$P^3$ APPROACH TO INTERSECTION-UNION TESTING OF HYPOTHESES [1]

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## Abstract

Recent applications of Statistics often leads one to encounter testing problems where the original hypothesis of interest comprises of the union of several sub-hypothesis. In the framework of such Intersection-Union testing of hypothesis, in contrast to the usual Union-Intersection framework, a sub-hypothesis therein may specify a parameter or a function of some of the parameters of the underlying distribution. The parameters may even be constrained to lie on the boundary of their parameter spaces. Even large-sample tests such as the usual Likelihood ratio. Lagrangian multiplier or the Wald's tests then do not apply as their usual asymptotic distribution theory remain no longer valid. An approach based on a Pivotal Parametric Product $P^3$ is enhanced here. It is shown that this approach often leads to appealing simple and elegant test statistics. The exact cut-off points and the power values can be computed by judicious use of numerical packages. L-optimality of such a test for the mixture problem is established. For multivariate multiparameter testing problems it is shown that such an approach leads to Union-Intersection Intersection-Union tests. Construction of such tests are exemplified through several real-life problems as in, e.g. testing for interval specifications in Acceptance Sampling, for Generalized Variance of structured correlation matrices in Generalized Canonical Variable, for agreement in Method Comparison Studies, for no contamination in multiparameter multivariate mixture models, etc. It is demonstrated for a real-life data set in an acceptance sampling problem that the proposed class of $P^3$ tests

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includes the intuitive one existing in the literature.

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**A brief running title:** P$^3$ tests for Intersection-Union problems

# 1 Introduction

S.N. Roy's principle of construction of tests for the case when the null hypothesis $H_0$ consists of the simultaneous occurrence of several disjoint subhypotheses and is represented as $H_0 = \cap_{i=1}^{s} H_{0i}$ is well known as the Union-Intersection (UI) principle (Roy, 1953). The reverse scenario, i.e. where $H_0$ holds when at least any one of $H_{0i}$ holds, ie. $H_0 = \cup_{i=1}^{s} H_{0i}$, referred to as the Intersection-Union (IU) testing of hypotheses problem in SenGupta (1991), is also faced in practice and is recently attracting quite some attention. While the celebrated Union-Intersection testing procedure of Roy is mainly enhanced for multiparameter testing problems in multivariate distributions, the Intersection-Union testing problems arise in important one-parameter situations also, e.g. in some recently emerging areas such as Bioequivalence or generally "equivalence" testing problems (see e.g., Choudhary and Nagaraja, 2005; Madallaz and Mau, 1981), Acceptance Sampling in Statistical Process Control (SPC) (Berger, 1982), Reliability and Multivariate Analysis. This problem also arises of course for multiparameter problems, of which special mention must be made of the test for no mixture in contaminated or mixture models. Sometimes the standard separate tests for each $H_{0i}$ may be combined, to yield a test for $H_0$ as, say, with the critical region given by the intersection of the separate critical regions, see e.g. Choudhary and Nagaraja (2005). However, the determination of the cut-off points there seems as to be often done in an intuitive manner. Also, as e.g. is exemplified below by the mixture models, it is not always even possible to have exact "separate" tests. This is so because the elimination of even location or scale, and of course a non location-scale, nuisance parameter poses non-trivial problems. An unified approach motivated by optimality considerations and based on Pivotal Parametric Product ($P^3$) (SenGupta, 1991) and its unbiased estimating function is pursued here and is shown to yield simple and elegant *exact* tests for a variety of situations including those mentioned above. An application of the $P^3$ test to yield a useful test that can be implemented in practice in lieu of the trivial UMP test (Lehmann, 1986, Additional Problems 53, p. 126) is also presented here. Examples also include tests for no contamination for linear random variables and for isotropy for circular random variables. Determination of exact cut-off points and derivation of exact power are also illustrated through an important real-life problem from Acceptance Sampling. Further, the fact that the class of $P^3$ tests can have attractive power performance is exemplified by the superiority of such a test over some ad-hoc ones and also by demonstrating that it includes the existing intuitive one for this problem through exact power computations. Additionally, such $P^3$ tests are shown to be capable of yielding even UMP and L-optimal tests. Finally, it is demonstrated interestingly that for the general situation of multivariate multiparameter IU testing problems, application of Roy's UI principle on the optimal $\mathrm{P}^3$ tests is a powerful method that can yield elegant tests. Such tests are named here as the Union-Intersection - Intersection-Union tests.

# 2 Definition and Construction of a $P^3$ Test

We note that for the IU testing problem, the likelihood ratio tests (LRTs) can be quite cumbersome, e.g. when some of the $H_{0i}$s constrain the parameter(s) in certain intervals. The LRTs may even lack their usual asymptotic properties , e.g. when one of the $H_{0i}$s, as written above, specifies a parameter value lying on the boundary of the parameter space. This is very common, e.g. in the framework of testing for no contamination in mixture models as will be taken up later. A simpler procedure which is based on an optimality approach is proposed here. The idea is to first recast the original multiple $H_{0i}$ representation of the null hypothesis in terms of only a single hypothesis involving an appropriately chosen parametric function. We will call such a parametric function a *Pivotal Parametric Product* ($\mathrm{P}^3$ in short).

**Definition 1.** A scalar parametric function $\eta$ of a possibly vector-valued parameter $\theta$, i.e. $\eta \equiv g(\theta)$, will be termed a Pivotal Parametric Product, $P^3$ in short, when the null hypothesis $H_0 \equiv \cup_{i=1}^{s} H_{0i}$ for $\theta$ holds if and only if $\eta = 0$.

After an appropriate $\mathrm{P}^3$ is chosen, the test for $H_0 : \eta = 0$, will be constructed on the basis of a suitable (usually unbiased and/or consistent) estimator, say $\hat{\eta}$, of it. The critical region will be defined based on how the original alternative translates, one-sided or two-sided, in terms of $\eta$. We will call such a test a $\mathrm{P}^3$ test. In general a IU testing problem will not admit of the existence of the UMP test. Attention will therefore be then focussed to locally optimal procedures, e.g. 'L-optimal' tests introduced by SenGupta (1991) and further pursued by Pal and SenGupta (2000a) in the context of mixture problems.

# 3 Case of One-parameter $H_0$

Let a random variable X follow the distribution $f(x;\theta)$, where $\theta$ is a scalar parameter. Here we consider testing

$H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$.

$H_0$ can be represented as $\cap_{i=1}^{2} H_{0i}$ where the subhypotheses $H_{01}$ and $H_{02}$ are given by, $H_{01} : \theta \leq \theta_1$ and $H_{02} : \theta \geq \theta_2$.

