Journal of Applied Statistics, Vol. 28, No. 1, 2001, 129–143

On optimal tests for isotropy against the symmetric wrapped stable–circular uniform mixture family

ASHIS SENGUPTA[1] **& CHANDRANATH PAL**[2], [1]*Applied Statistics Division, Indian Statistical Institute, Calcutta 700 035, India and* [2]*Department of Statistics, University of Kalyani, West Bengal 741 235, India*

ABSTRACT *The family of Symmetric Wrapped Stable (SWS) distributions can be widely used for modelling circular data. Mixtures of Circular Uniform (CU) with the former also have applications as a larger family of circular distributions to incorporate possible outliers. Restricting ourselves to such a mixture, we derive the locally most powerful invariant (LMPI) test for the hypothesis of isotropy or randomness of directions—expressed in terms of the null value of the mixing proportion, p, in the model. Global monotonicity of the power function of the test is established. The test is also consistent. Power values of the test for some selected parameter combinations, obtained through simulation reveal quite encouraging performances even for moderate sample sizes. The P^3 approach (SenGupta, 1991; Pal & SenGupta, 2000) for unknown p and ρ and the non-regular case of unknown a, the index parameter, are also discussed. A real-life example is presented to illustrate the inadequacy of the circular normal distribution as a circular model. This example is also used to demonstrate the applications of the LMPI test, optimal P^3 test and a Davies-motivated test (Davies, 1977, 1987). Finally, a goodness-of-fit test performed on the data establishes the plausibility of the above SWS-CU mixture model for real-life problems encountered in practical situations.*

1 Introduction

Symmetric Wrapped Stable (SWS) distributions (Mardia, 1972: 57) constitute a very large family of circular unimodal symmetric distributions useful in the analysis of directional data. The Circular Normal (CN) distribution, although extensively

Correspondence: Ashis SenGupta, Applied Statistics Division, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India. E-mail: ashis@isical.ac.in

ISSN 0266-4763 print; 1360-0532 online/01/010129-15
DOI: 10.1080/02664760120011653

used and probably the only well-known distribution to practitioners for modelling circular data is often not appropriate for modelling real-life data. A 'suitably chosen' member of the SWS family turns out to give a better fit than a CN distribution in many such situations. Also, it has been observed by Mardia (1972: 66) that a CN distribution can be well approximated by a SWS distribution for some specific parameter values. Mixtures of SWS and Circular Uniform (CU) distributions are also very useful and important for incorporating possible outliers in the data. By taking such a mixture, the symmetry in the distribution is still retained, whereas one has more latitude in the choice of an appropriate model because of the presence of the additional parameter p, the mixing proportion.

Let $f(\theta; a, \rho, \mu_0)$ be the density function of an SWS distribution given by

$$f(\theta; a, \rho, \mu_0) = \frac{1}{2\pi}\left\{1 + 2\sum_{n=1}^{\infty} \rho^{n^a} \cos n(\theta - \mu_0)\right\}.$$

Recall that a CN distribution with parameters κ and μ_0 has the p.d.f.

$$f^\star(\theta; \kappa, \mu_0) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa \cos(\theta - \mu_0)\}$$

where $\kappa \geqslant 0$ and $I_0(.)$ is the modified Bessel function of order 0 with a purely imaginary argument. Denote by $g(\theta; a, \rho, \mu_0, p)$ the density function of a p-mixture of SWS and CU distributions, with the parameters having their usual meanings. The density g, when the contaminating distribution is CU, occurs naturally in connection with experiments on the perception of a group of subjects (e.g. insects) for movements towards a given direction. A goodness-of-fit test, based on Watson's U^2-statistic incorporating a grouping correction introduced recently by Brown (1994), on Jander's ant data (Batschelet, 1981: 49) shows (SenGupta, 1998a) that a suitable SWS distribution gives a better fit than a CN distribution. It was also observed that a SWS-CU mixture gave a still better fit than a SWS distribution. Contamination of CU by SWS arises in human perception tests, e.g. in traffic engineering, where it is generally observed that most of the individuals tend to move randomly, save a few who have a rather strong perception, after undergoing some 'brain-washing' treatment. In this context a popular Indian game called 'Breaking the Pitcher' is worth mentioning, where the player is first shown the position of the target, then blindfolded and rotated randomly on the initial position and then asked to choose his/her own direction to break the pitcher with a stick.

The problem of testing isotropy is quite important and has received considerable attention. Several tests for this purpose under different set-ups (types of alternatives) exist in the literature. Beran (1968, 1969) considered this problem under a quite general set-up and derived a general form of the LMPI test. He noted that Ajne's test, Watson's U^2 test, Rayleigh's R^2 test etc, are different particular cases of the general test corresponding to different choices of the alternative hypotheses. Later, Gine (1975) developed the theory of Sobolev tests as a large class of tests containing many of the known ones. Chang (1991) and SenGupta & Chang (1996) have considered locally most powerful (location) invariant (LMPI) tests (with unknown μ_0) for isotropy, in terms of the hypothesis involving ρ, extensively under the model f and, the LMP test under g when p are known.

In this paper, we restrict ourselves to the density g and give a new derivation of the LMPI test for the null hypothesis $H_0: p = 0$ against $H_1: p > 0$, assuming ρ and a to be known. This derivation, based on an expansion of the Most Powerful

Invariant test statistic in powers of p (see, for example, Bhattacharyya & Johnson, 1969), is given in Section 2. It may be instructive to note the approach of Beran (1968) in its general formulation through group spaces for any 'arbitrary' density, together with the explicit derivation of ours, for the specific case of the general family of SWS-CU mixtures. Asymptotic distributions of the test statistic, both under the null and the alternative hypotheses, are also presented. Section 3 provides the global monotonicity of the power function and consistency of the test. Exact cut-off points and power values for some selected parameter combinations, obtained through extensive simulations, are presented in Section 4. The power computation reveals quite encouraging performances for reasonable parameter combinations even with moderate sample sizes. For the general situation when ρ is also unknown, we show in Section 5 that the location invariant P^3-test (SenGupta, 1991; Pal & SenGupta, 2000) reduces to Rayleigh's R^2-test and is L-optimal. This establishes the optimality robustness of the R^2-test in CN distribution against the extended SWS-CU general mixture family. The 'non-regular' situation when ρ is known and a is unknown is treated in Section 6. An example, and the associated summary results, in support of our proposed model are presented in Section 7. This example is also used to demonstrate the applications of the optimal tests obtained in Sections 2, 5 and 6. A large part of the computations of this section has been performed using the statistical package DDSTAP developed by the first author (SenGupta, 1998b). Finally, a rose diagram, also obtained by using DDSTAP, is displayed at the end of the paper.

