l. J. Statist., **29** (1), 1987, 49–59

TESTS FOR EQUICORRELATION COEFFICIENT OF A STANDARD SYMMETRIC MULTIVARIATE NORMAL DISTRIBUTION

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Summary

Sampson (1976, 1978) has considered applications of the standard mmetric multivariate normal (SSMN) distribution and the estimation its equi-correlation coefficient, ρ. Tests for ρ are considered here. he likelihood ratio test suffers from several theoretical and practical ortcomings. We propose the locally most powerful (LMP) test which globally (one-sided) unbiased, very simple to compute and is based n the best natural unbiased estimator of ρ. Exact null and non-null stributions of the test statistic are presented and percentage points e given. Statistical curvature (Efron, 1975) indicates that its perforance improves with mk (sample size × dimension) while exact power omputations show that even for reasonably small values of mk the erformance is quite encouraging. Recalling Brown's (1971) cautions e establish by local comparison with the LMP similar test for ρ in the MN (Rao, 1973) distribution, that here the additional information on he mean and variance is quite worthwhile.

ey *words:* Brown's rules; Kummer's function; locally most powerful similar test; andard symmetric multivariate normal distribution; statistical (Efron) curvature.

1. Introduction

A random vector follows a symmetric multivariate normal (SMN) distribution (Rao, 1973) if the components have equal means, equal variances and equal covariances—the common correlation coefficient, ρ, between any two components is termed the intraclass, equi-, uniform or familial correlation. Since they arise naturally in psychology, education, genetics etc., such models have recieved considerable attention. A random vector **Y** will be said to follow a standard symmetric multivariate normal (SSMN) distribution if it follows a SMN distribution and additionally the components have zero means and unit variances. Though the literature on the SMN distribution is quite extensive, no test for ρ has been proposed for SSMN distribution. However, such distributions can occur naturally in various ways, e.g., see Sampson (1978). Further, in many practical problems it is necessary to standardize the variables. The sample means and variances are usually employed for such standardizations and then the resulting variables behave asymptotically, by Slutsky's theorem, as standardized random variables. Such standardizations are always made and play important roles in the techniques for reduction of dimensionality, e.g. in canonical variables (Anderson, 1984) and generalized canonical variables analyses (SenGupta, 1981, 1983). Finally, if the means and variances are known and hence can be considered as zeros and ones respectively, then the SSMN distribution may be considered without loss of generality. This distribution is also quite interesting from several theoretical aspects. Firstly, it provides a practical example of a curved exponential family (Efron, 1975) and illustrates some associated difficulties and techniques in inference on them. Secondly, the components of **Y** are exchangeable and constitute an example of the mean-zero invariant model of Anderssen (1976). Thirdly, it indicates optimal methods for the construction of simple estimators of ρ in contrast to the iterative or sequential procedures leading to more complicated estimators (Sampson, 1976). Finally, it is demonstrated that, unlike the likelihood ratio tests (LRTs), small sample optimal test for the correlation coefficient may be conveniently derived in such models. Anderson (1963) noted '. . . the theory in the case of correlation matrices is much more complicated than for covariance matrices and no general result could be given in a simple form . . .'. However, for the above important and special structure of the correlation matrix, interesting and elegant results can be derived.

Noting that there does not exist an uniformly most powerful (UMP) test for ρ, the locally most powerful (LMP) globally unbiased test is presented. The test statistic $\bar{\rho}$ is very simple and turns out to be also the best (minimum variance) natural unbiased estimator (BNUE) of ρ. The exact distribution of $\bar{\rho}$, historically, is related to a problem attempted by Pearson *et al.* (1932, p. 341). The exact and asymptotic distributions of $\bar{\rho}$ are derived and some percentage points are given.

Observing that the SSMN distribution is a member of the one-parameter curved exponential family, its statistical curvature is computed. Exact power computations indicate that the LMP test performs reasonably well even for "large" curvatures. Brown (1971) has pointed out that additional information need not increase the efficiency of a test. So, as a final justification of the LMP unbiased test, we demonstrate that it compares favorably with the LMP similar unbiased test for ρ in the SMN distribution.