A $P^3$ can be easily identified as $\eta = (\theta - \theta_1)(\theta - \theta_2)$ and $H_0$ and $H_1$ translate in terms of $\eta$ to $H_0 : \eta \geq 0$ and $H_1 : \eta < 0$, respectively. A critical region (c.r.) of the test based on $\eta$ can then be suggested as $\omega :\ \hat{\eta} \leq K$, where $\hat{\eta}$ is an unbiased estimator of $\eta$ and $K$ is a constant to be determined such as to meet the desired size $\alpha$. However, noting that $\eta$ is a quadratic function of $\theta$, this test can be equivalently represented by the test $\phi(\hat{\theta})$ defined by the c.r. (assuming the non-randomized setup for simplicity)

$$\omega :\ K_1 \leq \hat{\theta} \leq K_2, \tag{1}$$

where $\hat{\theta}$ is an unbiased estimator of $\theta$. We specify the determination of $K_1, K_2$ by using the conditions:

$$E_{\theta_1}\phi(\hat{\theta}) = E_{\theta_2}\phi(\hat{\theta}) = \alpha. \tag{2}$$

## 3.1 Exponential Family

Consider (1) and (2) above. In particular, for the one-parameter Regular Exponential Family (REF), we have

**Theorem 1.** For the one-parameter testing problem stated above, a $P^3$ test can be constructed in the one-parameter REF to be the UMP test.
Proof: In (1) take the unbiased estimator $\hat{\theta}$ as the usual efficient estimator for the one-parameter exponential family based on the sufficient statistic. An application of Theorem 6 of Lehmann (1986), p. 101, then completes the proof.

We start with a very simple example, whose generalizations have applications in Reliability and Queuing theory, to illustrate a $P^3$ test.

**Example 1.** Let X follow an exponential distribution with mean parameter $\lambda$. Let us test $H_0 : \lambda \leq 0.2$ or $\geq 0.5$ vs $H_1$ : not $H_0$, based on a single observation, with $\alpha = .05$. In the context of queues, $H_1$ may be interpreted as neither a too short nor a too long waiting time (in hours, say) for a customer. From (1) the $P^3$ test is given by the critical region,

$$\omega :\ a \leq X \leq b.$$

Then (2) reduces to

$$.05 = e^{-.2a} - e^{-.2b} = e^{-.5a} - e^{-.5b}$$

$$\Rightarrow a = -5ln(.05 + e^{-.2b}),\ b = -2ln(e^{-.5a} - .05) \tag{3}$$

These give $b = f(b)$, for which the Newton-Raphson method can be conveniently employed to find $b$. $a$ can then be determined from any of the equations in (3).

Power of this $P^3$ test is given by

$$\beta_\lambda = e^{-\lambda a} - e^{-\lambda b}, \lambda = .2(.05).5$$

A plot of the power curve is given in Figure 1.

Note the (unusual) bell-shaped form of the power curve of this UMP test. This is to be expected from the nature of $H_1$ and is characteristic of the power curves of all reasonable tests for $H_0$ vs $H_1$ as defined for this section. For UMP tests, this fact follows from part (iii) of Theorem 6 of Lehmann (1986), p. 100.

**Example 2. (Acceptance Sampling.)** We now consider testing $H_0$ above (as specified at the outset of this section) for the normal distribution $N(\theta, \sigma^2)$. In the framework of Acceptance Sampling the intention here is to accept (rather than reject) the hypothesis that the $\theta$ lies within a specified (open) interval, which constitutes our $H_1$. Let $X_1, ., X_n$, be a random sample from this distribution. The MLE of $\theta$ under $H_0$ is given by

$$\begin{aligned} \hat{\theta} &= \bar{X},\ \bar{X} \leq \theta_1 \text{ or } \geq \theta_2 \\ &= argmin_{\theta_i}(\bar{X} - \mu_i)^2,\ \text{o.w.} \end{aligned}$$

The exact distribution of the LR test statistic is a mixed distribution and the usual $\chi^2$ large-sample approximation to even the null distribution of LRT does not hold here. Since $\sigma$ is unknown, a scale-invariant test is needed in general. This seems a non-trivial proposition here. Example 4.2 of Berger (1982), describes an example from textile industry where such a testing problem arises as a part of the acceptance sampling procedure and suggests an intuitive test.

For illustrative purposes, first let us assume $\sigma$ to be known, say equal to the upper bound of .05, the value as assumed by Berger. Also, we will be concerned with the last parameter $\theta_9$, the mean percentage change for upholstery fabric. It was of interest to test $H_0 : \theta_9 \geq .02$ or $\leq -.05$ vs $H_1$ : not $H_0$. Berger devoted detailed discussions on the derivation of his test, which we will refer to as Berger's test. Using $\alpha = .05$ we have,

$$\theta_1 = -.05, \theta_2 = .02, \sigma_0 = .05, m = 9; \theta_0 = \frac{\theta_1 + \theta_2}{2} = -.015.$$

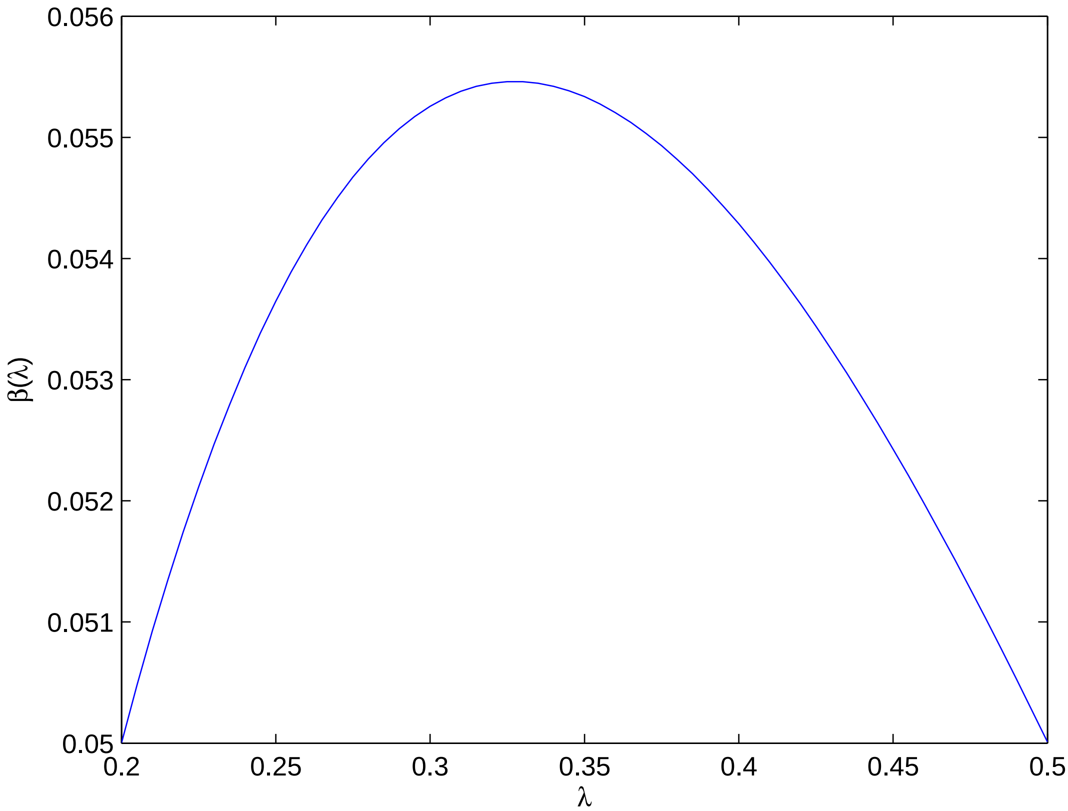


Figure 1: Power Curve for Example 1.