2 The LMPI test

The model under consideration is

$$g(\theta; a, \rho, \mu_0, p) = pf(\theta; a, \rho, \mu_0) + q(2\pi)^{-1} \tag{1}$$

where $f(\theta; a, \rho, \mu_0)$ is as given in Section 1. Here $0 \leqslant \theta < 2\pi$; $0 \leqslant \mu_0 < 2\pi$, $0 \leqslant \rho \leqslant 1$, $0 < a \leqslant 2$, $0 \leqslant p \leqslant 1$, $q = 1 - p$; ρ and a are known while μ_0 and p are unknown; μ_0 being the location parameter. It may be remarked in this context that, although ρ and a are assumed to be known, in practice either or both of them may be unknown. These aspects are discussed in subsequent sections. Suppose $(\theta_1, \theta_2, \ldots, \theta_m)$ is a random sample of size $m(\geqslant 2)$ from a population with density given by (1). We want to test the hypothesis H_0: $p = 0$ against H_1: $p > 0$. Denote by R_n^2 the quantity $(\sum_{j=1}^m \cos n\theta_j)^2 + (\sum_{j=1}^m \sin n\theta_j)^2$. The form of the LMPI test is derived in the following

Theorem 2.1
For known a and ρ, the LMPI test for H_0: $p = 0$ against H_1: $p > 0$ is given by the critical region $\omega: T = \sum_{n=1}^{\infty} (\rho^2)^{n^a} R_n^2 > C$, where the constant C is to be determined from the size condition.

Proof
First note that the problem of testing H_0 against H_1 remains invariant under the change of location $\theta_i \rightarrow \theta_i + c$ (mod 2π). A set of maximal invariant statistics is $(\theta_1 - \theta_m, \theta_2 - \theta_m, \ldots, \theta_{m-1} - \theta_m)$. Based on this maximal invariant, the most powerful invariant test for H_0 against a fixed $p > 0$ is given by the test statistic

$$T^\star(p) = \int_0^{2\pi} \prod_{i=1}^{m} \left[p(2\pi)^{-1}\left\{1 + 2\sum_{n=1}^{\infty} \rho^{n^a} \cos n(x+\theta_i)\right\} + q(2\pi)^{-1}\right] \mathrm{d}x. \tag{2}$$

To get the LMPI test we expand $T^\star(p)$ in powers of p and consider the lowest order random term (see, for example, Bhattacharyya & Johnson, 1969).

Now the right-hand side of (2) is

$$\left(\frac{1}{\pi}\right)^m \int_0^{2\pi} \prod_{i=1}^{m} \left\{\frac{1}{2} + pY_i(x)\right\} \mathrm{d}x \tag{3}$$

where

$$Y_i(x) = \sum_{n=1}^{\infty} \rho^{n^a} \cos n(x+\theta_i).$$

The coefficient of p in (3) is, apart from a multiplicative constant,

$$\sum_{i=1}^{m} \int_0^{2\pi} Y_i(x)\,\mathrm{d}x = \sum_{i=1}^{m} \left[\int_0^{2\pi} \left\{\sum_{n=1}^{\infty} \rho^{n^a} \cos n(x+\theta_i)\right\} \mathrm{d}x\right].$$

The signs of summation and integration in the square bracket of the above expression are seen to be interchangeable by virtue of the extended version of the Levi theorem for series of functions (see Theorem 10.26 of Apostol, 1974: 269). It then easily follows that the required coefficient of p is zero. The coefficient of p^2 on the other hand, barring again a multiplicative constant, is

$$\begin{aligned}
&\sum_{i<j} \int_0^{2\pi} Y_i(x) Y_j(x)\,\mathrm{d}x \\
&= \sum_{i<j} \int_0^{2\pi} \left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho^{n^a+k^a} \cos n(x+\theta_i)\cos k(x+\theta_j)\right] \mathrm{d}x \\
&= 2\pi \sum_{i<j}\sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n(\theta_i - \theta_j)
\end{aligned}$$

by the same reasoning as above for the interchangeability of summation and integration, followed by some algebraic manipulations. We can then write

$$T^\star(p) = k_1 + k_2 p^2 \sum_{i<j}\sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n(\theta_i - \theta_j) + o_P(p^2)$$

where k_1 and $k_2 (> 0)$ are constants. The critical region of the LMPI test is therefore

$$\omega^\star : T^\star = \sum_{i<j}\sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n(\theta_i - \theta_j) > C^\star \tag{4}$$

where $C^\star$ is to be determined from the size condition. This test is obviously equivalent to ω as given in the statement of the theorem. Hence the proof.

Remark 2.1

It is known that the Cardioid distribution can be viewed as obtained from an SWS distribution by retaining only the first term of the infinite series occurring in the expression of its p.d.f. f. Analogous results do, therefore, hold for the LMPI test statistic in the Cardioid-CU mixture family. We then have the following

Corollary 2.1

The LMPI test for $H_0: p = 0$ against $H_1: p > 0$ in the Cardioid-CU mixture family corresponds to the statistic T with $n = 1$ and therefore coincides with the Rayleigh's R^2 test.