2. Tests for Equi-correlation Coefficient ρ

Let **Y** be a $k \times 1$ random vector which is normally distributed with mean **0** and covariance matrix $\mathbf{\Sigma}_\rho = (\mathbf{I} - \rho)\mathbf{I} + \rho\mathbf{E}$, **I** and **E** being the identity matrix and the matrix with all elements equal to unity respectively. Then

$$\mathbf{\Sigma}_\rho^{-1} = (1-\rho)^{-1}\mathbf{I} - \rho[(1-\rho)\{1+(k-1)\rho\}]^{-1}\mathbf{E} = (c_{ij}(\rho))$$

where

$$c_{ii}(\rho) = \{1+(k-2)\rho\}/[(1-\rho)\{1+(k-1)\rho\}].$$
$$c_{ij}(\rho) = -\rho/[(1-\rho)\{1+(k-1)\rho\}], \quad i \neq j.$$

Hence the density function for non-singular $\mathbf{\Sigma}_\rho$ can be written as

$$f(\mathbf{Y};\rho) = \frac{1}{(2\pi)^{k/2}|\mathbf{\Sigma}_\rho|^{1/2}} \exp\left[-\frac{1}{2}\left\{\frac{(\Sigma y_i^2)}{(1-\rho)} + \frac{(\Sigma y_i)^2(-\rho)}{(1+(k-1)\rho)(1-\rho)}\right\}\right]$$
$$-\infty < y_i < \infty, \quad i = 1, \ldots, k, \quad -1/(k-1) < \rho < 1. \qquad (2.1)$$

The above representation is particularly useful because it shows that the density function constitutes a member of the curved exponential family and (which implies here) there does not exist any one-dimensional sufficient statistic for ρ. Also, $\Sigma(y_i - \bar{y})^2$ and $\bar{y} = \Sigma y_i/k$ are independent (Rao, 1973, p. 197). Suppose $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ is a random sample from $f(\mathbf{Y},\rho)$. Let

$$V_1 = k\sum_{j=1}^{m}(\bar{Y}_{.j})^2, \quad V_2 = \sum_{j=1}^{m}\sum_{i=1}^{k}(Y_{ij} - \bar{Y}_{.j})^2,$$

where $\bar{Y}_{.j} = \Sigma_{i=1}^{k} Y_{ij}/k$. Note that V_1, V_2 are jointly (minimal) sufficient statistics for ρ. Further, $V_1/[1+(k-1)\rho]$ and $V_2/(1-\rho)$ are independently distributed as χ^2 with m and $m(k-1)$ d.f. respectively. Also, by the reduction to the canonical form (Rao, 1973) for SSMN distribution, there exists an orthogonal transformation $\mathbf{Y} \rightarrow \mathbf{Z}$, such that $\Sigma Y_i^2 = \Sigma Z_i^2$ and $Z_1 = \Sigma Y_i/\sqrt{k}$ where Z_i, $i = 1, \ldots, k$ are all independent. It follows that Z_1 is normally distributed with $E(Z_1) = 0$, $\text{Var}(Z_1) = 1 + (k-1)\rho$ and Z_j are normally distributed with $E(Z_j) = 0$, $\text{Var}(Z_j) = 1 - \rho$, $j = 2, \ldots, k$. We want to test $H_0: \rho = \rho_0$ against $H_1: \rho < (>) \rho_0$ or against $H_2: \rho \neq \rho_0$.

2.1. Likelihood Ratio Test

For testing H_0 against H_2 the LRT is derived below and is non-trivial even for $m = 1$. From (2.1) above the maximum likelihood equation is given by,

$$g(\rho) = m(k-1)k\rho(1-\rho)\{1+(k-1)\rho\} - \sum_i\sum_j y_{ij}^2\{1+(k-1)\rho\}^2$$
$$+ \sum_j\left(\sum_i y_{ij}\right)^2\{1+(k-1)\rho^2\} = 0,$$

where $\mathbf{y}_j' = (y_{1j}, \ldots, y_{kj})$, $j = 1, \ldots, m$. It can be shown (e.g., see SenGupta, 1982, Section 2.1) that there exists a maximum likelihood estimator (MLE) of ρ, say $\hat{\rho}$, with $-1/(k-1) < \hat{\rho} < 1$.

