta k_{p,\lambda}(z^2) \sum_{u=1}^{p} \gamma_{s.uu}^{(5)} \\ &\quad + \tfrac{1}{6} \sum_{i=1}^{p} g_{iiii}^{(s)} \{3\Delta^2 k_{p,\lambda}(z^2) + 6\delta_i^2 \Delta k_{p+4,\lambda}(z^2) + \delta_i^4 k_{p+8,\lambda}(z^2)\} \end{aligned}$$

$$+\tfrac{1}{6}\sum_{i\neq u=1}^{p}\sum (g_{iiiu}^{(s)}+3g_{iiui}^{(s)})\{\delta_i^3\delta_u k_{p+8,\lambda}(z^2)+3\delta_i\delta_u\,\Delta k_{p+4,\lambda}(z^2)\}$$

$$+\tfrac{1}{2}\sum_{i\neq u=1}^{p}\sum g_{iiuu}^{(s)}\{\delta_i^2\delta_u^2 k_{p+8,\lambda}(z^2)$$

$$+(\delta_i^2+\delta_u^2)\,\Delta k_{p+4,\lambda}(z^2)+\Delta^2 k_{p,\lambda}(z^2)\}$$

$$+\tfrac{1}{2}\sum_{i\neq u\neq r=1}^{p}\sum\sum (g_{iiur}^{(s)}+g_{iuri}^{(s)})$$

$$\times\{\delta_i^2\delta_u\delta_r k_{p+8,\lambda}(z^2)+\delta_u\delta_r\,\Delta k_{p+4,\lambda}(z^2)\}$$

$$+\tfrac{1}{6}\sum_{i\neq u\neq r\neq v=1}^{p}\sum\sum\sum g_{iurv}^{(s)}\delta_i\delta_u\delta_r\delta_v k_{p+8,\lambda}(z^2)$$

$$+(w_{ss}'a_1)\,\Delta k_{p,\lambda}(z^2)+(w_{ss}'\delta)(a_1'\delta)\,\Delta k_{p+2,\lambda}(z^2)$$

$$-\tfrac{1}{2}a_{10}(w_{ss}'\delta)\,\Delta k_{p,\lambda}(z^2)$$

$$+\tfrac{1}{2}\sum_{u,r=1}^{p}\sum \delta_u\delta_r\{(w_{ss}'w_{ur})\,\Delta k_{p,\lambda}(z^2)$$

$$+(w_{ss}'\delta)(w_{ur}'\delta)\,k_{p+4,\lambda}(z^2)+(\gamma_{ss,ur}^{(4)}-\rho_{ur})\,k_{p,\lambda}(z^2)\}$$

$$+\tfrac{1}{6}\sum_{u,r,v=1}^{p}\sum\sum \delta_u\delta_r\delta_v\gamma_{urv}^{(2)}(w_{ss}'\delta)\,k_{p+2,\lambda}(z^2),\qquad 1\leqslant s\leqslant p, \tag{4.12}$$

$$B^*(\delta)=k_{p,\lambda}'(z^2)\left(\sum_{s,u=1}^{p}\sum\sigma_{su}\right)+k_{p+4,\lambda}'(z^2)\left(\sum_{s=1}^{p}w_{ss}'\delta\right)^2$$

$$+\{k_{p+2,\lambda}'(z^2)-k_{p,\lambda}'(z^2)\}\left(\sum_{s=1}^{p}w_{ss}\right)'\left(\sum_{s=1}^{p}w_{ss}\right). \tag{4.13}$$