The asymptotic null distribution of the test statistic may be obtained from Corollary 3.1 of Beran (1969) or Theorem 4.1 of Gine (1975).We, however, prove the following theorem, which establishes the asymptotic distribution explicitly by directly appealing to the multivariate Central Limit Theorem (CLT).

Theorem 2.2

Under H_0 the asymptotic distribution of $(2/m)T$ is the same as the distribution of $\sum_{n=1}^{\infty}(\rho^2)^{n^a}\chi^2_{(n)}$ where $\{\chi^2_{(n)}, n = 1, 2, \ldots\}$ is a sequence of independent χ^2 variables each with 2 d.f.

Proof

Straightforward calculations show that under circular uniformity of θ

$$\begin{aligned}
&\mathrm{E}(\sin n\theta) = \mathrm{E}(\cos n\theta) = 0;\\
&\mathrm{Var}(\sin n\theta) = \mathrm{Var}(\cos n\theta) = \tfrac{1}{2}, \text{ for each } n = 1, 2, \ldots;\\
&\mathrm{Cov}(\sin n\theta, \sin k\theta) = \mathrm{Cov}(\sin n\theta, \cos k\theta) = \mathrm{Cov}(\cos n\theta, \cos k\theta) = 0;
\end{aligned}$$

for each $n, k = 1, 2, \ldots; n \neq k$.

Consequently, for any n, by multivariate CLT

$$m^{-1/2}\begin{pmatrix} \sum_i \cos\theta_i \\ \sum_i \sin\theta_i \\ \sum_i \cos 2\theta_i \\ \sum_i \sin 2\theta_i \\ \cdots\cdots \\ \sum_i \cos n\theta_i \\ \sum_i \sin n\theta_i \end{pmatrix} \xrightarrow{\mathscr{L}} N_{2n}(\mathbf{0}, \mathbf{\Sigma})$$

where

$$\Sigma_{(2n\times 2n)} = \frac{1}{2}\begin{pmatrix} I_2 & \mathbf{O} & \ldots & \mathbf{O} \\ \mathbf{O} & I_2 & \ldots & \mathbf{O} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{O} & \mathbf{O} & \ldots & I_2 \end{pmatrix}.$$

This shows that the limiting distribution of $(2/m)T = (2/m)\sum_{n=1}^{\infty}(\rho^2)^{n^a}R_n^2$ is the same as the distribution of $\sum_{n=1}^{\infty}(\rho^2)^{n^a}\chi^2_{(n)}$ where $\{\chi^2_{(n)}, n = 1, 2, \ldots\}$ is a sequence of independent χ^2 variables each with 2 d.f. Hence the theorem.

To obtain the asymptotic non-null distribution of T, one may appeal to the general result in Theorem 1 of Beran (1969). However, the derivation of the distribution involves evaluations of some complicated integrals. We present here an alternative derivation that exploits multivariate CLT directly, as in the null case.

First note that since T is invariant under the change of location, one may take, without loss of generality, $\mu_0 = 0$. After some routine algebra, it follows that under the mixture alternative with density $g(\theta; a, \rho, 0, p)$ and for any $n = 1, 2, \ldots$,

$$\begin{aligned}
&\mathrm{E}(\cos n\theta) = p\rho^{n^a},\ \mathrm{E}(\sin n\theta) = 0,\\
&\mathrm{Var}(\cos n\theta) = \tfrac{1}{2}[1 + p\rho^{(2n)^a}] - p^2(\rho^2)^{n^a},\\
&\mathrm{Var}(\sin n\theta) = \tfrac{1}{2}[1 - p\rho^{(2n)^a}],\ \mathrm{Cov}(\cos n\theta, \sin n\theta) = 0.
\end{aligned}$$

It also follows that for any $n, l = 1, 2, \ldots; n \neq l$,

$$\begin{aligned}
&\mathrm{Cov}(\sin n\theta, \sin l\theta) = (p/2)\,[\rho^{|n-l|^a} - \rho^{(n+l)^a}],\\
&\mathrm{Cov}(\sin n\theta, \cos l\theta) = 0,\\
&\mathrm{Cov}(\cos n\theta, \cos l\theta) = (p/2)\,[\rho^{|n-l|^a} + \rho^{(n+l)^a}] - p^2\rho^{n^a + l^a}.
\end{aligned}$$

For $r, s = 1, 2, \ldots$, writing

$$\sigma_{rs}^{(CC)} = \mathrm{Cov}(\rho^{r^a}\cos r\theta, \rho^{s^a}\cos s\theta)$$

$$\sigma_{rs}^{(CS)} = \mathrm{Cov}(\rho^{r^a}\cos r\theta, \rho^{s^a}\sin s\theta)$$

$$\sigma_{rs}^{(SS)} = \mathrm{Cov}(\rho^{r^a}\sin r\theta, \rho^{s^a}\sin s\theta)$$

and noting that $\sigma_{rs}^{(CS)} = 0 \forall r$ and s, one sees again by multivariate CLT that, under the mixture alternative,

$$m^{-1/2}\begin{pmatrix}
\rho\sum_i \cos\theta_i - mp\rho^2\\
\rho\sum_i \sin\theta_i\\
\rho^{2^a}\sum_i \cos 2\theta_i - mp(\rho^2)^{2^a}\\
\rho^{2^a}\sum_i \sin 2\theta_i\\
\cdots\cdots\cdots\cdots\cdots\\
\rho^{n^a}\sum_i \cos n\theta_i - mp(\rho^2)^{n^a}\\
\rho^{n^a}\sum_i \sin n\theta_i
\end{pmatrix} \xrightarrow{\mathscr{L}} N_{2n}(\mathbf{0}, \boldsymbol{\Sigma}^\star)$$

where

$$\Sigma^\star_{(2n\times 2n)} = \begin{pmatrix}
\sigma_{11}^{(CC)} & 0 & \sigma_{12}^{(CC)} & 0 & \cdots & \sigma_{1n}^{(CC)} & 0\\
0 & \sigma_{11}^{(SS)} & 0 & \sigma_{12}^{(SS)} & \cdots & 0 & \sigma_{1n}^{(SS)}\\
\sigma_{21}^{(CC)} & 0 & \sigma_{22}^{(CC)} & 0 & \cdots & \sigma_{2n}^{(CC)} & 0\\
0 & \sigma_{21}^{(SS)} & 0 & \sigma_{22}^{(SS)} & \cdots & 0 & \sigma_{2n}^{(SS)}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\sigma_{n1}^{(CC)} & 0 & \sigma_{n2}^{(CC)} & 0 & \cdots & \sigma_{nn}^{(CC)} & 0\\
0 & \sigma_{n1}^{(SS)} & 0 & \sigma_{n2}^{(SS)} & \cdots & 0 & \sigma_{nn}^{(SS)}
\end{pmatrix}.$$

