There and Back Again

AEC 504 - Summer 2007

Fundamentals of Economics

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Bilbo Baggins enjoys eating muffins and smoking a pipe. Denote the number of muffins he eats a day by X and the number of pipes he smokes a day by Y. Bilbo's utility function is $U = \sqrt{X} + \sqrt{Y}$ and his daily budget is 8 silver coins. Coins, muffins and pipes are assumed to be infinitely divisible.

i. The story starts in the Shire, where pipes and muffins cost Bilbo one coin each. Compute his optimal consumption bundle.

$$\begin{cases} \max\limits_{X,Y} \sqrt{X} + \sqrt{Y} \\ s.t. X + Y = 8 \end{cases} \Rightarrow \begin{cases} \dfrac{MU_W}{MU_R} = \dfrac{p_W}{p_R} \\ X + Y = 8 \end{cases} \Rightarrow \begin{cases} \sqrt{\dfrac{R}{W}} = 1 \\ 2X = 8 \end{cases} \Rightarrow \begin{cases} X = 4 \\ Y = 4 \end{cases}$$

The utility Bilbo gets in the Shire is $\sqrt{4} + \sqrt{4} = 4$***.***

ii. Then Bilbo gets to the Lonely Mountain, where he can still smoke if he has time, but is clearly short of muffins. In terms of his budget constraint, it means that a pipe still costs him a coin, but a muffin costs him two coins instead of one. Compute his optimal consumption bundle.

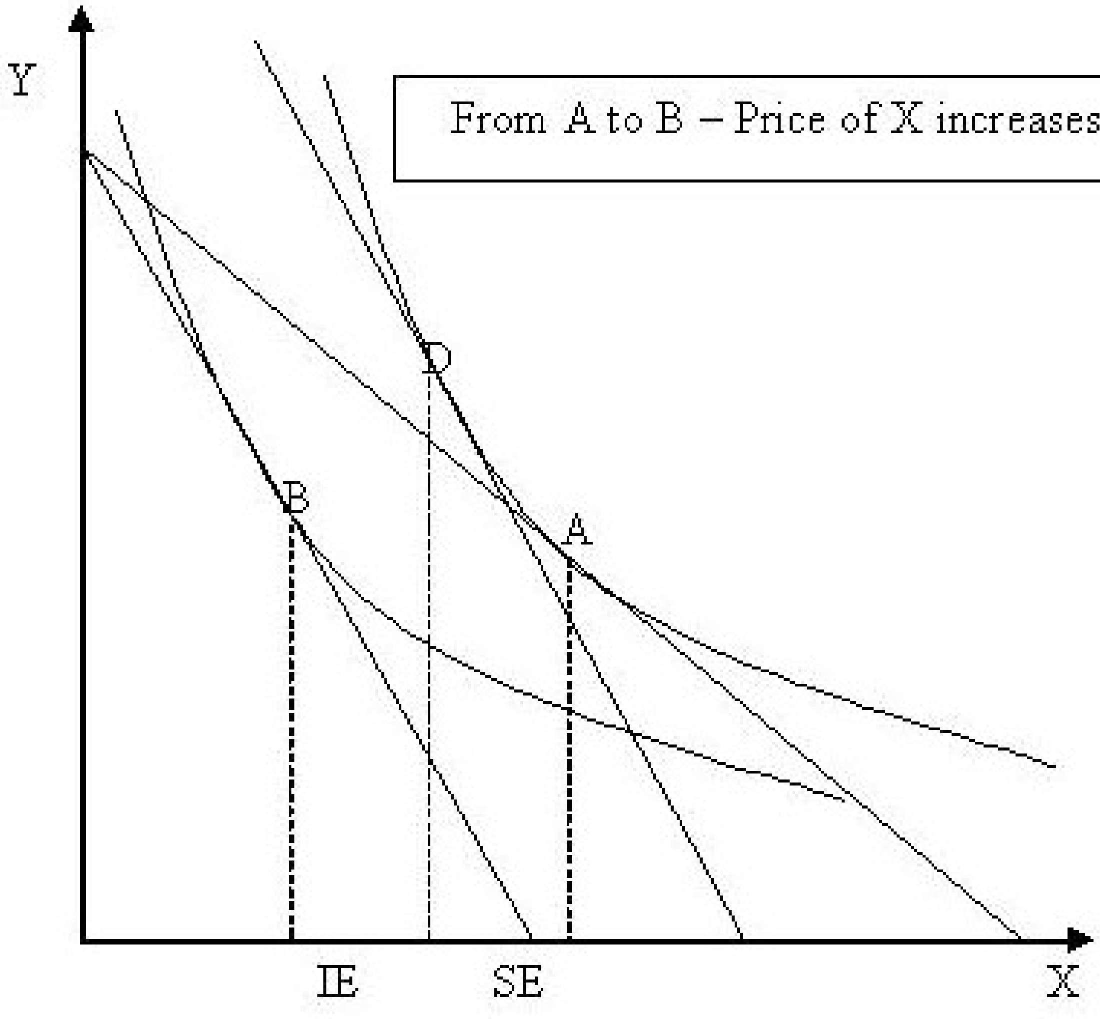
$$\begin{cases} \max\limits_{X,Y} \sqrt{X} + \sqrt{Y} \\ s.t. 2X + Y = 8 \end{cases} \Rightarrow \begin{cases} \dfrac{MU_W}{MU_R} = \dfrac{p_W}{p_R} \\ 2X + Y = 8 \end{cases} \Rightarrow \begin{cases} \sqrt{\dfrac{R}{W}} = 2 \\ 2X + 4X = 8 \end{cases} \Rightarrow \begin{cases} X = \dfrac{4}{3} \\ Y = \dfrac{16}{3} \end{cases}$$

