Theory Appendix to

"Turnover: Liquidity or Uncertainty?", AND *"Analyst Disagreement and Aggregate Volatility Risk"* AND *"Why Does Higher Turnover Variability Predict Lower Expected Returns?"*

Alexander Barinov*

Terry College of Business
University of Georgia

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Abstract

This document contains the formal derivation of the aggregate volatility story in "Turnover: Liquidity or Uncertainty?" and "Analyst Disagreement and Aggregate Volatility Risk". The document shows why aggregate volatility risk should explain the turnover effect (Datar, Naik, and Radcliffe, 1998), the analyst disagreement effect (Diether, Malloy, and Scherbina, 2002), and the turnover variability effect (Chordia, Subrahmanyam, and Anshuman, 2001). The aggregate volatility story suggests high uncertainty (high turnover) firms, high disagreement firms, and high turnover variability firms beat the CAPM in the periods of increasing aggregate volatility. Section 1 of this document presents the setup of the model and derives the main predictions. Section 2 collects the proofs of the propositions in Section 1. Section 3 presents simulations to back up the proofs and to evaluate the magnitude of the effects in the model.

*438 Brooks Hall, University of Georgia. Athens, GA 30602. Tel.: +1-706-542-3650. Fax: +1-706-542-9434. E-mail: abarinov@terry.uga.edu. Web: http://abarinov.myweb.uga.edu/

1 The Real Options Model

Johnson (2004) and Barinov (2010) develop and successfully test two related models that predict and explain the negative relation between uncertainty and expected returns. The main idea of these models is that real options can transform firm-specific uncertainty into lower systematic risk. In the Johnson setup, the natural empirical proxy for uncertainty is the dispersion of analyst forecasts, which measures analyst disagreement. In the Barinov model, the empirical proxy for uncertainty is idiosyncratic volatility. In this section, I use the Johnson setup, because it is more likely that disagreement creates trade and increases turnover, but the model can be restated in terms of idiosyncratic volatility. I extend the Johnson model by showing that high firm-specific uncertainty means lower aggregate volatility risk (as Barinov, 2010, also does in his setup) and by considering growth options in addition to leverage.

1.1 Leverage and Uncertainty

Consider a firm with unobservable true value of assets C_t that has issued risky debt with the face value K. Asssume that the true value of assets follows

$$dC_t = \mu_C C_t dt + \sigma_C C_t dW_C \tag{1}$$

Investors cannot observe the process C_t (the true value of assets) and observe U_t instead, which is C_t contaminated by a stationary noise process η_t. U_t is given by

$$U_t = C_t \cdot \exp(\eta_t) \tag{2}$$

The noise process η_t is an unobservable stationary diffusion process

$$d\eta_t = -\kappa\eta_t dt + \sigma_\eta dW_\eta \tag{3}$$

In addition to U_t, investors observe the stochastic discount factor Λ_t that follows

$$d\Lambda_t = -r\Lambda_t dt + \sigma_\Lambda \Lambda_t dW_\Lambda \tag{4}$$

Johnson (2004) shows that in this economy S_t, the observable price of unlevered claim on C_T (firm's assets)[1], and V_t, the observed value of equity, follow

$$dS_t = (r + \pi_S)S_t dt + \sigma_C S_t d\widetilde{W}_C \tag{5}$$

[1] Note that C_t is the true (unobservable) value of of the underlying asset, and S_t is the observable value of the underlying asset, which moves according to the information about the underlying asset investors are able to filter out of the price and the economy structure.