A version (to be referred to as BN1 test) of Berger's symmetrized test for the case of unknown $\sigma$ (see below) modified for this case of known $\sigma$, may be given by

$$\omega: \quad -a \leq \bar{X} \leq a,$$

where $a$ satisfies the size condition

$$.05 = \Phi(\frac{\sqrt{m}(a-\theta_0)}{\sigma_0}) - \Phi(\frac{\sqrt{m}(-a-\theta_0)}{\sigma_0}).$$

$a$ is trivially solved as, $a = \sigma_0 \tau_{.475}/m^{1/2} + \theta_0, \tau_{.475}$ being the upper 47.5 % of the standard normal distribution.
The power of this test is given by

$$\beta_\theta = \Phi(\frac{\sqrt{m}(a-\theta)}{\sigma_0}) - \Phi(\frac{\sqrt{m}(-a-\theta)}{\sigma_0}), \quad -.05 \leq \theta \leq .02$$

Now consider $P^3$ test for this situation, to be referred to as BN2 test. Using (1) and motivated by theorem 1, it is given by

$$\omega: \; a \leq \bar{X} \leq b,$$

where $a$ and $b$ are to be determined using (2).

$$.05 \;=\; \Phi(\frac{\sqrt{m}(b-\theta_1)}{\sigma_o}) - \Phi(\frac{\sqrt{m}(a-\theta_1)}{\sigma_o}) \;=\; \Phi(\frac{\sqrt{m}(b-\theta_2)}{\sigma_o}) - \Phi(\frac{\sqrt{m}(a-\theta_2)}{\sigma_o})$$

$$\Rightarrow b = \theta_1 + \frac{\sigma_0}{\sqrt{m}}\Phi^{-1}[.05 + \Phi(\frac{\sqrt{m}(a-\theta_1)}{\sigma_0})]$$

$$a = \theta_2 + \frac{\sigma_0}{\sqrt{m}}\Phi^{-1}[\Phi(\frac{(\sqrt{m}(\hat{b}-\theta_2)}{\sigma_0}) - .05] \qquad (4)$$

As in example 1, these give $b = f(b)$, for which the Newton-Raphson method can be conveniently employed to find $b$. $a$ can then be determined from any of the equations in (4).

Power of this $P^3$ test is given by

$$\beta_\theta = \Phi(\frac{\sqrt{m}(b-\theta)}{\sigma_0}) - \Phi(\frac{\sqrt{m}(a-\theta)}{\sigma_0}), -.05 \leq \theta \leq .02$$

Certainly, by virtue of theorem 1, this BN2 test being the $P^3$ test here will outperform the preceding Berger-type test. Here again, as in example 1, the $P^3$ test is the UMP test.

Next consider the case when $\sigma$ is unknown. Though $\sigma^2$ is unknown, Berger suggests a test that can be implemented with the assumption of its known upper bound, say $\sigma_0^2$. The test suggested by Berger is a symmetrized test, i.e. with symmetrized test statistic (around $\theta_0$) and symmetric cut-off points $a, -a$, given by :

$$\omega : \; -a \leq m^{1/2}(\bar{X} - \theta_0) \leq a, \; \theta_0 = (\theta_1 + \theta_2)/2.$$

The constant a is determined from the following equation

$$P(-a \leq T_{\delta_0} \leq a) = \alpha,$$

where $T_{\delta_0}$ follows a non-central t distribution with non-centrality parameter $\delta_0 = m^{1/2}(\theta_2 - \theta_1)/2\sigma_0$ and m-1 degrees of freedom, and $\alpha$ is the desired level of significance of the test.

Consider now the P$^3$ test for this problem. From (1), to invoke such a test we first need an unbiased estimator of $\theta$. $\bar{X}$ is of course such an estimator. We next need that the distribution of this statistics is free of the nuisance parameter under $H_0$. Since $\sigma$ is unknown, this implies that a scale-invariant test is needed in general. However, the distribution of $\bar{X}$ even under $H_0$ involves $\sigma$. We adopt the same assumption on $\sigma$ and the same argument of the monotonicity of the non-central t distribution with respect to its non-central parameter as given by Berger. The class of tests thus emerging based on the P$^3$ will be referred to as the P$^3-$ type tests.

It may be interesting, in the spirit of the P$^3$ tests, to study the behavior of the semi-symmetrized version of Berger's test, i.e. the class of tests with test statistics centered as before around $\theta_0$, but the cut-off points being now left arbitrary. We will refer to this test as P3S test. Observe that this class includes Berger's test, by taking $k_1 = -k_2$. The critical region of such a test, is then given by

$$\omega : \; k_1 \leq T = m^{1/2}(\bar{X} - \theta_0)/s \leq k_2, \; \theta_0 = (\theta_1 + \theta_2)/2,$$

where $s$ is the sample standard deviation, and $k_1$ and $k_2$ are determined using (2). The power function of this test is given by,

$$\beta_\theta = P_\theta[\frac{\sqrt{m}(\bar{X} - \theta_0)}{s} < k_2] - P_\theta[\frac{\sqrt{m}(\bar{X} - \theta_0)}{s} < k_1]$$

$$= F_{t'(\delta_\theta)}(k_2) - F_{t'(\delta_\theta)}(k_1), \; \delta_\theta = \frac{\sqrt{m}(\theta - \theta_0)}{\sigma_0} \qquad (5)$$

From (5) we get,

$$k_2 = F^{-1}_{t'(\delta_{\theta_1})}[.05 + F_{t'(\delta_{\theta_1})}(k_1)]$$

$$k_1 = F^{-1}_{t'(\delta_{\theta_2})}[F_{t'(\delta_{\theta_2})}(k_2) - .05]$$

The above is again in the same form as the previous ones leading to the invocation of Newton-Raphson method for a single variable, since we can combine these two equations to yield an equation of the form $k_1 = f(k_1)$. However, it turns out the solution is $k_1 = -k_2 = -a$, i.e. the resulting P3S test is the Berger's test. Hence we will denote Berger's test by P3S also.

Consider now a generalized/modified version of the P3S test, to be referred to as P3M test. In contrast to restricting to the symmetrized test statistic and symmetric cut-off points $-a$ and $a$ as done by Berger, our $P^3$ test from (1) (and motivated by theorem 1) is defined by the critical region

$$\omega: \; \{\sqrt{m}\frac{[\bar{X}-\theta_2]}{s} < k_2\} \cap \{\sqrt{m}\frac{[\bar{X}-\theta_1]}{s} > k_1\},$$

$$\Rightarrow \; \omega: \; \frac{sk_1}{\sqrt{m}} + \theta_1 < \bar{X} < \frac{sk_2}{\sqrt{m}} + \theta_2$$

Then,

$$\alpha = P_{\theta_1}[\frac{\sqrt{m}(\bar{X}-\theta_2)}{s} < k_2] - P_{\theta_1}[\frac{\sqrt{m}}{s}(\bar{X}-\theta_1) < k_1] = F_{t'(\delta_1)}(k_2) - F_{t_{m-1}}(k_1) \tag{6}$$

where, $\delta_1 = [\sqrt{m}(c-d)]/\sigma_0$, is the non-centrality parameter of the t distribution with (m-1) d.f. Similarly,

$$\alpha = F_{t_{m-1}}(k_2) - F_{t'(\delta_2)}(k_1), \tag{7}$$

where $\delta_2 = [\sqrt{m}(\theta_2 - \theta_1)]/\sigma_0 = -\delta_1$.