Theorem 1. *Consider a k-variate normal population with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}_\rho$. Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ be an independent random sample from the distribution* (2.1). *Then the likelihood ratio test for testing $H_0: \rho = \rho_0$ against the alternative $H_2: \rho \neq \rho_0$, is given by*

$$\textit{Reject } H_0 \textit{ iff } \lambda = [|\boldsymbol{\Sigma}_{\hat{\rho}}|/|\boldsymbol{\Sigma}_{\rho_0}|]^{m/2} \exp\left[-\tfrac{1}{2}\{(c_{11}(\rho_0) - c_{11}(\hat{\rho}))(V_2 + V_1) - (c_{12}(\rho_0) - c_{12}(\hat{\rho}))(V_2 - (k-1)V_1)\}\right] < K,$$

where $\hat{\rho}$ is the MLE *of ρ and K is a constant to be determined so that the level of the test meets the specified value.*

Under H_0, for large m, $-2 \ln \lambda$ is distributed as χ^2 with 1 d.f. It is clear that the LRT for ρ is cumbersome and the exact distributions of $\hat{\rho}$ and the LR statistics are nearly intractable. For one-sided alternatives, it will be even more complicated to study the small-sample behaviours of the LRT.

Owing to the above difficulties, we present below an alternative test statistic which is very simple to compute and which possesses some desirable optimal properties also.

2.2. LMPU Test For ρ

Note that there does not exist an UMP test for ρ.

Theorem 2. *The* LMP *test for $H_0: \rho = 0$ against $H_1: \rho > (<) 0$ is given by*

$$\textit{Reject } H_0 \textit{ iff } \tilde{\rho} = \sum_{\substack{i \\ i \neq i'}} \sum_{j} y_{ij} y_{i'j} / mk(k-1) > (< c')c, \qquad (2.2)$$

where $c(c')$ is determined to give the desired level of the test. The test is globally unbiased against one-sided alternatives.

Proof. Let $\dot{l}_\rho$ denote the first derivative of the log-likelihood function with respect to ρ. The LMP test for testing $H_0: \rho = 0$ against $H_1: \rho > (<) 0$ is given by

$$\text{Reject } H_0 \text{ iff, } \dot{l}_0 > (< k')k, \text{ i.e., } -\{\psi_1(0) + \psi_2(0)\} > (< c')c$$

where $\psi_1(\rho) = d\,|\boldsymbol{\Sigma}_\rho|/d\rho = -k(k-1)\rho(1-\rho)^{k-2}$,

$$\begin{aligned} \psi_2(\rho) &= d(\textstyle\sum \mathbf{Y}_j' \boldsymbol{\Sigma}_\rho^{-1} \mathbf{Y}_j)/d\rho \\ &= \sum_j \Big[\{1 + (k-1)\rho\}^2 \Big(\sum_i y_{ij}^2\Big) \\ &\qquad + \{-1 - (k-1)\rho^2\} \Big(\sum_i y_{ij}\Big)^2 \Big] \Big/ \{(1 + (k-1)\rho)(1-\rho)\}^2 \end{aligned}$$

and $c(c')$ is determined to provide the desired level of the test. Some simplifications yield (2.2).

For each j, use an orthogonal transformation (similar to that in Section 2) $\mathbf{Y}_j \to \mathbf{Z}_j$, $j = 1, \ldots, m$. Then

$$\bar{\rho} = \sum_j \left\{ \left(\sum_i y_{ij} \right)^2 - \left(\sum_i y_{ij}^2 \right) \right\} \Big/ mk(k-1)$$

$$= \sum_j \left\{ kz_{1j}^2 - \sum_i z_{ij}^2 \right\} \Big/ mk(k-1) \tag{2.3}$$

Hence, $\bar{\rho}$ is distributed as $\{(1 + (k-1)\rho)(k-1)\chi_m^2 - (1-\rho)\chi_{m(k-1)}^2\}/mk(k-1)$ where χ_m^2 and $\chi_{m(k-1)}^2$ are independent χ^2 variables with m and $m(k-1)$ d.f. respectively. Under H_0, the distribution of $\bar{\rho}$ is the same as above, with $\rho = 0$.

Consider $H_1 : \rho > 0$. To prove the unbiasedness in this case, it suffices to show that

$$(k-1)\chi_m^2 - \chi_{m(k-1)}^2 < (1 + (k-1)\rho)(k-1)\chi_m^2 - (1-\rho)\chi_{m(k-1)}^2$$

which is clearly true. The proof for $H_1 : \rho < 0$ follows similarly.