$$V_t = S_t\Phi(d_1) - \exp(r(T-t))K\Phi(d_2) \tag{6}$$

where $\pi_S = -\rho_{C\Lambda}\sigma_C\sigma_\Lambda$ is the risk premium, $\Phi(\cdot)$ is the normal cdf,

$$d_1 = \frac{log(S/K) + (r(T-t) + \widetilde{\sigma}^2/2)}{\widetilde{\sigma}},\ d_2 = d_1 - \widetilde{\sigma} \tag{7}$$

$$\widetilde{\sigma}^2 = \omega + \sigma_C^2(T-t) \tag{8}$$

$d\widetilde{W}_C$ is the posterior belief of investors about the process governing C_t given the signals they have received, and ω reflects firm-specific uncertainty. Johnson (2004) also shows that

$$d\widetilde{W}_C = F(X)(\sigma_C dW_C + \sigma_\eta dW_\eta) + G(X)dW_\Lambda, \tag{9}$$

where $F(X)$ and $G(X)$ are some functions of the model primitives κ, σ_V, σ_C, σ_Λ, $\rho_{C\Lambda}$. (Explicit expressions of $F(X)$ and $G(X)$ are given in Johnson, 2004).

The expression (8) for the total volatility shows that in the model the uncertainty is resolved all at once at the last instant, when S_T jumps to V_T. It is the reason why the firm-specific uncertainty, which represents the jump risk for the underlying asset, is priced only for the levered claim. Johnson (2004) notes that the fact that the risk measured by ω is resolved by a jump means that this risk is truly idiosyncratic.

Proposition 1. The risk premium of the firm equals

$$\pi_V = \pi_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{10}$$

and its derivatives with respect to firm-level uncertainty and the assets value have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{11}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial S} = \pi_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{12}$$

Proof: See Section 2.

The sign of (11) implies that firm-specific uncertainty is negatively related to expected returns, which is the main result established in Johnson (2004). The intuition for the sign is that the elasticity of the call option with respect to the underlying asset value, $\Phi(d_1)S_t/V_t$, decreases in the uncertainty above the underlying asset, because more uncertainty about the underlying asset means that its current value is less informative about the value of the option at the expiration date. Therefore, the current value of the option responds less to the same percentage change in the current value of the underlying asset if there is more uncertainty about its true value.

Johnson (2004) also notices that the effect of firm-specific uncertainty on the firm's expected returns should be stronger for highly levered firms. The intuition is that the uncertainty derives its pricing impact from the fact that a levered firm is a call option on the assets, and therefore volatility should matter more for more option-like firms. In algebraic terms it means that as the assets value increases (or as the face value of the debt decreases, or, equivalently, as leverage decreases), the first derivative (11) becomes less negative, i.e. (12) is positive. Evidently, as the face value of the debt and leverage reach zero, (11) also reaches zero.

In the rest of this subsection I extend the Johnson (2004) model to show that the expected returns effects he finds (Proposition 1 above) arise because changes in firm-specific uncertainty create changes in systematic risk, namely, in aggregate volatility risk.

Corollary 1. The equity beta equals

$$\beta_V = \beta_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{13}$$

and its derivatives have the following signs:

$$\frac{\partial \beta_V}{\partial \omega} = \beta_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{14}$$

$$\frac{\partial^2 \beta_V}{\partial \omega \partial S} = \beta_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{15}$$

Proof: See Section 2.

Corollary 1 stresses the fact that while firm-specific uncertainty creates only idiosyncratic risk at the level of the firm assets and does not change their systematic risk, this idiosyncratic risk reduces the systematic risk of equity, which is a call option of the assets, by making it less responsive to the changes in the assets value. This is the reason why the idiosyncratic risk created by firm-specific uncertainty is diversifiable at the level of the assets, but is not diversifiable at the level of firm equity.

Credit rating is an alternative measure of how valuable the real option created by leverage is. If credit rating is good, the firm is far from executing the option, and the convexity of the equity created by risky debt is small. If credit rating is bad, the option created by risky debt is close to being at the money, and therefore the equity is more option-like. Another advantage of using credit rating instead of leverage is that leverage is mechanically negatively related to market-to-book, the measure of growth options (see Section 1.2 for the discussion of the interaction between uncertainty and growth options). Because market cap is in the numerator of market-to-book and the denominator of leverage, high market-to-book implies low leverage and vice versa, which makes difficult studying the real option created by risky debt and growth options separately. The correlation of market-to-book and credit rating is weaker, thus suggesting that credit rating can be a cleaner measure of the real option created by the existence of risky debt.