The utility Bilbo gets in the Lonely Mountain is $\sqrt{4/3} + \sqrt{16/3} = 2\sqrt{3} < 4$***. Naturally, he is worse off in the Lonely Mountain because the price of muffins increased and all the rest is the same.***

Notice that even though Bilbo is clearly poorer after the price change, he smokes more than before (and probably even more than a respectable

hobbit should). That is to say that for pipes the substitution effect is much stronger than the income effect. It should not be surprising - the utility function shows that there is no complementarity between the two. The substitution effect for pipes would be exactly equal to the income effect if the utility function was Cobb-Douglas, which does have some complementarity effects.

iii. Compute the income effect and the substitution effect for both muffins and pipes as Bilbo travels from the Shire to the Lonely Mountain. What is the compensation Thorin Oakenshield has to offer him each day Bilbo stays away from the Shire to make him as happy in the Lonely Mountain as he was in the Shire?



In the graph above Bilbo's consumption in the Shire is point A, and his consumption in the Lonely Mountain is point B. His trip from A to B can be disaggregated into the trip from A to D and the trip from D to B. Going from A to D, he is riding up his Shire indifference curve because of the price change, so it represents the pure substitution effect. As he jumps down from D to B, his budget constraint moves parallel to the Lonely Mountain budget constraint, so this trip is the pure income effect.

The third budget constraint that touches the Shire indifference curve at point D shows the minimum compensation Thorin has to give him so that Bilbo is not sorry about the higher price of muffins in the Lonely Mountain. Therefore, point D can be found by solving

$$\begin{cases} \min\limits_{X,Y} 2X + Y \\ s.t. \sqrt{X} + \sqrt{Y} = 4 \end{cases} \Rightarrow \begin{cases} \dfrac{MU_W}{MU_R} = \dfrac{p_W}{p_R} \\ \sqrt{X} + \sqrt{Y} = 4 \end{cases} \Rightarrow \begin{cases} \sqrt{\dfrac{R}{W}} = 2 \\ \sqrt{X} + 2\sqrt{X} = 4 \end{cases} \Rightarrow \begin{cases} X = \dfrac{16}{9} \\ Y = \dfrac{64}{9} \end{cases}$$

The compensation Thorin has to give Bilbo is $2 \cdot \frac{16}{9} + \frac{64}{9} - 8 = \frac{24}{9}$***. Even though this compensation is required daily until Bilbo gets back to the Shire, it is clearly less than the value of the Arkenstone gem Bilbo steals to compensate himself.***