We solve (6) & (7) to get $k_1$ and $k_2$. Now, form (6)

$$k_1 = F^{-1}_{t_{m-1}}[F_{t'(\delta_1)}(k_2) - \alpha] \tag{8}$$

From (7) and (8) we get,

$$k_2 = F^{-1}_{t_{m-1}}[\alpha + F_{t'(\delta_2)}(k_1)] \equiv g(k_2), \qquad (9),$$

say. (9) appears in a tailor-made form for the implementation of, as for the previous examples, the Newton-Raphson method. Once $k_2$ is determined, $k_1$, can be obtained directly from (8).

The power function of this test is given by,

$$\beta_\theta = F_{t'(\delta_{\theta 2})}(k_2) - F_{t'(\delta_{\theta 1})}(k_1), \theta_1 \leq \theta \leq \theta_2,$$

where $\delta_{\theta i} = [\sqrt{m}(\theta - \theta_i)]/\sigma_0, i = 1, 2$, and $F_{t'(\delta)}(x)$ is the c.d.f of the non-central t-distribution with non-centrality parameter $\delta$ and degrees of freedom (m-1). We will refer to the version of P$^3-$ type incorporating the above modification as the P3M test.

Finally, suppose that we retain the original $\mathrm{P}^3$ test suggested for the known $\sigma$ case, referred to as the BN2 test, but simply replace $\sigma^2$ by its estimator $s^2$. We will refer to this test as the P3O test. It is defined by the critical region

$$\omega: \; A \leq \sqrt{m}\frac{\bar{X}}{s} \leq B,$$

where $A$ and $B$ are obtained from the size conditions as before. Then the power function of this test is given by,

$$\beta_\theta = F_{t'(\delta_\theta)}(B) - F_{t'(\delta_\theta)}(A),$$

where $\delta_\theta = \sqrt{m}\theta/\sigma_0$.

How do the two tests, BN1 and BN2, and the three tests P3S, P3M and P30 compare with each other ? As expected BN2 test, which is in fact the UMP test when $\sigma$, is known outperforms all others. Also, for the PS3 test here, $k_1$ turned out to be equal to $-k_2$, thereby coinciding with Berger's test, and outperforms the other two adhoc tests P3O and P3M. Care is thus needed to choose a good test here. These results are exhibited through Figure 2.

**Example 3. (Testing Bioequivalence.)** Consider two independent normal variables $X_i$, following distributions, $N(\mu_i, \sigma^2), i = 1, 2$. The problem of testing $H_0 : \mu_1 - \mu_2 \leq -\delta$ or $\geq \delta$ vs $H_1$ : not $H_0$ has received quite some attention. Schuirmann (1987) proposes combining *two one-sided tests procedure*, e.g. the corresponding one-sided t tests, for this *equivalence testing* problem, see e.g. Casella and Berger (2002), Exercise 8.47, p.411 and Berger and Hsu (1986). Westlake (1981) refers to a similar setup for the problem of testing Bioequivalence. Letting $\theta = \mu_1 - \mu_2, K_1 = -\delta$ and $K_2 = \delta$, we are back in the setup of a one-parameter problem as in example 2 above. Hence the $P^3$ test suggested there can be adapted for this situation also. The test can then based on the corresponding pooled t statistic and be given by

$$\omega: \; K_1 < \frac{\bar{X}_1 - \bar{X}_2}{[S_p^2(1/n_1 + 1/n_2)]^{1/2}} < K_2.$$

The cutoff points $K_1$ and $K_2$ are to be determined using the resulting noncentral t distributions under $H_i, i = 1, 2$, as before.

**Example 4. (Multidimensional Scatter)** The Generalized Variance (GV) $|\Sigma|$, the determinant of the population dispersion matrix $\Sigma = (\sigma_{ij})$ of a p-dimensional random vector variable $X$ plays an important role (see e.g. SenGupta, 2005a) in multivariate analysis as a scalar measure of multidimensional scatter. In one of its many and diverse applications, the reduction

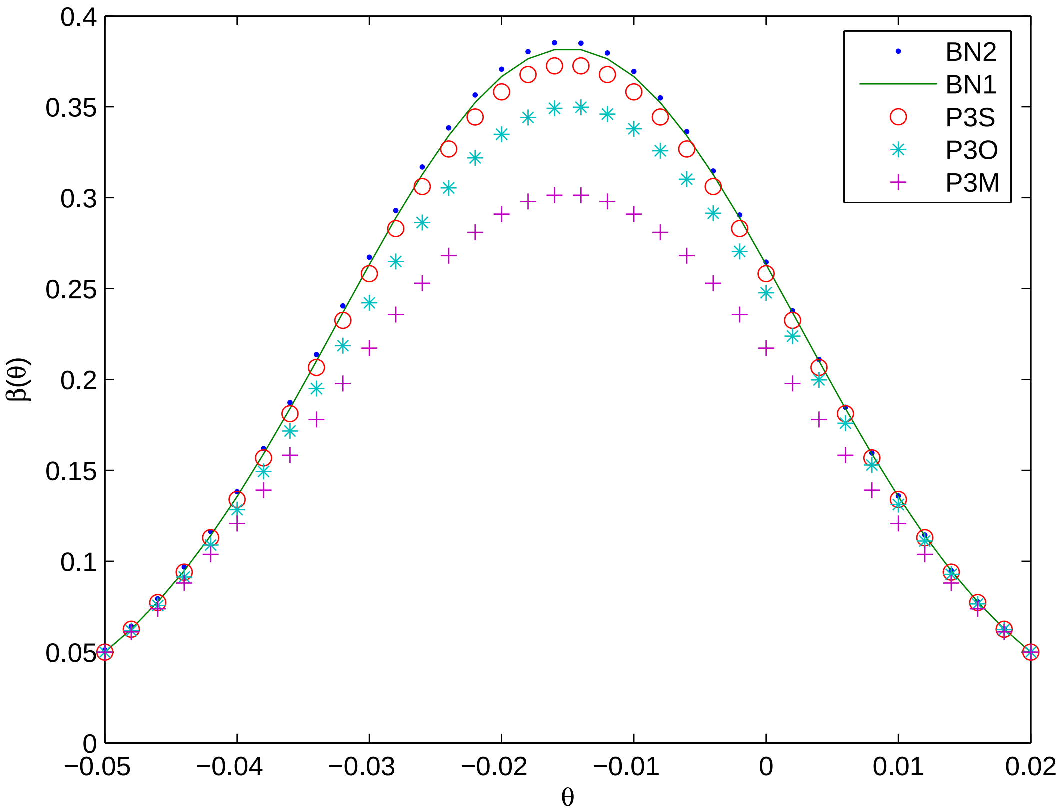


Figure 2: Superimposed Power Curves for Example 2.