If turnover (turnover variability) proxies for firm-specific uncertainty, we can formulate the following empirical hypothesis:

Hypothesis 1 *Future returns are negatively related to turnover (analyst disagreement, turnover variability). This effect is stronger for highly levered firms and firms with bad credit rating.*

The negative correlation between firm-specific uncertainty and the exposure of the firm's equity to systematic risk is useful during periods of high aggregate volatility. These periods usually coincide with the periods of high idiosyncratic volatility and high dispersion of analyst forecasts (see Barinov, 2009, and references therein). The next proposition shows that, all else equal, the increased uncertainty makes the risk premium of high uncertainty, high leverage firms increase less and makes their value drop less during the periods of aggregate volatility.

In untabulated results I show that the firm-specific uncertainty of low and high uncertainty firms respond to aggregate volatility movements by changing by the same percentage rather than by the same amount. Therefore, the key variable in the time-series dimension is the elasticity of risk premium with respect to uncertainty, instead of the derivative, which was the focus of the previous propositions.

Proposition 2 The elasticity of the equity risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\right) < 0 \qquad (16)$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is positive:

$$\frac{\partial^2}{\partial\omega\partial S}\Big(\frac{\partial\pi_V}{\partial\omega}\cdot\frac{\omega}{\pi_V}\Big) > 0 \tag{17}$$

Proof: See Section 2.

As Campbell (1993) and Chen (2002) show, investors require lower risk premium from the stocks that react less negatively to aggregate volatility increases. Hence, Proposition 2 implies that the firm's exposure to aggregate volatility risk decreases with firm-specific uncertainty. Because the uncertainty is transformed into lower aggregate volatility risk through the real option created by leverage, this effect is stronger for highly levered firms.

Another effect that ties real options and aggregate volatility risk comes from the fact that, all else equal, higher uncertainty means higher value of the real option[2]. When both aggregate volatility and firm-specific uncertainty increase, this effect makes the value of the real option created by leverage increase in value (holding other effects fixed). In Proposition 3, I show that, holding constant all other (usually negative) cash flow effects of the aggregate volatility increase, the positive effect of uncertainty on the real option value is larger for high uncertainty firms, especially if they also have high leverage.

Proposition 3 The elasticity of the equity value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial\omega}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) > 0 \tag{18}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is negative:

$$\frac{\partial^2}{\partial\omega\partial S}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) < 0 \tag{19}$$

Proof: See Section 2.

I define the aggregate volatility factor as the portfolio that tracks changes in expected aggregate volatility. The positive exposure to aggregate volatility factor is then desirable, because it means (relative) gains in response to aggregate volatility increases. If turnover (turnover variability) proxies for firm-specific uncertainty, I can formulate the following empirical hypothesis:

[2] A recent paper by Grullon, Lyandres, and Zhdanov (2007) presents supporting empirical evidence.

Hypothesis 2 *In cross-section, the exposure to aggregate volatility risk decreases as turnover (analyst disagreement, turnover variability) increases. This decrease in risk exposure is stronger for highly levered firms and firms with bad credit rating.*

I would like to stress that Propositions 2 and 3 are formulated in terms of partial derivatives. It is beyond doubt that, as almost all risky assets, real options lose value when the market goes down and aggregate volatility increases. During these periods, the risk premium of real options also increases. Moreover, since a real option is a levered claim on the underlying asset, the negative reaction of a real option to an increase in aggregate volatility can be stronger than average. What Propositions 2 and 3 state is that all else equal, firms with abundant real options and high uncertainty react to aggregate volatility increases less negatively than other firms. That is, firms with abundant real options and high uncertainty most likely have high market betas and, because changes in aggregate volatility and the market return are strongly negatively correlated, their reaction to aggregate volatility increases is very negative, but it is significantly less negative than the reaction of other firms with the same market beta.