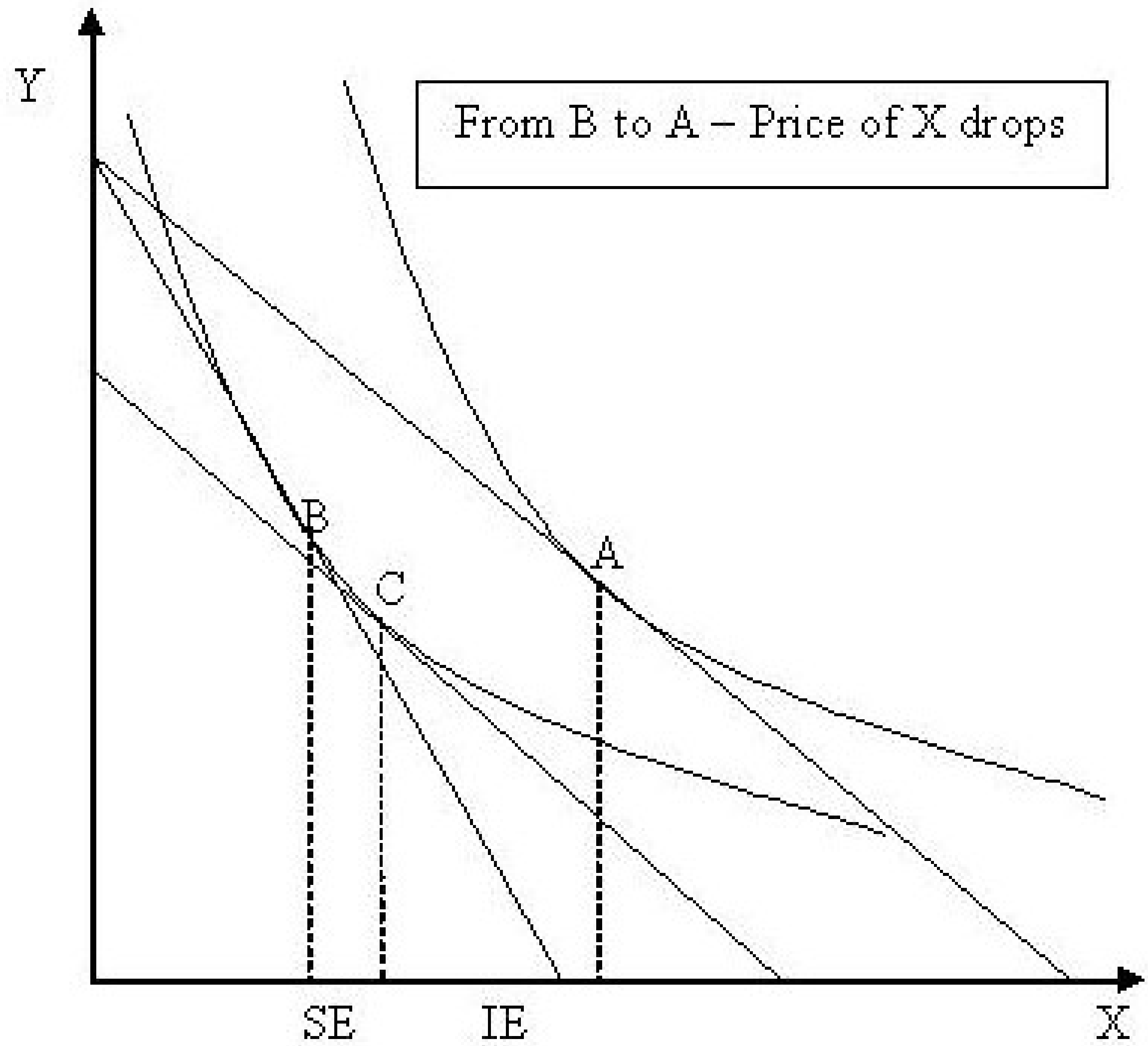
So, we have three points: A (4, 4), D (16/9, 64/9), and B (4/3, 16/3). The substitution effect is sliding up the original indifference curve from A to D:

$$SE_X = \frac{16}{9} - 4 = -\frac{20}{9}, \quad SE_Y = \frac{64}{9} - 4 = \frac{28}{9}$$

The income effect is jumping down from D to B (e.g. as Thorin stops compensating Bilbo after the theft of the Arkenstone is discovered):

$$IE_X = \frac{4}{3} - \frac{16}{9} = -\frac{4}{9}, \quad IE_Y = \frac{16}{3} - \frac{64}{9} = -\frac{16}{9}$$

iv. Compute the income effect and the substitution effect for both muffins and pipes as Bilbo travels back from the Lonely Mountain to the Shire. If Bilbo currently expects to get back to the Shire in 30 days, what is the maximum amount he would pay Gandalf, who offers to get him back to the Shire immediately using his magic?