of dimensionality by Generalized Canonical Variables (GCVs) (SenGupta, 2005b), a test for $H_0 : |\Sigma| \leq \sigma_0^{2p}$ vs $H_1 : |\Sigma| > \sigma_0^{2p}$, $\sigma_0^2$ known, is of importance. In the context of multivariate statistical process control (SPC) also such a test is quite useful as can be perceived by the increasing popularity of the $|S|$-chart. Let a p-dimensional random vector variable $\mathbf{X}$ follow a standard symmetric multivariate normal (SSMN) distribution, $N_p(\mu\mathbf{1}, \Sigma_\rho)$ so that $\sigma_{ii} = 1, \sigma_{ij} = \rho, i \neq j, i, j = i, , p$, i.e. each component has zero mean and unit variance and they are equicorrelated. This distribution, though not a member of the one-parameter REF, is however, a member of the (1,2) Curved Exponential Family (CEF), i.e. a CEF having a 2-dimensional sufficient statistic for a one-dimensional parameter. Consider now testing $H_0$ vs $H_1$ as stated in this example (for the usual two sided alternative, see SenGupta, 1982, 1987) for the GV of the SSMN distribution. Since $|\Sigma_\rho| = [1+(p-1)\rho](1-\rho)^{(p-1)}$, a concave function of $\rho$, the testing problem becomes that of testing $H_0 : \rho \leq \rho_1$ or $\geq \rho_2$, vs $H_1$ : not $H_0$, where $\rho_i, i = 1, 2$, are known (from $\sigma_0$). This problem then falls under the same general formulation as stated in the beginning of this section 3. The test suggested there can thus be implemented. However, theorem 1 cannot be invoked for this CEF. Note that

$$\hat{\rho} = \Sigma_{u=1}^{n} \Sigma_{i \leq j} X_{iu} X_{ju} / np(p-1)$$

is an unbiased (in fact best "quadratic" unbiased estimator) of $\rho$. Following our general procedure in (1) and (2), the corresponding $P^3$test here is then given by

$$\omega : \ K_1 \leq \hat{\rho} \leq K_2.$$

The exact null and non-null distributions of $\hat{\rho}$ needed for determining $K_1, K_2$ and power values are available from SenGupta (1987) in terms of confluent hypergeometric (or Kummer's) functions. It needs to be mentioned that the tests based on MLE and on the likelihood ratio are quite cumbersome here in contrast to this elegant $P^3$ test.

## 3.2 Non-exponential Families

Consider now non-exponential families. A mixture family, a particular member of the non-exponential families, constitutes an interesting example both from the theoretical as well as from the practical considerations. In general such a family does neither admit of any non-trivial sufficient statistic nor does there exist any test procedure based on invariance or similarity when the nuisance parameter is the mixing parameter. Also, since the mixing parameter may lie on the boundary of the parameter space, the usual large-sample distributional results for the maximum likelihood estimator and the likelihood ratio test do not hold here. Further, we recall (Lehmann, 1986, Additional

Problems 53, p.126) below a worrisome result on our above testing problem for a two-component mixture distribution.

**Result 1.** Let X follow the distribution

$$f(x;\theta) = \theta g_1(x) + (1-\theta)g_2(x),$$

where $g_1$ and $g_2$ are two probability densities with respect to $\mu$, and $0 \leq \theta \leq 1$. For testing $H_0$ vs $H_1$ as defined above, the trivial test $\phi(x) = \alpha$ is UMP at level $\alpha$.

However, in terms of the $P^3$ test the situation is not so bleak - this is demonstrated by the following constructiion.

**Example 5. (Mixture model)** Consider the popular example of the normal mixture model. Let $f_\theta(x) = \theta g_1(x)+(1-\theta)g_2(x), g_i(x)$ is $N(x;\mu_i,\sigma_i^2),\ \ \sigma_i,\mu_i, i = 1, 2,$ known. Then, based on a random sample $X_1, , X_n$, from this population, an unbiased estimator of $\theta$ is given by,

$$\hat{\theta} = [\bar{X} - \mu_2]/[\mu_1 - \mu_2]$$

We can construct the test in the same lines as described earlier. The cut-off points are obtained as follows.

$$\alpha = P_\theta[a < \sum_{i=1}^{n} X_i < b] = F_\theta(b) - F_\theta(a) \tag{10}$$

for $\theta = \theta_1$ and $\theta = \theta_2$, where

$$\begin{aligned} F_\theta(x) &= \sum_{m=0}^{n} \binom{n}{m} \theta^m (1-\theta)^{n-m} [G_1^{*m} * G_2^{*(n-m)}(x)] \\ &= \sum_{m=0}^{n} \binom{n}{m} \theta^m (1-\theta)^{n-m} [\Phi(\frac{x-(m\mu_1+(n-m)\mu_2)}{\sqrt{n}\sigma})] \qquad (11) \end{aligned}$$

which follows by the representation by Behboodian (1972) for the distribution of a symmetric statistic under an underlying mixture distribution.

Solve for a and b using (2) and (3) with $\theta = \theta_1$ and $\theta = \theta_2$. Then the exact power function $\beta_\theta$ can be obtained using these values in (2) and (3) with $\theta_1 < \theta < \theta_2$. We will return to the mixture families in more details when we consider the case of multi-parameter problems below.

# 4 Case of Multiparameter $H_0$

We continue with the two-component mixture distribution as an illuminating example of of a multiparameter non-exponential family to enhance the $P^3$ test where $H_0$ specifies several parameters. In particular we deal with the optimal testing problems for the hypothesis of 'no mixture' in two-component mixture distributions (with or without nuisance parameter(s)) in which the parameter of interest and the mixing proportion both are unknown.

Consider the mixture model with density

$$g(x|p, \theta, \vartheta) = pf(x|\theta, \vartheta) + (1-p)f(x|\theta_0, \vartheta) \tag{12}$$

where $0 \leq p \leq 1$, $\theta \in \Theta$, an interval of the real line; both $p, \theta$ are unknown and $\theta_0$ is a known point of $\Theta$; and $\vartheta$ is an unknown parameter (possibly vector-valued), to be interpreted as a nuisance parameter. The density $f(x|\theta, \vartheta)$ is assumed to be sufficiently 'regular'. We want to test the null hypothesis $H_0$: 'no contamination' against the alternative $H_1$: ' there is contamination'. Under the above setup, the null hypothesis of no contamination translates to the union of three parametric hypotheses : [ $H_{01}$ : $p = 0 \bigcup H_{02}$: $\theta = \theta_0 \bigcup H_{03}$ : $p = 0$ and $\theta = \theta_0$ ].

Durairajan (1980) addressed the problem of testing the hypothesis of no mixture in two-component mixtures of distributions in some generality. When both $p$ and the parameter (say $\theta$) are unknown, he obtained LMPI and LMPS tests for hypotheses involving $p$, treating $\theta$ as the nuisance parameter. The other problem, *i.e.*, testing for $\theta$ with $p$ as nuisance parameter can not, usually, be tackled by the Neyman-Pearson theory since $p$ can not be eliminated by any of the principles of sufficiency, similarity or invariance. The more general case when no mixture can arise by treating both $p$ and $\theta$ as parameters of interest was not considered by Durairajan. We now address this problem below through the $\mathrm{P}^3$ approach.