If turnover (turnover variability) proxies for firm-specific uncertainty, I can formulate the following empirical hypothesis:

Hypothesis 2a *When aggregate volatility increases, high turnover (high analyst disagreement, high turnover variability) firms lose value, but beat the CAPM. This effect is stronger for highly levered firms and firms with bad credit rating.*

1.2 Growth Options and Uncertainty

Consider now an all-equity firm that consists of growth options, P_t, and assets in place, B_t. The firm value V_t is, by definition, equal to $P_t + B_t$. The growth options are represented by a European call option, which gives the right to receive G_T for price I. Both G_t and B_t follow geometric Brownian motions:

$$dG_t = \mu_G G_t dt + \sigma_G G_t dW_G \tag{20}$$

$$dB_t = \mu_B B_t dt + \sigma_B B_t dW_B \tag{21}$$

Investors observe B_t, but cannot observe the process G_t (the true value of the asset behind growth options) and observe U_t instead, which is G_t contaminated by a stationary noise process η_t. U_t is given by

$$U_t = G_t \cdot \exp(\eta_t) \tag{22}$$

The noise process η_t is an unobservable stationary diffusion process

$$d\eta_t = -\kappa \eta_t dt + \sigma_\eta dW_\eta \tag{23}$$

In addition to B_t and U_t, investors observe the stochastic discount factor Λ_t that follows

$$d\Lambda_t = -r\Lambda_t dt + \sigma_\Lambda \Lambda_t dW_\Lambda \tag{24}$$

The setup of the model is identical to the one in the previous subsection, except for the fact that the previous subsection assumes that it is the whole firm true value that is unobservable, and this subsection only makes unobservable the value of the asset behind growth options. The results of the previous subsection generalize to this setup with several changes.

Proposition 4. The risk premium of the firm equals

$$\pi_V = \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{P}{V} + \pi_B \cdot \frac{B}{V} = \pi_B - \Big(\pi_B - \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big)\Big) \cdot \frac{P}{V} \tag{25}$$

and its derivatives with respect to firm-level uncertainty and the value of assets in place have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_G \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) + \Big(\pi_B - \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big)\Big) \cdot \frac{\partial}{\partial \omega}\Big(\frac{P}{V}\Big) < 0 \tag{26}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial B} > 0 \tag{27}$$

Proof: See Section 2.

As in the previous subsection, more firm-specific uncertainty makes the real option, in this case the growth option, less sensitive to the value of the underlying asset and therefore less risky. The new effect is that more firm-specific uncertainty also increases the value of the growth option and makes the firm more growth-like. The sufficient (though not necessary) condition for the firm risk premium to increase in firm-specific uncertainty is that growth options are less risky than assets in place[3].

[3]There are currently three strands of the value effect literature that make this prediction. A good example of the first strand is Zhang (2005) that argues that assets in place are riskier in recessions

If turnover (turnover variability) proxies for firm-specific uncertainty, I can now reformulate my empirical Hypothesis 1 replacing leverage with market-to-book, a measure of growth options:

Hypothesis 3 *Future returns are negatively related to turnover (analyst disagreement, turnover variability). This effect is stronger for growth firms.*

Propositions 2 and 3, which look at the effects of the time-series changes in firm-specific uncertainty, also go through in the new setup:

Proposition 5 The elasticity of the firm's risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{28}$$

The second cross-derivative of the elasticity with respect to uncertainty and the value of assets in place is positive:

$$\frac{\partial^2}{\partial \omega \partial B}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) > 0 \tag{29}$$

Proof: See Section 2.

Proposition 6 The elasticity of the firm's value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial \omega} \cdot \frac{\omega}{V}\Big) > 0 \tag{30}$$

The second cross-derivative of the elasticity with respect to uncertainty and the value of assets in place is negative:

$$\frac{\partial^2}{\partial \omega \partial B}\Big(\frac{\partial V}{\partial \omega} \cdot \frac{\omega}{V}\Big) < 0 \tag{31}$$

Proof: See Section 2.