Now applying [(ii), 6a.2] of Rao (1973: 387) and taking $n \to \infty$, one can see that

$$m^{-\frac{3}{2}}\left(T-(pm)^2\sum_{k=1}^{\infty}(\rho^4)^{k^a}\right)\xrightarrow{\mathscr{L}}N(0,\sigma^{\star 2})\text{ as }m\to\infty \tag{5}$$

where

$$\sigma^{\star 2}=4p^2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(\rho^2)^{r^a+s^a}\sigma_{rs}^{(CC)}. \tag{6}$$

We then have the following

Theorem 2.3
For any $p>0$ under the alternative hypothesis H_1, the asymptotic distribution of T is given by (5) with $\sigma^{\star 2}$ given by (6).

Remark 2.2
Exact distributions of the test statistic T under both the null and the alternative hypotheses are analytically intractable. That is why we take recourse to extensive simulations (in Section 4) to obtain exact cut-off points and power values of the LMPI test for different parameter and sample size combinations.

Remark 2.3
The asymptotic distribution of $(2/m)T$ under H_0, as given by Theorem 2.2, is not convenient to carry out the test in practice. We, therefore give an approximation for the above distribution. The approximation of Satterthwaite (1946) seems not to be appropriate for this purpose, as it yields very bad results for some parameter values. We, therefore, adopt a different approach based on characteristic functions. Note that under circular uniformity, the asymptotic characteristic function of $(2/m)T$ is $\varphi(t)=\{\prod_{n=1}^{\infty}(1-2(\rho^2)^{n^a}it)\}^{-1}$, a first-order approximation of which is $(1-2Kit)^{-1}$ where $K=\sum_{n=1}^{\infty}(\rho^2)^{n^a}$. This is clearly the characteristic function of $K.Z$ where Z has a chi square distribution with 2 d.f. The values of the cut-off points obtained using this approximation are quite close to those obtained by simulation, at least for some parameter combinations. For example, for $m=20$, $\alpha=0.05$ and for $\rho=0.5$ and $a=1.5$, the cut-off point is calculated as 16.21, which compares favourably with the simulated value 16.08 of Table 1.

3 Monotonicity of the power function and consistency of the test

It is convenient to work with the test statistics $T^\star$ given in (4). Monotonicity and consistency follow from the following

Theorem 3.1
For any fixed ρ and a, the test given in (4) possesses a monotone power function in $p\in[0,1]$. Further, the test is also consistent.

Proof
To prove the first part of the theorem we need the following results:

Result 3.1 (Wintner, 1947: 591)
The function $1+2\sum_{n=1}^{\infty}\rho^{n^a}\cos n\theta$ is decreasing in $0\leqslant\theta<\pi$ and, by symmetry, increasing in $\pi\leqslant\theta<2\pi$ irrespective of $0<\rho<1$ as long as $0<a\leqslant 2$.

Result 3.2 (Wintner, 1947: 591)
For $0< \delta< \pi$, a set of sufficient conditions for the series $\Sigma_{n=1}^{\infty} b_n \sin n\delta$ to be positive is

$$nb_n \to 0 \text{ as } n \to \infty \text{ and}$$

$$nb_n > (n+1)b_{n+1} \forall n = 1, 2, \ldots$$

Result 3.3 (SenGupta & Chang, 1996)
Let (X, Y) have absolutely continuous joint distribution depending on a single parameter θ such that each of X and Y has stochastic ordering property in θ. Then $X+Y$ has also the same stochastic ordering property.

Note that the p.d.f. of $\gamma_{ij} = \theta_i - \theta_j \pmod{2\pi}$, where θ_i and θ_j are independently distributed as (1), can be written as

$$h(\gamma_{ij}; p) = \frac{1}{2\pi}\left\{1 + 2\sum_{n=1}^{\infty} p^2(\rho^2)^{n^a} \cos n\gamma_{ij}\right\}, 0 \leqslant \gamma_{ij} < 2\pi.$$

By Result 3.1, for given c, $\exists$ a $\delta \in (0, \pi)$

$$\sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n\gamma_{ij} > c \Leftrightarrow 0 < \gamma_{ij} < \delta \text{ or } 2\pi - \delta < \gamma_{ij} < 2\pi.$$

Hence for any $i, j (i< j)$,

$$\begin{aligned} P(p) &= P\left\{\sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n\gamma_{ij} > c \,\middle|\, p\right\} \\ &= 2\int_0^{\delta} h(\gamma_{ij}; p)\, \mathrm{d}\gamma_{ij} \\ &= \frac{\delta}{\pi} + \frac{2p^2}{\pi}\int_0^{\delta} \sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n\gamma_{ij}\, \mathrm{d}\gamma_{ij} \end{aligned}$$

and on differentiation w.r.t. p

$$\begin{aligned} \frac{\partial P}{\partial p} &= \frac{4p}{\pi}\int_0^{\delta} \sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n\gamma_{ij}\, \mathrm{d}\gamma_{ij} \\ &= \frac{4p}{\pi}\sum_{n=1}^{\infty} b_n \sin n\delta \end{aligned}$$

with $b_n = (1/n)(\rho^2)^{n^a}$, by interchanging again the order of summation and integration.