The main idea of the $\mathrm{P}^3$ approach, stemming from definition 1, is to characterize a single parametric function $\eta \equiv \eta(p, \theta, \theta_0)$ so that $\eta = 0$ holds iff $H_0$ is true. In the context of the mixture distribution given in (12), one may take (SenGupta, 1991) $\eta = p(\theta - \theta_0)$. Clearly several such characterizations are possible.

Consider the setup of a mixture model given in (12). In the following development we focus our attention to obtaining such a $\mathrm{P}^3$ as will lead to a locally optimal procedure by considering what is called (SenGupta, 1991) a 'L-optimal' test. The following definitions are slightly generalized versions, to incorporate nuisance parameters, of those given in SenGupta (1991) (for the case of no nuisance parameter).

Suppose $\theta$ is the parameter of interest and $\vartheta$ is the nuisance parameter, which may be vector-valued.

**Definition 2.** A parametric function $\eta = \eta(p, \vartheta, \theta, \theta_0)$ is said to be a $\mathrm{P}^3$ if $\eta = 0$ iff $p = 0$ or $\theta = \theta_0$ or $p = 0, \theta = \theta_0$ for all $\vartheta$, *i.e.*, iff 'no mixture' holds for all $\vartheta$.

**Definition 3.** A (randomized) test $\psi$ based on an unbiased and/or consistent estimator T of $\eta$ is said to be L-optimal similar unbiased (LSU) or L-optimal invariant unbiased (LIU) or L-optimal $C(\alpha)$ (L-$C(\alpha)$) for testing $H_0 : \eta = 0$ against $H_1 : \eta \neq 0$ according as $\psi$ is LMPSU or LMPIU or $C(\alpha)$ for testing $H_0 : \theta = \theta_0$ against $\theta \neq \theta_0$ for each given $p \in (0, 1)$.

The power of an L-optimal test, therefore, matches that of the corresponding locally best test for the parameter for each given $p$. An L-optimal test is admissible, *i.e.*, there does not exist any other test which performs at least equally well (in terms of power function) at all the points under the alternative hypothesis and actually better than this test at some point(s). Furthermore, this test is consistent.

The approach suggested is that of first deriving a locally optimal test with $p$ assumed to be known. We shall choose $\eta$ so as to ensure that L-optimal test for the hypothesis $H_0' : \eta = 0$ (which is now equivalent to $H_0$) can be constructed based on an unbiased and consistent estimator of $\eta$. The (locally) optimal $\mathrm{P}^3$ test, if it exists and is unique, can then be established to be the L-optimal test.

## 4.1 No Nuisance Parameter

Consider first the case when there is no nuisance parameter $\vartheta$. A set of sufficient conditions on the class of densities such that the associated class of mixtures admits an appropriate $\mathrm{P}^3$ is formulated.

Denote by $\mathcal{G}$ the class of density functions $g$, given by (12), of mixture distributions obtained by restricting $f$ to a certain class $\mathcal{F}$ of the component density functions. The following lemma gives a set of general conditions on $\mathcal{F}$ under which each member of $\mathcal{G}$ admits a $P^3$, along with its appropriate general form and the structure of the corresponding L-optimal test for members of $\mathcal{G}$. Let $X_1, \ldots, X_n$ be $n$ i.i.d. observations drawn from a population with density $g$.

Then a characterization of a general form of the appropriate $\mathrm{P}^3$ and the structure of the corresponding L-optimal test can be given. This approach is formalized in the following

**Theorem 2.** (Lemma 2.1 of Pal and SenGupta, 2000) Let $\mathcal{F} = \{f\}$ be a class with any member $f$ of this class being a one-parameter density function, with respect to an appropriate $\sigma$-finite measure $\mu$, of a possibly multidimensional random variable $X$. Assume that $f$ satisfies the following conditions :

(C1) The parameter $\theta$ belongs to the parameter space $\Theta$ which is a non-degenerate open, semi-open or closed interval of the real line containing $\theta_0$ as an interior or a boundary point.

(C2) The support $\mathcal{X}$ is independent of the parameter $\theta$.

(C3) The (one- or two-sided) derivative $\partial f(x|\theta)/\partial\theta|_{\theta_0}$ exists and is finite for all $x \in \mathcal{X}$.

(C4)
$$E_{f_\theta}\left(\left.\frac{\partial \log f(X|\theta)}{\partial\theta}\right|_{\theta_0}\right)^2 < \infty \text{ for all } \theta \in \Theta.$$

(C5)
$$E_{f_\theta}\left(\left.\frac{\partial \log f(X|\theta)}{\partial\theta}\right|_{\theta_0}\right) \stackrel{\text{def}}{=} \gamma(\theta,\theta_0) = 0 \text{ if and only if } \theta = \theta_0.$$

Then $\eta = p\gamma(\theta,\theta_0)$ or any monotone function of it may serve as an appropriate $P^3$. A test appropriately based on an unbiased and consistent estimator $T$ of $\eta$ will be $L$-optimal for testing $H_0' : \eta = 0$ against either of the one-sided alternatives, if $T$ coincides, a.e., with the average score statistic $\frac{1}{n}\sum_{i=1}^{n} \left.\frac{\partial \log f(X_i|\theta)}{\partial\theta}\right|_{\theta=\theta_0}$. Furthermore, this test is consistent.

A large class of (component) distributions is covered by the above lemma. In particular, we exhibit the following examples.

**Examples 6. (REF components.)** Consider the one-parameter REF given by the density (w.r.t. a $\sigma$-finite measure $\mu$) in the canonical form

$$h(x|\theta) = \exp[\theta W(x) - A(\theta)].$$

Then $h \in \mathcal{F}$ and $\eta$ in this case is given by $p(A'(\theta) - A'(\theta_0))$.

**Examples 7. (General components.)** Examples of some other common component distributions $f(x|\theta)$ are: lognormal, inverse Gaussian with location or dispersion parameter, bivariate exponential conditionals (Arnold and Strauss, 1988) with dependency parameter, bivariate inverse Gaussian with dependency parameter, folded normal, Weibull with shape parameter, etc., see Pal and SenGupta (2000).

**Remark.** There are situations in practice where the nuisance parameter appears only under the alternative. In that case a $P^3$ may be constructed and estimated as done above for the case of no nuisance parameter.

**Example 8. (Test for Isotropy.)** An example of the preceding remark is provided by the test for isotropy (or uniformity) on the circle against the alternative of an underlying distribution from the symmetric wrapped stable-circular uniform (SWS-CU)mixture family. The nuisance parameter in point is the index parameter or the characteristic exponent of the stable distribution.