When Bilbo gets to the Shire, he becomes richer because muffins are twice cheaper there than in the Lonely Mountain. Therefore, he is willing to reduce his budget up to when it gets him to the Lonely Mountain indifference curve at the Shire prices (point C). So, we can compute point C by solving

$$\begin{cases} \min\limits_{X,Y} X+Y \\ s.t. \sqrt{X}+\sqrt{Y}=2\sqrt{3} \end{cases} \Rightarrow \begin{cases} \dfrac{MU_W}{MU_R}=\dfrac{p_W}{p_R} \\ \sqrt{X}+\sqrt{Y}=2\sqrt{3} \end{cases} \Rightarrow \begin{cases} \sqrt{\dfrac{R}{W}}=1 \\ 2\sqrt{X}=2\sqrt{3} \end{cases} \Rightarrow \begin{cases} X=3 \\ Y=3 \end{cases}$$

For each additional day in the Shire Bilbo can give up at most $8-3-3=2$***, so for the 30 days Gandalf can count only on 60 silver coins. Notice that even though Bilbo travels between the same A and B points in (iii) and (iv), the compensation for getting from A to B (24/9) is more than the compensation for getting from B to A (2=18/9).***

The intuition is that the utility function is "symmetric" (X ***and*** Y ***are equally important to Bilbo), so Bilbo prefers a balanced consumption bundle. Points A (4,4) and C (3,3) offer him that, and points B (4/3,16/3) and D (16/9, 64/9) clearly do not. The compensation in (iii), 24/9, is what is needed to keep him as happy with the unbalanced consumption at D as he was with the balanced consumption at A. The compensation in (iv), 2 or 18/9, is what is needed to keep him as happy with the balanced consumption at C as he was with the unbalanced consumption at B. So, the compensation in (iii) is increased by the need to make Bilbo bear the unbalanced consumption of muffins and pipes at D, and the compensation in (iv) is reduced by the fact that Bilbo is willing to move to the balanced Shire-type consumption in C.***

So, we have three points: A (4, 4), C (3, 3), and B (4/3, 16/3). The substitution effect is sliding down the new indifference curve from point B to point C:

$$SE_X = 3-\frac{4}{3}=\frac{5}{3}, \quad SE_Y = 3-\frac{16}{3}=-\frac{7}{3}$$

The income effect is jumping up from C to A (e.g. as Gandalf remembers he is a good and kind wizard and sends Bilbo home for free):

$$IE_X = 4-3=1, \quad IE_Y = 4-3=1$$

The signs of the substitution and the income effects are naturally opposite to their signs in (iii). What is important, their absolute magnitudes

are nowhere close to the magnitudes in (iii), while one might think they should be equal, because (iii) and (iv) is traveling between the same pair of points - A and B.

The reason (iii) and (iv) give different estimates of the income and the substitution effects is again that Bilbo gets from A to B through the unbalanced point D, and gets back from B to A through the balanced point C.

Based on that, we can develop the intuition for the difference in the effects magnitude.

$|IE_X^{(iii)}| < |IE_X^{(iv)}|$ ***and*** $|IE_Y^{(iii)}| > |IE_Y^{(iv)}|$***, because the price of muffins is 2 at point D in (iii) and 1 at point C in (iv). It is cheaper to compensate Bilbo with more pipes in (iii) and with more muffins in (iv).***

$|SE_X^{(iii)}| > |SE_X^{(iv)}|$ ***and*** $|SE_Y^{(iii)}| > |SE_Y^{(iv)}|$***, because the absolute value of the substitution swings is greater at the higher indifference curve with higher income.***