Thus, $P(p)$ is increasing globally in $p \in [0, 1]$, by Result 3.2. Repeated applications of Result 3.3 on the variables $\Sigma_{n=1}^{\infty} (\rho^2)^{n^a} \cos n\gamma_{ij}$ for all i, j $(i< j)$ then ensure the monotonicity of the power function.

To prove consistency, it suffices (from Theorem 4 of Beran, 1969) to verify that the function $b(\theta; a, \rho, \mu_0, p)$ defined by

TABLE 1. Simulated cut-off points of the LMPI test based on T

		a			
m	ρ	0.5	1.0	1.5	2.0
10	0.25	(2.25, 3.30)	(1.86, 2.74)	(1.81, 2.73)	(1.78, 2.59)
	0.50	(12.94, 16.81)	(8.40, 11.48)	(7.53, 10.73)	(7.26, 10.56)
	0.75	(53.74, 65.48)	(25.12, 35.99)	(19.78, 27.92)	(17.00, 24.90)
	0.90	(124.97, 157.95)	(64.86, 79.08)	(39.55, 52.67)	(30.52, 43.53)
15	0.25	(3.36, 4.67)	(2.84, 4.17)	(2.69, 4.05)	(2.61, 4.00)
	0.50	(19.33, 25.85)	(12.96, 17.84)	(11.74, 17.64)	(11.35, 16.26)
	0.75	(81.41, 96.91)	(37.46, 50.37)	(28.29, 40.89)	(26.78, 39.99)
	0.90	(189.15, 225.82)	(97.02, 119.74)	(58.53, 81.79)	(46.15, 60.80)
20	0.25	(4.44, 6.29)	(3.81, 5.84)	(3.77, 5.69)	(3.59, 5.45)
	0.50	(25.82, 35.90)	(17.48, 24.73)	(16.08, 23.05)	(14.77, 21.71)
	0.75	(110.06, 133.34)	(51.53, 65.28)	(37.80, 51.91)	(35.61, 50.21)
	0.90	(250.27, 300.33)	(127.81, 159.79)	(78.08, 101.63)	(63.26, 85.76)
30	0.25	(6.54, 9.42)	(5.66, 8.35)	(5.29, 8.09)	(5.27, 7.67)
	0.50	(40.17, 52.45)	(24.66, 36.26)	(23.44, 35.44)	(21.13, 30.38)
	0.75	(163.61, 199.44)	(78.37, 105.04)	(62.08, 94.85)	(53.25, 82.42)
	0.90	(376.02, 442.52)	(196.16, 240.38)	(119.42, 156.11)	(94.53, 131.99)

(The figures inside the brackets denote the cut-off points at 5% and 1% levels respectively.)

$$b(\theta; a, \rho, \mu_0, p) = \int_0^{2\pi} \left[f(\theta; a, \rho, x) - \frac{1}{2\pi} \right] g(x; a, \rho, \mu_0, p)\, \mathrm{d}x$$

corresponding to any density $g(\theta; a, \rho, \mu_0, p)$ under the alternative hypothesis, is non-zero.

Now

$$b(\theta; a, \rho, \mu_0, p) = \frac{1}{\pi} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \rho^{n^a} \cos n(\theta - x) \right] \left[\frac{1}{2\pi} + \frac{p}{\pi} \sum_{n=1}^{\infty} \rho^{n^a} \cos n(x - \mu_0) \right] \mathrm{d}x$$

$$= \frac{p}{\pi} \sum_{n=1}^{\infty} (\rho^2)^{n^a} \cos n(\theta - \mu_0)$$

after simplification, followed by some algebraic manipulation. Thus, $b(\theta; a, \rho, \mu_0, p) \neq 0$, proving the consistency of the test. Hence the theorem.

4 Simulation and computation

In this section we present exact cut-off points (Table 1) and power values (Table 2) of the LMPI test, obtained through simulations, for some selected parameter combinations. Simulation from a symmetric stable distribution have been done using the RNSTA subroutine of IMSL. An observation X from a symmetric stable distribution with 'scale factor' 1, when multiplied by $d^{1/\alpha}$ gives another, say Y, with scale factor d, having the characteristic function