Observe that due to the equicorrelated structure of our model in (2.1), for the jth observation $\mathbf{Y}_j$, $j = 1, \ldots, m$,

$$E(Y_{ij} Y_{i'j}) = \rho, \quad i \neq i', \quad i, i' = 1, \ldots, k.$$

Hence

$$E\left[\sum_{i \neq i'} Y_{ij} Y_{i'j} / k(k-1) \right] = \rho.$$

Based on the entire sample, it is then natural to consider unbiased estimators, to be called natural unbiased estimators (NUEs), of the form

$$\sum_j a_j \left[\sum_{i \neq i'} Y_{ij} Y_{i'j} / k(k-1) \right].$$

Lemma 1. *The test statistic, $\bar{\rho}$, of Theorem* 2 *is* BNUE *of ρ.*

Proof. *Note that,* $\bar{\rho} = \{(k-1)V_1 - V_2\}/mk(k-1)$.

So $\bar{\rho}$ is a function of the minimal sufficient statistic (V_1, V_2) and Lemma 1 follows by an application of the Rao–Blackwell theorem. [Observe that besides $\bar{\rho}$, no other BNUE is a function of (V_1, V_2).]

Table 1 presents some percentage points for the distribution of $\bar{\rho}$, when $\rho = 0$. More detailed tables are given in Gokhale and SenGupta, 1986. Note that, $-\infty < \rho < \infty$ and it may be desirable to base the test on a bounded statistic, say, $\bar{\bar{\rho}}$, where

$$\bar{\bar{\rho}} = \begin{cases} -1/(k-1), & \bar{\rho} \leq -1/(k-1) \\ \bar{\rho}, & -1/(k-1) < \bar{\rho} < 1 \\ 1, & \bar{\rho} \geq 1 \end{cases}$$

However, it is reassuring to find from Table 1 that only one percentage

TABLE 1

Percentage points of $\bar{\rho}$

	$m = 5$			$m = 10$			$m = 15$		
k	$\alpha = 0{\cdot}10$	0·05	0·01	$\alpha = 0{\cdot}10$	0·05	0·01	$\alpha = 0{\cdot}10$	0·05	0·01
2	0·54220	0·73200	1·14180	0·39315	0·51819	0·77570	0·32403	0·42340	0·62330
3	0·32866	0·45390	0·72560	0·23508	0·31630	0·48520	0·19247	0·25626	0·38620
4	0·23636	0·32977	0·53280	0·16832	0·22846	0·35400	0·13750	0·18450	0·28080
5	0·18466	0·25910	0·42120	0·13122	0·17897	0·27890	0·10708	0·14430	0·22080
6	0·15155	0·21345	0·34830	0·10756	0·14716	0·23020	0·08772	0·11853	0·18198
7	0·12852	0·18149	0·29700	0·09114	0·12497	0·19600	0·07430	0·10059	0·15480
8	0·11158	0·15786	0·25880	0·07908	0·10860	0·17060	0·06445	0·08738	0·13469
9	0·09858	0·13968	0·22940	0·06985	0·09603	0·15110	0·05691	0·07723	0·11920
10	0·08830	0·12526	0·20590	0·06254	0·08607	0·13558	0·05095	0·06920	0·10690

point, and that too for the smallest m, k and α, exceeded 1. Thus the probability that $\bar{\rho}$ will exceed 1 is quite small.

Kallenberg (1981) concludes that the shortcomings of the LMP test in a curved exponential family, under suitable conditions, tend to zero at the rate $m^{-1}|\log \alpha_m|^{3/2}$ where $\alpha_m \in (0, 1)$ is the level of significance. This result holds good for our test based on $\bar{\rho}$.

3. Exact Null and Non-null Distributions of $\bar{\rho}$

The exact distribution of $\bar{\rho}$ is that of the weighted difference of two independent χ^2 variables with different weights and arbitrary d.f.s. Now, historically, this problem was attempted by Pearson *et al.* (1932, p. 341) and later solved only partly for the very special case of equal weights and equal d.f.s. by Pachares (1952). It was also encountered by Anderson (1963, p. 139) who conjectured a possible approximation. The distribution is presented below in terms of Kummer's function.