The SWS-CU mixture family for a circular random variable is defined by the probability density functions

$$g(x|a,\rho,\mu_0,p) = \frac{p}{2\pi}\left\{1+2\sum_{r=1}^{\infty}\rho^{r^a}\cos r(x-\mu_0)\right\}+\frac{1-p}{2\pi}. \qquad (13)$$

Here $H_0$ : X has a circular uniform distribution, is equivalent to $(p=0)\cup(\rho=0)$ and a $\mathrm{P}^3$ is given by $\eta = p\rho$. The two nuisance parameters are thus $\mu_0$ and $a$. $\mu_0$ may be considered to be a "location parameter." First, let us assume that $\mu_0$ is known. Then, it turns out that $\eta$ is estimated unbiasedly by $\bar{C} \equiv \Sigma_{i=1}^n \cos x_i/n$, which is infact the LMP test statistic for $H_0$ when $p>0$ is known. Thus the $\mathrm{P}^3$ test here is robust, whatever be $a$, L-optimal test. For further details see SenGupta and Pal (2001).

## 4.2 Integrated likelihood approach and the optimal $P^3$ test

Let $L(\theta,p)$ denote the likelihood function of $\theta$ and $p$ and let $\pi(p)$ be the density of a prior distribution of $p$ (with respect to Lebesgue measure on [0,1]). The integrated likelihood of $\theta$, $\tilde{L}(\theta)$, is then obtained by integrating $L(\theta,p)$ with respect to $\pi(p)dp$ (see Aitkin and Rubin, 1985). We then have

**Theorem 3** (Theorem 3.1 of Pal and SenGupta, 2000) The locally most powerful test for $H_0 : \theta = \theta_0$ against either of the one-sided alternatives, based on the integrated likelihood $\tilde{L}(\theta)$, is equivalent to the optimal $\mathrm{P}^3$ test.

## 4.3 Nuisance Parameters Present

Nuisance parameters may appear in the context of $\mathrm{P}^3$ as ones involved in functions of several parameters defining some of the $H_{0i}$s. By transformations, each such function may be represented by a new parameter and further result in parameters $\nu$ no longer involved in the $\mathrm{P}^3$. We will refer to such nuisance parameters $\nu$ as *induced* nuisance parameters. There may also be parameters which are not involved in $\mathrm{P}^3$ before any transformation as referred to above. Such parameters will be referred to as ordinary or simply nuisance parameters. In general we will denote a nuisance parameter, possibly vector-valued, by $\vartheta$. The following examples will illustrate these definitions.

**Examples 9.** We present three general mixture distributions below.
Example 9.1. $g(x|p,\mu,\sigma^2) = \dfrac{p}{\sigma}f\left(\dfrac{x-\mu}{\sigma}\right)+\dfrac{1-p}{\sigma}f\left(\dfrac{x-\mu_0}{\sigma}\right)$, $\mu_0$ known.
A $\mathrm{P}^3$ is given by $\eta_1 = p(\mu_1-\mu_0)$. Here $\sigma$ is an ordinary nuisance parameter.
Example 9.2. $g(x|p,\mu_1,\mu_2) = pf(x-\mu_1)+f(x-\mu_2)$.

A $\mathrm{P}^3$ is given by $\eta_2 = p(\mu_1 - \mu_2)$. However, the second factor of $\eta_2$ involves two parameters. A simple transformation $\mu_1 = \mu_2 + \theta, \mu_2 = \mu$ shows that now a $\mathrm{P}^3$ can be written in terms of the product of the two single parameters p and *theta*, $\eta_2' = p\theta$. Then $\mu \equiv \mu_2$ can be seen to become, what we refer to as, an induced nuisance (location) parameter. It often turns out that to obtain a L-optimal (LMP S,I,$C_\alpha$, etc.) test, it will greatly facilitate to adopt a $\mathrm{P}^3$ with a(some) induced nuisance parameter(s).

Example 9.3. $g(x|p,\mu,\sigma^2) = \dfrac{p}{\sigma} f\left(\dfrac{x-\mu_1}{\sigma}\right) + \dfrac{1-p}{\sigma} f\left(\dfrac{x-\mu_2}{\sigma}\right)$.

A $\mathrm{P}^3$ can be given by $\eta_3 = p(\mu_1 - \mu_2)$. Transform the parameters by letting $\mu_1 = \mu + \delta, \mu_2 = \mu$ and denote $\delta/\sigma = \theta$. We can now define a $\mathrm{P}^3$ as $\eta_3' = p\theta$. Then, here $\mu$ and $\sigma$ are induced and ordinary nuisance parameters respectively.

**Method of Transformation of Parameters:** Observe that transformation of parameters can be a powerful tool to obtain appropriate $\mathrm{P}^3$s which lead to L-optimal tests in various classes. In example 9.2 above, a simple such transformation has been employed. This shows that under $H_0$ the nuisance parameter is a location parameter. Hence, for such a p.d.f. $g(.)$ as admitting a boundedly complete sufficient statistics for its location parameter $\mu$ under $H_0 : \theta = 0$ (and some mild conditions, see Sp$\phi$jtvoll, 1968), one can construct locally most powerful similar test for $H_0$. It is well known that the class of such p.d.f.s is quite a rich one.

**Example 10. (Example 9.2 continued.)** Specializing to $f(.) \equiv \phi(.)$, the normal p.d.f., we have $\eta_1 = p(1-p)(\mu_1 - \mu_2)^2$. In terms of the reparameterized model, we have $\eta_1 = p\theta$ with an unbiased estimator of it being given by $T_1 = S^2 - 1$, where $S^2 = \dfrac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Further, it can be shown using Basu's theorem that an unconditional LMPSU test obtains here and its critical function is given by:

$$\psi_0^{(1)}(x) \;=\; 1 \;\text{ if }\; S^2 \;>\; c.$$

Note that the above test is also LMPIU and its cut-off points can be found easily using the $\chi^2_{n-1}$ distribution.

We now consider a distribution where random variable $\mathbf{X}$ is a vector, but the parameter of interest $\theta$ is a scalar and the nuisance parameter $\vartheta$ is possibly vector-valued. The mixture of SSMN distributions (see example 4) is an example of such a distribution. It is given by

$$\begin{aligned} g_{(p)}(x|m,\sigma^2,\rho) &= p(2\pi\sigma^2)^{-k/2}|\mathbf{\Sigma}_\rho|^{-1/2}\exp\left\{-(x-\mathbf{M})'\mathbf{\Sigma}_\rho^{-1}(x-\mathbf{M})/2\sigma^2\right\} \\ &+ q(2\pi\sigma^2)^{-k/2}\exp\left\{-(x-\mathbf{M})'(x-\mathbf{M})/2\sigma^2\right\} \end{aligned}$$

Then $g(.)$ is the density of a mixture of two $k$-variate SMN distributions with mean vector $\mathbf{M} = m\mathbf{1}, \sigma^2$ being the common variance and $\rho$, the equicorrelation coefficient defining the correlation matrix $\Sigma_\rho$. Here $H_0$ : no mixture, corresponds to having a $\mathrm{P}^3$ defined by $\eta = p\rho$. Optimal similar tests for known $p$ have been derived in SenGupta and Pal (1993) ( see this also for examples of real-life applications) and therein the approach of $\mathrm{P}^3$ for unknown $p$ to derive L-optimal similar tests has also been suggested.

# 5 Multivariate Multiparameter problems

We now enhance our $\mathrm{P}^3$ approach to the encompass the generalized situation of multivariate multiparameter models possibly involving nuisance parameters also. We will restrict our discussions here to the case of such mixture models as generalizations to the univariate setups dealt with above.