$$\phi_Y(t) = \exp(-d|t|^{\alpha}).$$

TABLE 2. Simulated power of the LMPI test at 5% level

			p									
a	ρ	m	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.5	0.25	10	0.051	0.054	0.055	0.068	0.077	0.082	0.106	0.119	0.138	0.156
		20	0.054	0.061	0.072	0.098	0.108	0.145	0.165	0.200	0.256	0.300
	0.50	10	0.053	0.063	0.090	0.145	0.193	0.251	0.337	0.413	0.518	0.605
		20	0.062	0.095	0.168	0.248	0.356	0.482	0.635	0.734	0.828	0.921
	0.75	10	0.072	0.120	0.237	0.363	0.502	0.645	0.783	0.871	0.942	0.972
		20	0.078	0.214	0.390	0.638	0.827	0.913	0.971	0.993	1.000	1.000
	0.90	10	0.074	0.215	0.396	0.592	0.760	0.877	0.944	0.986	0.998	1.000
		20	0.142	0.400	0.689	0.898	0.963	0.993	0.999	1.000	1.000	1.000
1.0	0.25	10	0.054	0.057	0.058	0.060	0.068	0.093	0.094	0.112	0.130	0.144
		20	0.056	0.058	0.073	0.079	0.099	0.123	0.133	0.183	0.236	0.282
	0.50	10	0.055	0.063	0.077	0.100	0.146	0.191	0.251	0.335	0.419	0.502
		20	0.058	0.067	0.105	0.169	0.247	0.359	0.473	0.604	0.723	0.858
	0.75	10	0.062	0.088	0.157	0.262	0.367	0.522	0.635	0.783	0.874	0.949
		20	0.065	0.140	0.255	0.442	0.644	0.826	0.911	0.973	0.994	1.000
	0.90	10	0.074	0.134	0.283	0.449	0.628	0.779	0.897	0.958	0.992	1.000
		20	0.106	0.295	0.535	0.766	0.912	0.978	0.998	1.000	1.000	1.000
1.5	0.25	10	0.052	0.053	0.060	0.067	0.081	0.084	0.088	0.108	0.131	0.149
		20	0.053	0.057	0.068	0.078	0.089	0.117	0.141	0.168	0.221	0.271
	0.50	10	0.055	0.061	0.082	0.114	0.150	0.207	0.272	0.336	0.440	0.532
		20	0.056	0.070	0.106	0.158	0.235	0.364	0.477	0.609	0.744	0.838
	0.75	10	0.056	0.074	0.127	0.214	0.310	0.425	0.589	0.734	0.847	0.946
		20	0.073	0.145	0.259	0.401	0.599	0.773	0.904	0.974	0.993	1.000
	0.90	10	0.073	0.115	0.212	0.368	0.524	0.702	0.846	0.929	0.987	0.999
		20	0.091	0.200	0.423	0.661	0.866	0.945	0.989	0.999	1.000	1.000
2.0	0.25	10	0.053	0.056	0.062	0.071	0.075	0.080	0.100	0.126	0.132	0.157
		20	0.054	0.062	0.071	0.087	0.109	0.128	0.157	0.201	0.235	0.288
	0.50	10	0.054	0.057	0.078	0.111	0.158	0.199	0.269	0.353	0.443	0.539
		20	0.055	0.081	0.127	0.169	0.289	0.429	0.520	0.663	0.786	0.881
	0.75	10	0.060	0.098	0.140	0.205	0.335	0.463	0.615	0.762	0.900	0.981
		20	0.063	0.123	0.223	0.386	0.574	0.746	0.891	0.975	0.997	1.000
	0.90	10	0.068	0.109	0.198	0.339	0.492	0.652	0.830	0.934	0.988	1.000
		20	0.093	0.206	0.329	0.584	0.816	0.939	0.982	1.000	1.000	1.000

Given ρ, one can take $d = -\log_e \rho$ and wrap the resulting Y over $(0, 2\pi)$ to get an observation from $\mathrm{SWS}(0, \rho, \alpha)$ distribution.

For each parameter and sample size combination, the cut-off points and the power values have been computed on the basis of 5000 and 1000 observations, respectively, on the statistic T calculated from samples generated from respective distributions. It may be noted that, although the statistic T appears in the form of an infinite series, it usually suffices to consider only a finite number of terms in practice. We have taken the first 30 terms of the series for the computation of T. Power computations of Table 2 show encouraging performances for 'reasonable' parameter combinations, even for samples of size 20. It is also worth noting that the power is increasing with ρ for each fixed m, a and p. This is expected because, if a and p are fixed, the larger the value of ρ, the more the deviation of the density g from circular uniformity and this fact should be reflected by any reasonable test.

5 The P^3-test

In Section 2, we have assumed both ρ and a to be known and have derived the LMPI test for the null value of p. The situation becomes more complicated if either or both of ρ and a are unknown. The experimenter usually has the choice of 'a' for which a mixture model might give the best fit to the data. We therefore assume that ρ alone is unknown. However, even then the problem cannot be reduced by any of the principles of similarity, unbiasedness or invariance with respect to the nuisance parameters μ_0 and ρ. Invoking location invariance, one can search for an optimal P^3-test (see SenGupta, 1991; Pal & SenGupta, 2000, for further details) in such a situation. Note that for any pair $(i,j), i< j$,

$$E_g(\cos(\theta_i - \theta_j)) = (p\rho)^2 \equiv \eta, \text{ say}$$

where η plays the role of the appropriate Pivotal Parametric Product (P^3) in this case. Based on $(\theta_1, \theta_2, \ldots, \theta_m)$, an unbiased (and consistent) estimator of η is then

$$U = \frac{2}{m(m-1)} \sum_{i<j} \cos(\theta_i - \theta_j) = \frac{R^2}{m(m-1)} - \frac{1}{m-1}.$$

A test for isotropy (which is now equivalent to H_0': $\eta = 0$) may be based on U or equivalently on R^2. Chang (1991) has shown that for $p = 1$ but unknown, the LBI test for H_0: $\rho = 0$ against H: $\rho > 0$ is the Rayleigh's test, i.e. one based on R^2. Note that this test does not depend on a, and is therefore robust against the SWS ρ_0 family. From Chang (1991), we then have the following.

Theorem 5.1
When $p = 1$ but ρ_0 is unknown, Rayleigh's R^2-test is (location invarient) robust optimal against the SWS family.

6 ρ known, a unknown: non-regular case

When ρ is known but a is unknown, we encounter the non-regular problem of having a only under the alternative. Motivated by Davies (1977, 1987), we enhance a technique of constructing the optimal test for this case. To apply this technique, assume that $a \in [\varepsilon, 2]$, where ε is a known small positive number.

Observe that for each a, under H_0

$$T'(a) \equiv \frac{2}{m} T(a) = \frac{2}{m} \sum_{n=1}^{\infty} (\rho^2)^{n^a} R_n^2 \xrightarrow{\mathscr{L}} \sum_{n=1}^{\infty} (\rho^2)^{n^a} \chi^2_{(n)} \text{ as } m \to \infty \tag{7}$$

where $\{\chi^2_{(n)}, n = 1, 2, \ldots\}$ is a sequence of independent χ^2 variables each with 2 d.f. The test then consists in rejecting H_0 for large values of $\sup_{a \in [\varepsilon,2]} T'(a)$. Also observe that $T'(a)$ is monotonically decreasing in a and hence

$$\sup_{a \in [\varepsilon,2]} T'(a) = \frac{2}{m} \sum_{n=1}^{\infty} (\rho^2)^{n^\varepsilon} R_n^2 \tag{8}$$

The significance probability of the test determined by the statistic (8), however, cannot be directly obtained from the results given in Davies (1987), since the asymptotic distribution considered there is that of a single χ^2. However the required significance probability can be calculated using the asymptotic distribution given in (7) with a replaced by ε. To obtain this, we recall (2.10) of Beran (1969). It then follows that the required probability is

$$\sum_{n=1}^{\infty} a_n \exp[-t/2(\rho^2)^{n\varepsilon}] \quad (9)$$

where $a_n = \prod_{k \neq n} [1 - (\rho^2)^{k\varepsilon - n\varepsilon}]^{-1}$ and t is the observed value of $\sup_{a \in [\varepsilon, 2]} T'(a)$ in (8).