Let $U(a, b; z)$ give independent solutions to the confluent hypergeometric differential equation of Kummer:

$$zd^2w/dz^2 + (b - z)dw/dz - aw = 0.$$

Then in terms of an integral

$$U(a, b; z) = [\Gamma(a)]^{-1}\int_0^\infty \exp(-zV)V^{a-1}(1+V)^{b-a-1}dV, \quad a>0, \quad z>0$$

and in terms of the ${}_1F_1$ hypergeometric function,

$$U(a, b; z) = \frac{\pi}{\sin \pi b}\left\{\frac{{}_1F_1(a, b; z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b}{}_1F_1(1+a-b, 2-b; z)}{\Gamma(a)\Gamma(2-b)}\right\}$$

Theorem 3. *Let $V = \alpha_1 V_1 - \alpha_2 V_2$ where $\alpha_1, \alpha_2 > 0$ and V_1, V_2 are independent χ^2 variables with ξ_1 and ξ_2* d.f. *respectively. Then, the probability density function of V is given by,*

$$f(v) = [C(\xi_1, \xi_2)/\Gamma(\xi_1/2)]v^{(\xi_1+\xi_2-2)/2} \exp(-v/2\alpha_1)$$

$$.\ U[\xi_2/2, (\xi_1+\xi_2)/2; (\alpha_1+\alpha_2)/2\alpha_1\alpha_2 v], \quad v \geq 0$$

$$= [C(\xi_1, \xi_2)/\Gamma(\xi_2/2)](-v)^{(\xi_1+\xi_2-2)/2} \exp(v/2\alpha_2)$$

$$.\ U[\xi_1/2, (\xi_1+\xi_2)/2; -(\alpha_1+\alpha_2)/2\alpha_1\alpha_2 v], \quad v \leq 0$$

where $C^{-1}(\xi_1, \xi_2) = 2^{(\xi_1+\xi_2)/2}\alpha_1^{\xi_1/2}\alpha_2^{\xi_2/2}$.

Proof. Let us expand the density $f(v)$ of V as,

$$\alpha_1 f(v) = \int_{-\infty}^{\infty} g_1\{(v + \alpha_2 v_2)/\alpha_1\} g_2(v_2)\, dv_2$$

where g_i represents the probability density function of V_i, $i = 1, 2$. For $V \geq 0$, noting that the limits of the above integral reduce to 0 and ∞, simplifications yield the form of $f(v)$ as in the theorem. For $v \leq 0$, in order to represent $f(v)$ in terms of $U(.)$ consider the following. First note that the exponent for $U(.)$ must be negative. An initial transformation, $\alpha_2 v_2 = -vy$ and then a further transformation $y - 1 = z$ yields the claimed result.

Using the above theorem and (2.3) we have the following Corollary.

Corollary 1. *The exact non-null distribution of $\bar{\rho}$ is given by $f(v)$ of Theorem* 3 *with $\xi_1 = m$, $\xi_2 = m(k-1)$, $\alpha_1 = \{1 + (k-1)\rho\}/mk$ and $\alpha_2 = (1-\rho)/mk(k-1)$, for which the null distribution is obtained by substituting $\rho = 0$.*

4. Asymptotic Null and Non-null Distributions of $\bar{\rho}$

Recalling that $\bar{\rho} = \alpha_1\chi^2_{\xi_1} - \alpha_2\chi^2_{\xi_2}$, where ξ_1, ξ_2, α_1 and α_2 are given in Corollary 1,

$$E(\bar{\rho}^h) = \sum_{j=0}^{h} (-1)^{h-j}\binom{h}{j}\alpha_1^j\alpha_2^{h-j}\mu'_{1,j}\mu'_{2,h-j}, \quad h = 1, 2, \ldots$$

where $\mu'_{1,s} = E(\chi^{2s}_{\xi_1})$ and $\mu'_{2,s} = E(\chi^{2s}_{\xi_2})$, $s = 0, 1, \ldots, h$.