Now observe that a critical region of the form $T_n'T_n>z^2+n^{-1/2}a_{10}+n^{-1}a_{20}$, where $T_n=H_1+n^{-1/2}a_1+n^{-1}a_2$, gives a test belonging to the class considered in Mukerjee [12] and, therefore, as in Section 2 of Mukerjee [12] (rectifying a minor error in Equation (2.11f) of that paper),

$$P_{\theta(n)}(T_n'T_n>z^2+n^{-1/2}a_{10}+n^{-1}a_{20})$$

$$=\bar{K}_{p,\lambda}(z^2)+n^{-1/2}[\Omega(\delta,z^2)-a_{10}k_{p,\lambda}(z^2)$$

$$-(a_1'\delta)\,\Delta K_{p,\lambda}(z^2)]$$

$$+n^{-1}\left[\sum_{j=0}^{3}h_j(\delta)+\sum_{j=1}^{4}W_j^*(\delta)\right]+o(n^{-1}), \tag{4.14}$$

where

$$\Omega(\delta, z^2) = \sum\sum_{i,u,s=1}^{p}\sum \gamma_{ius}^{(3)} \delta_i \delta_u \delta_s (\tfrac{1}{6} \Delta^3 \bar{K}_{p,\lambda}(z^2) + \tfrac{1}{2} \Delta^2 \bar{K}_{p,\lambda}(z^2))$$

$$+ \tfrac{1}{2} \sum_{i,u=1}^{p}\sum \gamma_{iiu}^{(3)} \delta_u (\Delta^2 \bar{K}_{p,\lambda}(z^2) + \Delta \bar{K}_{p,\lambda}(z^2))$$

$$+ \tfrac{1}{2} \sum\sum_{i,u,s=1}^{p}\sum (\gamma_{ius}^{(3)} + \gamma_{i.us}^{(1)}) \delta_i \delta_u \delta_s \Delta \bar{K}_{p,\lambda}(z^2), \tag{4.15}$$

$$\begin{aligned} h_1(\delta) &= -\{a_{20} k_{p,\lambda}(z^2) + \tfrac{1}{2} a_{10}^2 k'_{p,\lambda}(z^2)\}, \\ h_2(\delta) &= a_{10}\{\partial\Omega(\delta, z^2)/\partial z^2\}, \\ h_3(\delta) &= -a_{10}(a_1'\delta)\, \Delta k_{p,\lambda}(z^2), \end{aligned} \tag{4.16}$$

$$W_j^*(\delta) = \int_S \cdots \int W_j(-D, \delta) \prod_{i=1}^{p} \phi(y_i - \delta_i)\, dy \qquad (1 \leqslant j \leqslant 4) \tag{4.17}$$

$$S = \{ y = (y_1, ..., y_p)' : y'y > z^2 \},$$

$$\begin{aligned} W_1(c, \delta) &= (c'a_1) \sum\sum_{i,u,s=1}^{p}\sum \gamma_{ius}^{(3)} c_i c_u (\tfrac{1}{6} c_s + \tfrac{1}{2} \delta_s), \\ W_2(c, \delta) &= \tfrac{1}{2}(c'a_1)^2, \\ W_3(c, \delta) &= c'a_2, \\ W_4(c, \delta) &= \tfrac{1}{2}(c'a_1) \sum\sum_{i,u,s=1}^{p}\sum c_i \delta_u \delta_s (\gamma_{ius}^{(3)} + \gamma_{i.us}^{(1)}), \end{aligned} \tag{4.18}$$

$c = (c_1, ..., c_p)'$, $D = (\partial/\partial y_1, ..., \partial/\partial y_p)'$ is a vector of partial differentiation operators, $\phi(\cdot)$ is the standard univariate normal density, and $h_0(\delta)$ does not involve $a_{1i}$, $a_{2i}$ $(0 \leqslant i \leqslant p)$.