7 Example

We present here the analysis of Jander's Ant data to demonstrate the applications of the LMPI test and the other optimal tests considered in Sections 5 and 6. We also recall (SenGupta, 1998a) that the CN distribution does not fit satisfactorily or is even unsuitable for this example whereas the SWS-CU mixture distribution gives a good fit.

The tests are applied to the data assuming that SWS-CU mixture model holds. The significance probability corresponding to the LMPI test has been computed using (9) of Section 6 with the determined value of a in place of ε. For the Davies-motivated test, the same formula has been used with $\varepsilon = 0.1$, whereas the (asymptotic) significance probability for the Rayleigh's test has been calculated on the basis of a χ^2 distribution with 2 d.f. It may be remarked in this context that, although the expression for the significance probability in (9) and that of a_n therein involve an infinite number of terms, for numerical computation it sufficed to retain at most 25 terms. Once rejection is obtained, the goodness-of-fit test is carried out using Watson's U^2-statistic, incorporating a grouping correction recently introduced by Brown (1994). The parameters for fitting CN are estimated by the m.l. method and, for the SWS and SWS-CU cases, following SenGupta (1998a), the method of moments has been used for estimating μ_0 and ρ while p and a have been determined adaptively (using DDSTAP) to select the 'best' member of the respective family. In the following, although the parameters have been estimated from the samples, we have assumed, for the sake of simplicity, that the estimates are actually the true values of the respective parameters (because the estimates are consistent and samples are of large sizes). For the purpose of comparison of the U^2-values with the tabulated ones, we have used the appropriate figure corresponding to case 0 of Table 1 of Lockhart & Stephens (1985: 649) since these figures are universal whatever the 'known' distribution we fit.

7.1 *Jander's ant data*

Batschelet (1981: 49, Fig. 2) depicted the orientation of ants towards a black target when released in a round arena—an experiment originally conducted by Jander (1957). We adapt the data to construct a grouped frequency distribution of angles with 36 classes of equal widths; the total frequency being 146. The rose diagram of the data is shown in Fig. 1. Summary results of the applications of the optimal tests are presented in Table 3.

From Table 3 it is seen that each of the three optimal tests discussed in Sections 2, 5 and 6 leads to rejection of the null hypothesis of circular uniformity of the data. We therefore carry out the goodness-of-fit test. Summary results for fitting CN (μ_0, κ), SWS (μ_0, ρ, a) and SWS-CU mixture are presented in Table 4.

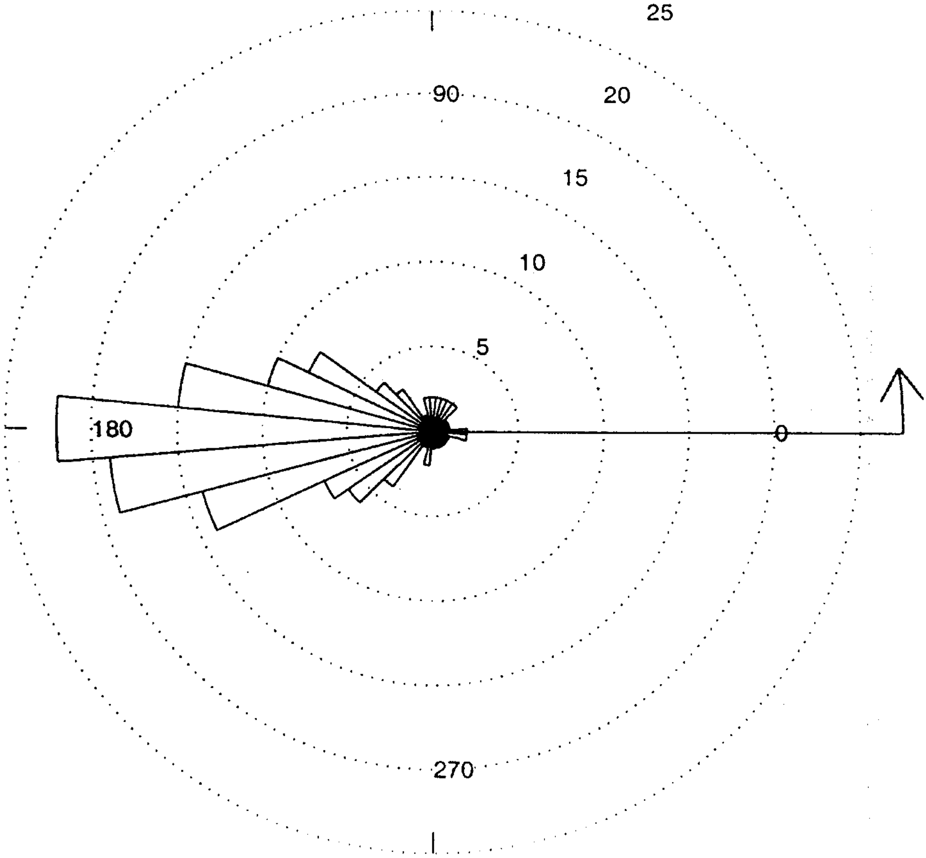

FIG. 1. Rose diagram of Ant data.