By central limit theorem, we have

Theorem 4. *$k(m/2)^{1/2}(\bar{\rho} - \rho)$ is asymptotically distributed as a normal variable with mean* 0 *and variance $\{1 + (k-1)\rho\}^2 + (1-\rho^2)/(k-1)$.*

Theorem 3. *Let $V = \alpha_1 V_1 - \alpha_2 V_2$ where $\alpha_1, \alpha_2 > 0$ and V_1, V_2 are independent χ^2 variables with ξ_1 and ξ_2* d.f. *respectively. Then, the probability density function of V is given by,*

$$\begin{aligned} f(v) &= [C(\xi_1, \xi_2)/\Gamma(\xi_1/2)]v^{(\xi_1+\xi_2-2)/2} \exp(-v/2\alpha_1) \\ &\quad . \, U[\xi_2/2, (\xi_1+\xi_2)/2; (\alpha_1+\alpha_2)/2\alpha_1\alpha_2 v], \quad v \geq 0 \\ &= [C(\xi_1, \xi_2)/\Gamma(\xi_2/2)](-v)^{(\xi_1+\xi_2-2)/2} \exp(v/2\alpha_2) \\ &\quad . \, U[\xi_1/2, (\xi_1+\xi_2)/2; -(\alpha_1+\alpha_2)/2\alpha_1\alpha_2 v], \quad v \leq 0 \end{aligned}$$

where $C^{-1}(\xi_1, \xi_2) = 2^{(\xi_1+\xi_2)/2}\alpha_1^{\xi_1/2}\alpha_2^{\xi_2/2}$.

Proof. Let us expand the density $f(v)$ of V as,

$$\alpha_1 f(v) = \int_{-\infty}^{\infty} g_1\{(v + \alpha_2 v_2)/\alpha_1\} g_2(v_2)\, dv_2$$

where g_i represents the probability density function of V_i, $i = 1, 2$. For $V \geq 0$, noting that the limits of the above integral reduce to 0 and ∞, simplifications yield the form of $f(v)$ as in the theorem. For $v \leq 0$, in order to represent $f(v)$ in terms of $U(.)$ consider the following. First note that the exponent for $U(.)$ must be negative. An initial transformation, $\alpha_2 v_2 = -vy$ and then a further transformation $y - 1 = z$ yields the claimed result.

Using the above theorem and (2.3) we have the following Corollary.

Corollary 1. *The exact non-null distribution of $\tilde{\rho}$ is given by $f(v)$ of Theorem 3 with $\xi_1 = m$, $\xi_2 = m(k-1)$, $\alpha_1 = \{1 + (k-1)\rho\}/mk$ and $\alpha_2 = (1-\rho)/mk(k-1)$, for which the null distribution is obtained by substituting $\rho = 0$.*

4. Asymptotic Null and Non-null Distributions of $\tilde{\rho}$

Recalling that $\tilde{\rho} = \alpha_1\chi^2_{\xi_1} - \alpha_2\chi^2_{\xi_2}$, where ξ_1, ξ_2, α_1 and α_2 are given in Corollary 1,

$$E(\tilde{\rho}^h) = \sum_{j=0}^{h} (-1)^{h-j}\binom{h}{j}\alpha_1^j\alpha_2^{h-j}\mu'_{1,j}\mu'_{2,h-j}, \quad h = 1, 2, \ldots$$

where $\mu'_{1,s} = E(\chi^{2s}_{\xi_1})$ and $\mu'_{2,s} = E(\chi^{2s}_{\xi_2})$, $s = 0, 1, \ldots, h$.

By central limit theorem, we have

Theorem 4. *$k(m/2)^{1/2}(\tilde{\rho} - \rho)$ is asymptotically distributed as a normal variable with mean 0 and variance $\{1 + (k-1)\rho\}^2 + (1-\rho^2)/(k-1)$.*

5. Statistical Curvature and the Global Performance of the LMP Test for ρ

In a regular exponential family, at least in large samples, the LMP test (being an approximation to the LRT) is expected to perform reasonably well. On the other hand, there are specific examples (e.g. Cox and Hinkley, 1979, pp. 119–120) which demonstrate that the choice of the LMP test can be disastrous. It will be thus desirable to have an idea of the performance of the LMP test in our case before proceeding with the actual details of the computations for its exact power. This is achieved through the criterion of statistical curvature γ_θ (Efron, 1975, p. 1193).

Consider the LMP test for $H_0: \theta = \theta_0$ against one-sided alternatives. Efron suggests that a value of $\gamma_{\theta_0}^2 \geq \frac{1}{8}$ is "large" and one can expect linear methods to work "poorly" in such a case. In repeated sampling situations, the curvature $\hat{m}\gamma_{\theta_0}^2 = {}_1\gamma_{\theta_0}^2/m$. Hence one can determine the sample size which reduces the curvature below 1/8.