By (4.11), (4.14), the power function of the LMMPU test under contiguous alternatives $\theta(n) = \theta_0 + n^{-1/2}\delta$ is given by

$$P^{(2)}(\delta) = P_0(\delta) + n^{-1/2} P_1^{(2)}(\delta) + n^{-1} P_2^{(2)}(\delta) + o(n^{-1}), \tag{4.19}$$

where $P_0(\delta)$, $P_1^{(2)}(\delta)$, $P_2^{(2)}(\delta)$ are free from $n$,

$$P_0(\delta) = \bar{K}_{p,\lambda}(z^2), \tag{4.20a}$$

$$P_1^{(2)}(\delta) = \Omega(\delta, z^2) - a_{10} k_{p,\lambda}(z^2) - (a_1' \delta)\, \Delta K_{p,\lambda}(z^2)$$
$$+ \frac{1}{2}\left(\sum_{s=1}^{p} w_{ss}' \delta\right) \Delta \bar{K}_{p,\lambda}(z^2), \tag{4.20b}$$

$$P_2^{(2)}(\delta) = \sum_{j=0}^{3} h_j(\delta) + \sum_{j=1}^{4} W_j^*(\delta) + \sum_{s=1}^{p} C_s^*(\delta) - \tfrac{1}{2} B^*(\delta). \tag{4.20c}$$

In order to simplify (4.19) further, we proceed to determine $a_{10}, a_{20}, a_1, a_2$ from the conditions of size and local unbiasedness up to $o(n^{-1})$, namely,

$$P_1^{(2)}(0) = 0, \qquad \{\partial P_1^{(2)}(\delta)/\partial \delta_i\}_{\delta=0} = 0 \ (1 \leqslant i \leqslant p), \tag{4.21a}$$

$$P_2^{(2)}(0) = 0, \qquad \{\partial P_2^{(2)}(\delta)/\partial \delta_i\}_{\delta=0} = 0 \ (1 \leqslant i \leqslant p). \tag{4.21b}$$

Exactly as in Mukerjee [12], from (4.15), (4.20b), (4.21a), after some simplification, the unique solutions for $a_{10}$ and $a_1 = (a_{11}, ..., a_{1p})'$ are given by

$$a_{10} = 0, \qquad a_{1i} = -\tfrac{1}{2} \sum_{s=1}^{p} \{\gamma_{i.ss}^{(1)} + (p+2)^{-1} z^2 \gamma_{iss}^{(3)}\}. \tag{4.22}$$

Similarly, from (4.21b), $a_{20}$ and $a_2 = (a_{21}, ..., a_{2p})'$ can be determined uniquely. The detailed expression for $a_2$ will not, however, be required for the present purpose. From (4.15), (4.20b), (4.22), one can check that

$$P_1^{(2)}(\delta) = P_1(\delta), \tag{4.23}$$

where $P_1(\delta)$ is as in the statement of Theorem 1.

By (4.12), (4.13), (4.16), (4.20c), (4.22), for $\lambda \geqslant 0$,

$$\bar{P}_2^{(2)}(\lambda) = \bar{h}_0(\lambda) - a_{20} k_{p,\lambda}(z^2) + \sum_{j=1}^{4} \bar{W}_j^*(\lambda) + \sum_{s=1}^{p} \bar{C}_s^*(\lambda) - \tfrac{1}{2} \bar{B}^*(\lambda), \tag{4.24}$$

where the averages are as defined in (2.1), and

$$\bar{C}_s^*(\lambda) = \tfrac{1}{2}\{\lambda p^{-1} k_{p+4,\lambda}(z^2) + \Delta k_{p,\lambda}(z^2)\} \sum_{u=1}^{p} \gamma_{s.uu}^{(5)}$$
$$+ \tfrac{1}{2}\{2\lambda p^{-1} \Delta k_{p+4,\lambda}(z^2) + \Delta^2 k_{p,\lambda}(z^2)\} \sum\sum_{i,u=1}^{p} g_{iiuu}^{(s)}$$
$$+ (w_{ss}' a_1)\, \Delta k_{p,\lambda}(z^2) + \lambda p^{-1} (w_{ss}' a_1)\, \Delta k_{p+2,\lambda}(z^2)$$
$$+ \tfrac{1}{2} \lambda p^{-1} \sum_{u=1}^{p} \{(w_{ss}' w_{uu})\, \Delta k_{p,\lambda}(z^2) + \sigma_{su} k_{p,\lambda}(z^2)\}$$
$$+ O(\lambda^2), \qquad 1 \leqslant s \leqslant p, \tag{4.25}$$