If turnover (turnover variability) proxies for firm-specific uncertainty, I also reformulate Hypotheses 2 and 2a replacing leverage with market-to-book:

because of costly divesture. The second strand starts with Campbell and Vuolteenaho (2004) that shows that value firms have higher cash flow betas and growth firms have low cash flow betas, and the cash flow risk earns a much higher risk premium. Lastly, Barinov (2010) shows that when both aggregate volatility and idiosyncratic volatility increase in recession, the risk exposure of growth options declines and their value increases holding other effects of recessions fixed. Hence, growth options beat the CAPM when aggregate volatility increases and thereby provide a hedge against aggregate volatility risk.

Hypothesis 4 *In cross-section, the exposure to aggregate volatility risk decreases as turnover (analyst disagreement, turnover variability) increases. This decrease in risk exposure is stronger for growth firms.*

Hypothesis 4a *When aggregate volatility increases, high turnover (high analyst disagreement, high turnover variability) firms lose value, but beat the CAPM. This effect is stronger for growth firms.*

1.3 Leverage and Growth Options in One Model

The vast majority of firms have both some debt and some growth options. Therefore, for a typical firm the firm-specific uncertainty should create the hedge against aggregate volatility risk both through the leverage channel (Section 1.1) and the growth options channel (Section 1.2). Leverage and market-to-book are strongly negatively correlated in the data, both mechanically via market cap and economically via the free cash flow story and the underinvestment story. Thus, one of the two channels should be rather weak compared to the other. Still, one may wonder if both channels can coexist in my model. In this subsection, for the sake of brevity, I will provide the intuition of why the conclusions will be the same if we treat both real options simultaneously (formal proofs are available upon request).

There are two ways to introduce both growth options and leverage into one model. First, one can assume that only assets in place are pledgable and leverage creates a call option on their value only. If this is the case, the firm will consist of two separate real options. All else equal, when aggregate volatility and the firm-specific uncertainty increase, the risk of both will decrease and the value of both will increase, hence, more firm-specific uncertainty will mean less aggregate volatility risk. More leverage will mean higher relative weight of the option created by leverage and make the option more sensitive to the changes in uncertainty. Thus, highly levered firms with large firm-specific uncertainty will have lower aggregate volatility risk. Also, if we assume that the levered claim on assets is riskier than growth options, more growth options will both make the firm less risky and make the hedges created by growth options more important. Thus, growth firms with large firm-specific uncertainty will have lower aggregate volatility risk.

The second way to model jointly leverage and growth options is to assume that the whole firm value is pledgable and leverage creates a call option on the sum of growth options and assets in place. Assume for a minute that the firm has no assets in place and the call option created by leverage is written on growth options only. All else equal, if uncertainty increases, the risk of the growth options decreases and their value increases. The same happens to the value of the firm, because growth options are in turn an underlying asset for the call option created by leverage, and an increase in their value and decrease in their risk means the same for the call option on them. On top of that, all else equal, higher uncertainty would lower the risk of the call option created by leverage and increase its value even if nothing happened to the growth options.

If I add assets in place back to this version of the model, nothing changes compared to the model in Section 1.2. If, all else equal, an increase in uncertainty lowers the risk of the mix of assets in place and growth options and increases the total value of the mix (as Section 1.2 shows), the equity value, i.e. the value of the call option on the mix, will go up, and the equity risk will go down. On top of that, all else equal, the equity value would increase with uncertainty and the equity risk would decrease with uncertainty even if the risk and the value of the mix stayed constant.

It is easy to see that one can introduce the recovery rate into the model without changing the results. For example, it is possible to assume that only half of the growth options value is recovered after the default, and the other half is either destroyed, or accrues to shareholders. The asset-pricing implications of my model will still be the same under both scenarios.