Detailed computations give,

$${}_1\gamma_0^2(k) = \left[\begin{matrix} 1 & -2(k-2) \\ & 4[(k-1)+(k-2)^2] \end{matrix}\right] \Big/ \left[\frac{k(k-1)}{2}\right] = 8/k.$$

Hence ${}_1\gamma_0^2(k)$ decreases with increase in the dimension k. Further, we note that for a sample of size m, by Efron's rule, we would need $mk > 64$ to reduce the curvature below the "worrisome point" of $\frac{1}{8}$. We next investigate how poor the performance of the test is for small mk. However, note from Table 2, that even for $m = 10$, $k = 2$, i.e. $mk = 20$, the power is reasonable. The performance becomes better as mk increases. With $m = 10$, $k = 4$, i.e. $mk = 40$ only, the power is globally quite encouraging. The LMPU test thus merits serious consideration as a test for ρ.

TABLE 2

Power of LMPU *test for certain m and k*. ($\alpha = 0{\cdot}05$)

		$m = 10$			$m = 15$			$m = 25$		
ρ	k	2	3	4	2	3	4	2	3	4
0·1		0·09234	0·13855	0·19001	0·10428	0·16539	0·23440	0·12568	0·21461	0·3156
0·2		0·15284	0·27037	0·38876	0·18769	0·34640	0·49852	0·25307	0·48034	0·6703
0·3		0·23144	0·42304	0·58218	0·29923	0·54649	0·72389	0·42389	0·73050	0·88731
0·4		0·32547	0·57215	0·73493	0·43096	0·71961	0·86685	0·60915	0·88943	0·9703
0·5		0·42987	0·70092	0·84126	0·56943	0·84434	0·94217	0·77253	0·96364	0·9937
0·6		0·53795	0·80210	0·90934	0·69943	0·92197	0·97702	0·88928	0·99036	0·9988
0·7		0·64250	0·87572	0·95037	0·80833	0·96459	0·99160	0·95613	0·99794	0·99984
0·8		0·73695	0·92583	0·97392	0·88937	0·98547	0·99717	0·98624	0·99965	0·9999
0·9		0·81649	0·95793	0·98685	0·94260	0·99462	0·99913	0·99665	0·99995	1·0000
1·0		0·87870	0·97732	0·99366	0·97323	0·99821	0·99976	0·99937	1·00000	1·0000

6. Brown's Rules and Comparison of LMPU Tests

Brown's paper points out that, in many problems use of "extra" ation does not lead to better tests. It would thus be desirable to t evidence that the $\bar{\rho}$-test has indeed greater efficiency against st" tests for ρ. From Gokhale and SenGupta (1982) it is known the LMPU similar test for ρ in the non-standard SMN distribu- with unknown common marginal mean μ and variance σ^2, is d on the sample intra-class correlation coefficient r (Rao, 1973, p. It is then natural to compare the $\bar{\rho}$ and the r-tests, both being U tests for ρ.

Let $\mathbf{X}_1, \ldots, \mathbf{X}_m$ be an independent sample from a k-variate SMN bution. Let

$$B = k \sum_{j=1}^{m} (\bar{x}_j - \bar{x})^2, \quad W = \sum_{j=1}^{m} \sum_{i=1}^{k} (x_{ij} - \bar{x}_j)^2 \quad \text{and} \quad T = B + W,$$

re

$$\bar{x}_j = \sum_i x_{ij}/k, \quad \bar{x} = \sum_i \sum_j x_{ij}/mk.$$

$r = (kB - T)/(k-1)T$. Further, $B/[1+(k-1)\rho]\sigma^2$ and $W/$ $-\rho)\sigma^2$ are independently distributed as χ^2 with $(m-1)$ and $-1)$ d.f. respectively. The LMPU similar test for $H_0: \rho = 0$ $H_1: \rho > 0$ is given by

Reject H_0 iff $r > r_0$, where r_0 is a constant to be determined
to give the desired level of significance.