$$\bar{B}^*(\lambda)=k'_{p,\lambda}(z^2)\left(\sum_{s,u=1}^{p}\sum \sigma_{su}\right)$$
$$+\{\lambda p^{-1}k'_{p+4,\lambda}(z^2)+k'_{p+2,\lambda}(z^2)-k'_{p,\lambda}(z^2)\}$$
$$\times\left(\sum_{s=1}^{p} w_{ss}\right)'\left(\sum_{s=1}^{p} w_{ss}\right). \tag{4.26}$$

By the first condition in (4.21b), $\bar{P}_2^{(2)}(0)=0$ so that by (4.24)–(4.26),

$$a_{20}=\left[\bar{h}_0(0)+\sum_{j=1}^{4}\bar{W}_j^*(0)+\tfrac{1}{2}\Delta k_{p,0}(z^2)\sum_{s,u=1}^{p}\sum \gamma_{s.uu}^{(5)}\right.$$
$$+\tfrac{1}{2}\Delta^2 k_{p,0}(z^2)\sum_{s,i,u=1}^{p}\sum\sum g_{iiuu}^{(s)}$$
$$+\Delta k_{p,0}(z^2)\sum_{s=1}^{p}(w'_{ss}a_1)-\tfrac{1}{2}k'_{p,0}(z^2)\left(\sum_{s,u=1}^{p}\sum \sigma_{su}\right)$$
$$\left.-\tfrac{1}{2}\{k'_{p+2,0}(z^2)-k'_{p,0}(z^2)\}\left(\sum_{s=1}^{p} w_{ss}\right)'\left(\sum_{s=1}^{p} w_{ss}\right)\right]\Big/k_{p,0}(z^2). \tag{4.27}$$

Also, as in Section 3 of Mukerjee [12], by (4.17), (4.18),

$$\sum_{j=1}^{4}[\bar{W}_j^*(\lambda)-\{k_{p,\lambda}(z^2)/k_{p,0}(z^2)\}\,\bar{W}_j^*(0)]$$
$$=\lambda p^{-1}k_{p+2,0}(z^2)\left\{\sum_{u=1}^{p}(w'_{uu}a_1)-a'_1a_1\right\}+O(\lambda^2). \tag{4.28}$$

From (4.22), (4.24)–(4.28), after some algebra, one obtains

$$\bar{P}_2^{(2)}(\lambda)=\bar{h}_0(\lambda)-\{k_{p,\lambda}(z^2)/k_{p,0}(z^2)\}\,\bar{h}_0(0)$$
$$+\lambda k_{p+2,0}(z^2)\left\{(4z^2)^{-1}(1'\Sigma^*1)\right.$$
$$\left.+(4p)^{-1}\sum_{s,u=1}^{p}\sum w'_{ss}w_{uu}-p^{-1}a'_1a_1\right\}+O(\lambda^2)$$
$$=\bar{h}_0(\lambda)-\{k_{p,\lambda}(z^2)/k_{p,0}(z^2)\}\,\bar{h}_0(0)$$
$$+\lambda k_{p+2,0}(z^2)\{(4z^2)^{-1}(1'\Sigma^*1)+p^{-1}\varepsilon'_1(\varepsilon_2-\varepsilon_1)\}+O(\lambda^2), \tag{4.29}$$

where $\varepsilon_1$ is a $p\times 1$ vector with $i$th element $-\tfrac{1}{2}z^2(p+2)^{-1}\sum_{s=1}^{p}\gamma_{iss}^{(3)}$ $(1\leqslant i\leqslant p)$ and $\varepsilon_2=\sum_{s=1}^{p}w_{ss}$.