To sum up, I conclude that it is possible to unite the models in Section 1.1 and Section 1.2 in one model without changing any of the conclusions.

2 Proofs

Proposition 1. The risk premium of the firm equals

$$\pi_V = \pi_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{32}$$

and its derivatives with respect to firm-level uncertainty and the assets value have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{33}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial S} = \pi_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{34}$$

Proof: The risk-premium equation (32) follows from a straightforward application of Ito's lemma and no-arbitrage condition to the Black-Scholes (1973) formula (6), which gives the value of equity for a levered firm. The Ito's lemma and no-arbitrage condition applied to (6) yield

$$dV_t/V_t = (r + \pi_S \cdot \Phi(d_1)\frac{S_t}{V_t})dt + \sigma_C \Phi(d_1)\frac{S_t}{V_t} d\widetilde{W}_C \tag{35}$$

and the drift less the risk-free rate is, by definition, the risk premium.

The fact that the first derivative of the risk premium (33) is negative is well-known. The exact expression and the reason why it is positive were first given in appendix to Galai and Masulis (1976).

The expressions for the second derivative (34) is complicated and can be signed only by simulations. The simulations in Section 3 show that for all plausible parameter values (34) is positive.

Corollary 1. The equity beta equals

$$\beta_V = \beta_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V}, \tag{36}$$

and its derivatives have the following signs:

$$\frac{\partial \beta_V}{\partial \omega} = \beta_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{37}$$

$$\frac{\partial^2 \beta_V}{\partial \omega \partial S} = \beta_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{38}$$

Proof: The Ito's lemma implies that in discrete time

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial S^2} \cdot \sigma^2 S^2 \Delta t + \frac{\partial V}{\partial t}\Delta t \tag{39}$$

Dividing both sides of (39) by V and using the definition of returns, I obtain that when $\Delta t \to 0$

$$R_V = \frac{\Delta V}{V} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \frac{\Delta S}{S} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot R_S \tag{40}$$

By definition of beta

$$\beta_V = \frac{cov(R_V, R_M)}{Var(R_M)} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \frac{cov(R_S, R_M)}{Var(R_M)} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \beta_S \tag{41}$$

The rest of the proof copies the proof of Proposition 1.

Proposition 2 The elasticity of the equity risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{42}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is positive:

$$\frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) > 0 \tag{43}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for all reasonable parameter values the signs of (42) and (43) are negative and positive, respectively.

Proposition 3 The elasticity of the equity value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial \omega} \cdot \frac{\omega}{V}\Big) > 0 \tag{44}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is negative:

$$\frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial \omega} \cdot \frac{\omega}{V}\Big) < 0 \tag{45}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for all reasonable parameter values the signs of (44) and (45) are negative and positive, respectively.

Proposition 4. The risk premium of the firm equals

$$\pi_V = \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{P}{V} + \pi_B \cdot \frac{B}{V} = \pi_B - \Big(\pi_B - \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big)\Big) \cdot \frac{P}{V} \tag{46}$$

and its derivatives with respect to firm-level uncertainty and the value of assets in place have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_G \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) + \Big(\pi_B - \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big)\Big) \cdot \frac{\partial}{\partial \omega}\Big(\frac{P}{V}\Big) < 0 \tag{47}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial B} > 0 \tag{48}$$

Proof: Applying the Ito's lemma and the no-arbitrage condition to the value of the firm, $V_t = P_t + B_t$, I find that the value of the firm follows

$$dV/V = \Big(\Big(r + \pi_G \frac{\partial P}{\partial G}\Big)\frac{P}{V} + \pi_B \frac{B}{V}\Big)dt + \sigma_G \cdot \frac{\partial P}{\partial G} \cdot \frac{P}{V} d\widetilde{W}_C + \sigma_B \frac{B}{V} dW_B \tag{49}$$