, $P_\rho[r > r_0] = P_\rho[\beta < \beta_\rho]$ where β is distributed as a Beta variable $m(k-1)/2$, $(m-1)/2$ d.f. and

$$\beta_\rho^{-1} = 1 + \left(\frac{k}{(k-1)(1-r_0)} - 1\right)\left(\frac{1-\rho}{(1+(k-1)\rho)}\right)$$

r H_0, $\beta_\rho \equiv \beta_0$ is the lower cut-off point of the Beta distribution. cut-off points and the powers for the r-test are obtained through dard packages for computing incomplete Beta integrals.

For the $\bar{\rho}$-test, note that for $\alpha_1, \alpha_2 > 0$, χ_1^2 and χ_2^2 independent,

$$P[\alpha_1\chi_1^2 - \alpha_2\chi_2^2 < v] = \int_t^\infty G_1[(v + \alpha_2 v_2)/\alpha_1] g_2(v_2)\, dv_2 \qquad (6.1)$$

$t = \max(0, -v/\alpha_2)$ and $G_i(\cdot)$ and $g_i(\cdot)$ represent the c.d.f. and p.d.f. respectively of χ_i^2. Corollary 1 gives specific values of α_1, α_2 Under H_0 the constant v in (6.1) is obtained through iteration. ·) is available from MDGAM of IMSL. The integral in (6.1) is ated through use of Gauss–Laguerre quadrature formula or atively through tabulated values of Kummer's function and

TABLE 3

Comparison of powers of $\bar{\rho}$ and r-tests ($\alpha = 0{\cdot}05$)

	$m=5, k=2$		$m=5, k=3$		$m=10, k=2$		$m=10, k=3$	
ρ	$\bar{\rho}$	r	$\bar{\rho}$	r	$\bar{\rho}$	r	$\bar{\rho}$	r
0·03	0·057731	0·055940	0·065088	0·062623	0·060885	0·059481	0·071244	0·06931
0·06	0·066169	0·062485	0·082300	0·077178	0·073299	0·070406	0·097110	0·09313
0·09	0·075316	0·069691	0·101513	0·093748	0·087302	0·082933	0·127482	0·12171
0·12	0·085178	0·077622	0·122561	0·112395	0·102945	0·097229	0·162024	0·15512
0·15	0·095756	0·086345	0·145253	0·133160	0·120256	0·113465	0·200237	0·19325
0·18	0·107048	0·095935	0·169383	0·156059	0·139246	0·131812	0·241503	0·23585
0·21	0·119049	0·106474	0·194734	0·181083	0·159903	0·152442	0·285133	0·28245
0·24	0·131748	0·118053	0·221089	0·208198	0·182194	0·175517	0·330404	0·33245
0·27	0·145133	0·130768	0·248236	0·237346	0·206064	0·201191	0·376600	0·38512
0·30	0·159188	0·144726	0·275970	0·268443	0·231437	0·229595	0·423041	0·43963

standard numerical integration techniques. The powers can be evaluated similarly. Exact comparison of local powers for small sample sizes are presented in Table 3.

7. Conclusions

From Table 2, we had noted that for moderate/large sample sizes and dimensions, at even somewhat distant alternatives, e.g. $m \geq 10$, $k \geq 4$, $\rho \geq 0{\cdot}4$, the performance of the $\bar{\rho}$-test is quite satisfactory for practical purposes. It becomes even better, rapidly, with increase in m, k and ρ. For example, with $m = 25$, $k = 4$, $\rho = 0{\cdot}5$, the power exceeds 0·99. We thus restrict ourselves to small m and k to compare two locally optimal tests in terms of their local powers. For example, let us consider the alternatives, say $\rho \leq 0{\cdot}3$, to see how "local" the dominance of the $\bar{\rho}$-test is over the r-test. This dominance is over the entire range of comparison (with $m = 10$, $k = 3$ for $\rho \leq 0{\cdot}21$) as exhibited in Table 3. Recall that there does not exist in the literature any test for ρ in the SSMN distribution. The proposed $\bar{\rho}$-test, by virtue of its optimal properties and satisfactory power performance, seems to be a reasonable choice.

Acknowledgements

The author is grateful to Profs. T. W. Anderson, I. Olkin and C. R. Rao for encouragements and helpful comments. Constructive criticisms of the editor and a referee, which have improved the presentation, are greatly appreciated.

Research started and partially supported by NSF Grant SE 579-13976 and ONR Contract N00014-75-0442 at Stanford University and completed at Indian Statistical Institute—Calcutta.

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ived July 1985; revised April 1986