We now consider Rao's test given by (3.1) which belongs to the class considered in Mukerjee [12]. Following [12], if $b_{10}, b_{20}, b_1, b_2$ in (3.1) are chosen subject to the conditions of size and local unbiasedness up to $o(n^{-1})$, then it can be seen that the power function of Rao's test, under contiguous alternatives $\theta(n) = \theta_0 + n^{-1/2}\delta$, is given by

$$P^{(1)}(\delta) = P_0(\delta) + n^{-1/2} P_1^{(1)}(\delta) + n^{-1} P_2^{(1)}(\delta) + o(n^{-1}), \tag{4.30a}$$

where $P_0(\delta)$, $P_1^{(1)}(\delta)$, $P_2^{(1)}(\delta)$ are free from $n$,

$$P_0(\delta) = \bar{K}_{p,\lambda}(z^2), \qquad P_1^{(1)}(\delta) = P_1(\delta), \tag{4.30b}$$

$P_1(\delta)$ is as in the statement of Theorem 1, and $P_2^{(1)}(\delta)$ is such that for $\lambda \geqslant 0$,

$$\begin{aligned} \bar{P}_2^{(1)}(\lambda) = \bar{h}_0(\lambda) - \{k_{p,\lambda}(z^2)/k_{p,0}(z^2)\}\, \bar{h}_0(0) \\ + \lambda k_{p+2,0}(z^2)\, p^{-1} \varepsilon_1'(\varepsilon_2 - \varepsilon_1) + O(\lambda^2). \end{aligned} \tag{4.30c}$$

Theorem 1 now follows from (4.19), (4.20a), (4.23), (4.29), (4.30a)–(4.30c).

The derivation of the local power function of the LMMPU test (see (4.19), (4.20a)–(4.20c)), with the help of some results from Mukerjee [12] was an important step in the above proof. Comparing with [12], it can be seen that we are essentially approximating $E_{\theta(n)}\{\exp(\xi Z_n^{(2)})\}$ by

$$\begin{aligned} E_{\theta(n)}[\{1 + n^{-1/2}\xi(2a_1'H_1 + 1'H_2) + n^{-1}(2\xi a_2' H_1 + 2\xi^2(a_1'H_1)^2 \\ + 2\xi^2(1'H_2)(a_1'H_1) + \xi a_1' a_1 + \tfrac{1}{2}\xi^2(1'H_2)^2\} \exp(\xi H_1' H_1)], \end{aligned}$$

as computed up to $o(n^{-1})$ from the assumed multivariate Edgeworth expansion of $(H_1', H_2')'$ under $\theta(n)$, where $H_2 = (H_{211}, ..., H_{2pp})'$, and then inverting this approximation using (4.10a), (4.10b) whenever necessary. Alternatively, one can consider the more laborious procedure of directly integrating the multivariate Edgeworth expansion, under $\theta(n)$, of $(H_1', H_2')'$ over the critical region represented by (3.4) by first making the transformation

$$\zeta_i = H_{1i}\ (1 \leqslant i \leqslant p), \qquad \zeta_{p+1} = \sum_{j=1}^{p} (H_{2jj} - w_{jj}' H_1),$$
$$\zeta_{p+i} = H_{2ii} - w_{ii}' H_1\ (2 \leqslant i \leqslant p),$$

then integrating $\zeta_{p+2}, ..., \zeta_{2p}$ out, and finally integrating with respect to $\zeta_1, ..., \zeta_p, \zeta_{p+1}$, using relations analogous to (4.10b). It can be shown that the contribution of each term in the multivariate Edgeworth expansion under the above procedure of direct integration agrees, up to $o(n^{-1})$, with that under the procedure employed above in proving Theorem 1. Further-more, up to $o(n^{-1})$, the contribution of each term in the multivariate Edgeworth expansion to $E_{\theta(n)}\{\exp(\xi Z_n^{(2)})\}$, is of the form