Then I rearrange the expression for the drift and subtract the risk-free rate to get the risk premium

$$\pi_V = \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{P}{V} + \pi_B \cdot \frac{B}{V} = \pi_B - \Big(\pi_B - \pi_G \cdot \Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big)\Big) \cdot \frac{P}{V} \tag{50}$$

Determining the sign of (47), the risk premium derivative with respect to firm-level uncertainty, is now simple and intuitive. The term in the square brackets is positive if assets in place earn higher returns than growth options, which is a sufficient condition to derive the value effect. An increase in uncertainty, ω, increases the price of growth options, P_t, and their fraction in the value of the firm, P_t/V_t. The increase in firm-specific uncertainty also leads to a decrease in the option elasticity with respect to the price of the underlying asset. Therefore, both parts of the last term in (50) increase as uncertainty increases, and expected returns decrease. Algebraically,

$$\frac{\partial \pi_V}{\partial \omega} = \pi_G \frac{\partial}{\partial \omega}\Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{P_t}{V_t} - \Big(\pi_B - \pi_G \frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{B_t}{V_t^2} \cdot \frac{\partial P_t}{\partial \omega} < 0\ , \tag{51}$$

where the first term captures the effect of uncertainty on the option elasticity, and the second term captures the increase in the relative weight of growth options.

$$\frac{\partial^2 \pi_V}{\partial \omega \partial B} = -\pi_G \frac{\partial}{\partial \omega}\Big(\frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{P_t}{V_t^2} + \Big(\pi_B - \pi_G \frac{\partial P}{\partial G} \cdot \frac{G}{P}\Big) \cdot \frac{B - P}{V^3} > 0 \tag{52}$$

The first term is always positive, and the second term is positive if $B > P$ and negative otherwise. However, for small B the first term becomes relatively large. The simulations in Sections 3 show that the derivative is positive for almost all reasonable parameter values.

Proposition 5 The elasticity of the firm's risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial\omega}\Big(\frac{\partial\pi_V}{\partial\omega}\cdot\frac{\omega}{\pi_V}\Big) < 0 \tag{53}$$

The second cross-derivative of the elasticity with respect to uncertainty and the value of assets in place is positive:

$$\frac{\partial^2}{\partial\omega\partial B}\Big(\frac{\partial\pi_V}{\partial\omega}\cdot\frac{\omega}{\pi_V}\Big) > 0 \tag{54}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for reasonable parameter values the signs of (53) and (54) are negative and positive, respectively.

Proposition 6 The elasticity of the firm's value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial\omega}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) > 0 \tag{55}$$

The second cross-derivative of the elasticity with respect to uncertainty and the value of assets in place is negative:

$$\frac{\partial^2}{\partial\omega\partial B}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) < 0 \tag{56}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for reasonable parameter values the signs of (55) and (56) are negative and positive, respectively.

3 Simulations

3.1 The Model with Leverage

Parameter Values I fix two sets of parameter values to calibrate my model. First, I look at the risk premium and the risk-free rate. I set the risk free rate to 5% per year, close to its long-term average in the data. I set the risk premium of the assets of the representative firm, π_S, at 5% per year also. This is at the lower end of the long-term average for the equity premium, but in the model the correct counterpart of the equity premium will be $\Phi(d_1) \cdot S/V \cdot \pi_S$, which is larger than 5% because equity is a levered claim on the assets.

I achieve $\pi_S = 5\%$ by fixing the volatility of the pricing kernel, σ_Λ, at 40% per year, the volatility of the assets, σ_S, at 20% per year, and the correlation between the assets and the pricing kernel, $\rho_{S\Lambda}$, at -0.625, which yields the risk premium $\pi_S = -\rho_{S\Lambda}\sigma_\Lambda\sigma_S = 5\%$. All simulations produce similar results for other combinations of the parameters values that yield the risk premiums of 5%.