TABLE 3. Results of the application of optimal tests on Jander's Ant data

Test applied	p-value (significance probability)	Remark
LMPI	0.0	Highly significant
Rayleigh's R^2	0.0	Highly significant
Davies-motivated ($\in = 0.1$)	0.0	Highly significant

Table 4. Goodness-of-fit tests for Jander's Ant data

Distribution fitted	U^2	Remark
CN	0.4799	Not satisfactory
SWS ($a = 0.8$)	0.0492	Satisfactory
MIX ($a = 1.59$, $p = 0.7$)	0.039	More satisfactory

8 Concluding remarks

The test procedures we used in the earlier sections are the locally best tests or their different modifications. One reason for considering such tests is that they are easy to obtain and, as is seen, enjoy nice properties, namely monotonicity of the power function and consistency. In addition, it should be borne in mind that it is, in general, difficult to detect small departures from the null hypotheses while large departures can be detected quite easily by any reasonable test. The LRTs in all these situations are very computation-intensive and difficult to apply since they cannot be written in any closed form. Note that no non-trivial sufficient statistic exists for our mixture family and hence this is expected. The exact distribution of the LRT statistic is intractable. Further, the standard result yielding the χ^2 as the asymptotic null distribution of the LRT statistic is not valid for our case. This is so due to the non-regular nature of our problem where the parameter space is no longer open and where the parameter lies on the boundary under H_0. For these reasons, the LRT approach is not to be pursued, and any numerical comparison, which can be possibly explored at most via simulations, seems unappealing and uninstructive. However, it may be worthwhile to note that the LMP test can be viewed as a first-order approximation of the LRT.

The problem of detection of outliers is currently drawing the attention of many researchers. The review paper by Jupp & Mardia (1989) contains several references on this interesting area. Guttorp & Lockhart (1988) provide a Bayesian solution of the problem of detecting the location of a downed aircraft from distress signals transmitted by it and received by different search-and-rescue stations. Outliers in the data may occur due to irregularities in the readings caused by distorting objects near the receptor site. One can, therefore, either detect and reject the outliers or can assume a mixture model in order to incorporate them for the analysis. Thus, if one intends to 'accommodate' outliers in the model, it is common practice to assume a mixture distribution for it. Otherwise, one may assume a non-mixture model like CN (Collett, 1980) for outlier detection. The SWS distribution introduced in Section 2 may also be used for this purpose and this may be an interesting problem for further research.

Acknowledgements

The authors would like to thank Sri Subhasish Pal, CSSC, Indian Statistical Institute, for help with the computer-aided work in this paper. Thanks are also due to an anonymous referee for some valuable comments that led to a substantial improvement of the presentation of the paper.

REFERENCES

Apostol, T. (1974) *Mathematical Analysis* (Addison Wesley).

Batschelet, E. .(1981) *Circular Statistics in Biology* (Academic Press).

Beran, R. J. (1968) Testing for uniformity on a compact homogeneous space, *Journal of Applied Probability*, 5, pp. 177–195.

Beran, R. J. (1969) Asymptotic theory of a class of tests for uniformity of a circular distribution, *Annals of Mathematical Statistics*, 40, pp. 1196–1206.

Bhattacharya, G. K. & Johnson, R. A. (1969) On Hodge's bivariate sign test for uniformity of a circular distribution, *Biometrika*, 56, pp. 446–449.

Brown, D. (1994) Grouping correction of Watson's U^2 statistic, *Journal of the Royal Statistical Society, Series B*, 56, pp. 275–283.

CHANG, H. (1991) Some optimal tests in directional data, *PhD dissertation, Department of Statistics and AP, University of California at Santa Barbara, USA.*

COLLETT, D. (1980) Outliers in circular data, *Applied Statistics*, 29, pp. 50–57.

DAVIES, R. B. (1977) Hypothesis testing when a nuisance parameter is present only under the alternative, *Biometrika*, 64, pp. 247–254.

DAVIES, R. B. (1987) Hypothesis testing when a nuisance parameter is present only under the alternative, *Biometrika*, 74, pp. 33–43.

GINE, E. (1975) Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms, *Annals of Statistics*, 3, pp. 1234–1266.

GUTTORP, P. & LOCKHART, R. A. (1988) Finding the location of a signal: a Bayesian analysis, *Journal of the American Statistical Association*, 83, pp. 322–330.

JANDER, C. (1957) Die optiche Richtungsorientierung der roten Waldameise (*Formica rufa* L.), *Z. vergl. Physiologie*, 40, pp. 162–238.

JUPP, P. E. & MARDIA, K. V. (1989) A unified view of the theory of directional statistics, *International Statistical Review*, 57, pp. 261–294.

LOCKHART, R. A. & STEPHENS, M. A. (1985) Tests of fit for the von Mises distribution, *Biometrika*, 72, pp. 647–652.

MARDIA, K. V. (1972) *Statistics of Directional Data* (Academic Press).

PAL, C. & SENGUPTA, A. (2000) Optimal tests for no contamination in reliability models. Accepted for publication in *Lifetime Data Analysis*, 6, pp. 281–290.

RAO, C. R. (1973) *Linear Statistical Inference and its Applications*, 2nd edn (Wiley).

SATTERTHWAITE, F. E. (1946) An approximate distribution of estimates of variance components, *Biometrics Bulletin*, 2, pp. 110–114.

SENGUPTA, A. (1991) A review of optimality of multivariate tests. *In:* S. R. JAMMALAMADAKA & A. SENGUPTA (guest eds), *Special Issue: Multivariate Optimality and Related Topics, Statistics and Probability Letters*, 12, pp. 527–535.

SENGUPTA, A. (1998a) Optimal parametric statistical inference. *In:* A. SENGUPTA (ed), *Analysis of Directional Data*, Indian Statistical Institute, Calcutta, pp. 23–52.

SENGUPTA, A. (1998b) DDSTAP—Statistical package for the analysis of directional data, *Applied Statistics Division, Indian Statistical Institute, Calcutta*.

SENGUPTA, A. & CHANG, H. (1996) A robust optimal test for uniformity for directional data against wrapped stable mixture family, *Private communication*.

WINTNER, A. (1947) On the shape of the angular case of Cauchy's distribution curves, *Annals of Mathematical Statistics*, 18, pp. 589–593.