$$\sum_{j=1}^{d} \Gamma_{0j}\eta(j, \lambda, \xi) + n^{-1/2}\left\{\sum_{j=1}^{d} \Gamma_{1j}\eta(j, \lambda, \xi) + \xi \sum_{j=1}^{d} \Gamma_{2j}\eta(j, \lambda, \xi)\right\}$$
$$+ n^{-1}\left\{\sum_{j=1}^{d} \Gamma_{3j}\eta(j, \lambda, \xi) + \xi \sum_{j=1}^{d} \Gamma_{4j}\eta(j, \lambda, \xi) + \xi^2 \sum_{j=1}^{d} \Gamma_{5j}\eta(j, \lambda, \xi)\right\},$$

where $d$ is a positive integer, the $\Gamma_{ij}$'s are free from $n$ and $\xi$ (but may depend on $\delta$), and some of the $\Gamma_{ij}$'s are possibly zeros; one can check that if two expressions like the above are identical in $\xi$ (and $\delta$) then their inversions, as done here, will also be identical up to $o(n^{-1})$. These considerations, under the assumption of existence of a valid Edgeworth expansion under $\theta(n)$, up to $o(n^{-1})$, for $(H_1', H_2')'$ in the $L_1$-sense, provide a justification for the calculations in this section.

## Acknowledgments

The authors are grateful to Professor J. K. Ghosh, Indian Statistical Institute, for very constructive suggestions. The work of the first author was supported by a grant from the Centre for Management and Development Studies, Indian Institute of Management Calcutta.

## References

[1] Amari, S. (1985). *Differential Geometric Methods in Statistics.* Springer-Verlag, New York.

[2] Bhattacharya, R. N., and Ghosh, J. K. (1978). On the validity of formal Edgeworth expansions. *Ann. Statist.* **6** 434–451.

[3] Chandra, T. K., and Ghosh, J. K. (1979). Valid asymptotic expansions for the likelihood ratio statistic and other perturbed chi-square variables. *Sankhyā Ser. A* **41** 22–47.

[4] Chandra, T. K., and Joshi, S. N. (1983). Comparison of the likelihood ratio, Rao's and Wald's tests and a conjecture of C. R. Rao. *Sankhyā Ser. A* **45** 226–246.

[5] Efron, B. (1975). Defining the curvature of a statistical problem (with applications to second-order efficiency). *Ann. Statist.* **3** 1189–1242.

[6] Ghosh, J. K. (1991). Higher order asymptotics for the likelihood ratio, Rao's and Wald's tests. *Statist. Prob. Lett.*, to appear.

[7] Ghosh, J. K., Sinha, B. K., and Joshi, S. N. (1980). A property of maximum likelihood estimator. *Sankhyā Ser. B* **42** 143–152.

[8] Hayakawa, T. (1975). The likelihood ratio criterion for a composite hypothesis under a local alternative. *Biometrika* **62** 451–460.

[9] Kumon, M., and Amari, S. (1985). Differential geometry of testing hypothesis—A higher order asymptotic theory in multi-parameter curved exponential family, preprint, Tokyo University.

[10] MUKERJEE, R. (1989). Third-order comparison of unbiased tests: A simple formula for the power difference in the one-parameter case. *Sankhyā Ser. A* **51** 212–232.

[11] MUKERJEE, R. (1990a). Comparison of tests in the multiparameter case. I. Second-order power. *J. Multivariate Anal.* **33** 17–30.

[12] MUKERJEE, R. (1990b). Comparison of tests in the multiparameter case. II. A third-order optimality property of Rao's test. *J. Multivariate Anal.* **33** 31–48.

[13] MUKERJEE, R., AND CHANDRA, T. K. (1987). Comparison between the locally most powerful unbiased and Rao's tests. *J. Multivariate Anal.* **22** 94–105.

[14] PEERS, H. W. (1971). Likelihood ratio and associated test criteria. *Biometrika* **58** 477–487.

[15] SENGUPTA, A., AND VERMEIRE, L. (1986). Locally optimal tests for multiparameter hypotheses. *J. Amer. Statist. Assoc.* **81** 819–825.