Consider a multivariate mixture density with possibly vector parameters $\boldsymbol{\theta_i}, i = 1, 2$, and a nuisance parameter $\vartheta$, of the form

$$g(\boldsymbol{x}|p, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \vartheta) = p\ f(\boldsymbol{x}|\boldsymbol{\theta_1}, \vartheta) + (1-p)\ f(\boldsymbol{x}|\boldsymbol{\theta_2}, \vartheta). \tag{14}$$

In some cases, $\boldsymbol{\theta_2}$ may be known, say specified as $\boldsymbol{\theta_0}$. We will propose two approaches here, one based on LMMPU tests and the other based on Roy's UI principle, to be referred to as UI-IU tests.

## 5.1 LMMPU based tests

Proceeding as for the univariate case, let us first consider the case when $p$ is known. Suppose further, for simplicity, that $\boldsymbol{\theta_2} = \boldsymbol{\theta_0}$, known. We seek a multiparameter optimal test here. This can be taken as Locally Most Mean Powerful Unbiased (LMMPU) test when there is no nuisance parameter or LMMPU Similar (LMMPUS) test in the presence of a nuisance parameter $\vartheta$, as given by theorems 2 and 3 respectively of SenGupta and Vermeire (1986)- the reader is referred to this paper for basic definitions and derivations also. For the original problem of testing for no mixture, i.e. with $p$ unknown, now an appropriate $\mathrm{P}^3$ may be defined which may also be "indicated" by the LMMPU(S) test statistic. We illustrate this approach by an example below.

**Example 11. (Example 9.1 continued.)** Consider the multivariate (k-dimensional) version of example 9.1 specializing to the multivariate normal distribution with $\Sigma = I$ and $\boldsymbol{\theta_0} = \mathbf{0}$. Then the underlying mixture density is given by,

$$g(x|p, \boldsymbol{\theta}, \mathbf{0}) = pN_k(x|\boldsymbol{\theta}, I) + (1-p)N_k(x|\mathbf{0}, I)$$

Then for $p$ known, we have $H_0 : \boldsymbol{\theta} = \mathbf{0}$. Let us adopt for our mixture setup the LMMPU test (using $n_i = n \forall i$ and $\Sigma = I$) derived (see Result 5 of SenGupta and Vermeire, 1986) for the component distribution $N_k(x|\boldsymbol{\theta}, I)$ for the same $H_0$. We have seen earlier that this has been usually the case for the univariate mixture problems with known $p$. (It remains to be established that this test is indeed the LMMPU test under the mixture setup with known $p$ - this problem will be taken up elsewhere.) The test statistic reduces to $T = n\Sigma_{i=1}^{k}\bar{X}_i^2$.

For $p$ unknown, an obvious choice for a P$^3$ is $\eta_0 = p\boldsymbol{\theta}$. However, a choice of P$^3$ which is most convenient (as suggested by the test statistic) for us here is given by $\eta = [p + (n-1)p^2]\ \Sigma_{i=1}^{k}\theta_i^2$. Then a little algebra shows that an unbiased estimator of $\eta$ is given by $T - k$, and hence the P$^3$ test coincides with the LMMPU test based on T. Thus this test is a multivariate mutiparameter 'optimal' test.

## 5.2 Union-Intersection - Intersection-Union based tests

For known p, observe that the null hypothesis of no mixture reduces to the vector (multi) parameter hypothesis $H_0 : \boldsymbol{\theta_1} = \boldsymbol{\theta_2}$, which is representable in the equivalent form $\underset{l \neq 0}{\cap} H_{0l}$, where $H_{0l}$ denotes a scalar parametric function (possibly a scalar function of the corresponding components in the parameters $\boldsymbol{\theta_i}$, i = 1,2,.) Assume that an optimal test is available for testing the scalar hypothesis $H_{0l}$ for each $l$. We are then back in the classical framework for the application of the $\cup - \cap$ principle of Roy (1953) to get a UI test for $H_0$. When $p$ is unknown, the IU test obtained from the P$^3$ approach may form the required optimal test for the scalar hypothesis $H_{0l}$. We may then apply the $\cup - \cap$ principle to yield, what we will call, the UI-IU test for $H_0$. This approach is very useful for many multivariate testing problems, in particular those involving the multivariate normal distribution. Here we illustrate it by considering the multivariate extension to example 9.2, which involves nuisance (induced) parameters also.

**Example 12. (Example 9.2 continued.)** Consider the multivariate analogue of the distribution in example 9.2. Let X be a random vector variable following the distribution

$$g(x|p, \boldsymbol{\mu_1}, \boldsymbol{\mu_2}) = pN_k(x|\boldsymbol{\mu_1}, I) + (1-p)N_k(x|\boldsymbol{\mu_2}, I),$$

or equivalently,

$$g(x|p, \boldsymbol{\theta}, \boldsymbol{\mu}) = pN_k(x|\boldsymbol{\mu} + \boldsymbol{\theta}, I) + (1-p)N_k(x|\boldsymbol{\mu}, I).$$

A P$^3$ for no mixture is given by the vector $\boldsymbol{\eta} = p\boldsymbol{\theta}$. A more convenient representation of it is in terms of a scalar $\eta' \equiv p(1-p)(\boldsymbol{\theta}'\boldsymbol{\theta})$.

Following Roy's UI approach, define $Y_i = \boldsymbol{l}'X_i,\ i = 1, 2, \ldots, n$, where $\boldsymbol{l} \in \boldsymbol{L}$, the collection of all $k$-vectors $\boldsymbol{l}$ of unit norm. Then it is easily seen

(Pal and SenGupta, 2006) that the UI approach leads to the test which rejects $H_0$ if $T_2' = \sup_{l \in L} \boldsymbol{l}' A \boldsymbol{l} > c$ *i.e.*, if $\lambda_M(A) > c$, where $A = \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})'$ and $\lambda_M(A)$ is the largest characteristic root of $A$. The cut-off point $c$ can be found from Sugiyama (1972). This test is thus an example of UI-IU test described above.

# 6 Concluding Remarks

We have considered in this paper a testing problem, referred to as the Intersection-Union testing problem, which may be looked upon as the problem complementary to the one considered by Roy and which led to his Union-Intersection principle. For such problems the usual likelihood ratio or Wald type tests are not convenient to implement and their usual asymptotic distributional results are in general not valid. It has been shown here that many intersection-union testing problems can be conveniently formulated in terms of testing a single parametric function, denoted here by a Pivotal Parametric Product, or notationally by $\mathrm{P}^3$. The choice of a $\mathrm{P}^3$ is motivated by optimal tests. It is demonstrated that such an approach leads to UMP tests in regular exponential families and L-optimal tests in several non-exponential families. Variety of examples were presented. Further it was seen that $\mathrm{P}^3-$tests can be conveniently constructed even in the presence of nuisance parameters. Extensions to multivariate multiparameter situations naturally led to the construction of Union-Intersection - Intersection Union tests. There are many intersection-union testing problems in real-life and it is expected that the $P^3$ approach presented here may be enhanced to solve such problems.

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