Second, I set the maturity (time to expiration) of the call option that represents equity to four years. The four years can be thought of as the assumed average duration of debt. Changing the time to expiration does not alter my results. Since my model is scale-invariant, I fix the value of assets in place, S, at 100 and let the value of debt, K, which is the strike price of the option, to vary from 0 to 120. When $K = 0$, leverage is zero. When $K = 120$, leverage (debt over debt plus equity) is about 0.8. The leverage is roughly proportional to K with some concavity if K is between 0 and 120.

The second parameter that I let vary is ω, the measure of uncertainty about the firm's assets. As ω varies from 1% to 64% per annum, the idiosyncratic volatility varies between 15% and 100% per annum.

The Magnitude of the Uncertainty Effect The graph in the top left corner of Figure 1 shows the variation in the expected return as a function of the uncertainty parameter, ω, and the value of the assets, K. First, I notice that uncertainty (the theoretical counterpart of turnover, or analyst disagreement, or turnover variability) is always negatively related to returns. The uncertainty effect varies from 6% per year for highly levered firms ($K = 100$, leverage is 0.7) to only a few basis points per year for low leverage firms ($K = 15$, leverage is 0.1).

The magnitude of the the effect of uncertainty on expected returns (the uncertainty effect) is comparable to my empirical results on the turnover effect, the analyst disagreement effect, and the turnover variability effect. Consistent with the empirical results, the uncertainty effect is small unless the firm is very highly levered.

Simulations for Proposition 1 In Proposition 1 I claim that the uncertainty effect is stronger for the highly levered firms (or, alternatively, for the firms with bad credit rating, which can be a cleaner empirical proxy for the importance of the real option created by leverage). Algebraically, it means that the second cross-derivative of the expected return with respect to uncertainty, ω, and the value of the assets, S, is positive. The more valuable are the assets (relative to the value of debt), the less levered the firm is and the less negative is the uncertainty effect, which works through the option created by leverage.

In the top left corner of Figure 1, I look at the cross-derivative graph and, expectedly, find that the derivative is positive everywhere. Hence, the effect of uncertainty on expected returns is more negative for highly levered firms. I predict that the empirical counterparts of the uncertainty effect - the turnover effect, the analyst disagreement effect, and the turnover variability effect - are stronger for highly levered firms and firms with bad credit rating.

Simulations for Proposition 2 Proposition 2 asserts that the elasticity of the risk premium with respect to uncertainty decreases in uncertainty. I use this fact to state that the increase in the expected risk premium in recessions, when uncertainty is high, is the smallest for high uncertainty firms. Proposition 2 implies that these firms have lower betas in recessions and the value of these firms decreases the least when the economy slides into recession. In the papers mentioned in the title of this document, I use this fact to predict that high uncertainty firms (high turnover firms, high analyst disagreement firms, high turnover variability firms) hedge against aggregate volatility risk and have procyclical market betas.

In the simulations, I need to determine the sign of the derivative of the elasticity with respect to uncertainty. The left graph in the middle of Figure 1 shows that the elasticity indeed declines (increases in the absolute magnitude) in uncertainty for all values of ω and K. Also from simulations (unplotted) I learn that the elasticity can reach -0.7. Given that the 20% increase in uncertainty is not uncommon in recessions (see Table 1 in Barinov, 2009), the expected risk premium of high uncertainty growth firms can easily be cut by 15% in bad times compared to what it could have been in the static CAPM.

The second assertion in Proposition 2 is that the elasticity of the risk premium with respect to uncertainty decreases in both uncertainty and leverage (and increases in credit rating, which is an alternative empirical proxy for the importance of the real option created by leverage). That is, the second cross-derivative of the elasticity with respect to the value of the assets, S, and uncertainty, ω, is positive (low S means high leverage).

In the right graph in the middle of Figure 1 I plot the cross-derivative of the elasticity with respect to uncertainty and the value of the assets. I see in the graph that the derivative is positive almost everywhere, but it turns negative for the firms with very high uncertainty (higher than 50% per annum) and very high leverage (higher than 0.5), clearly a small set of firms. Even then, the graph of the elasticity itself (unreported) shows that high uncertainty, high leverage firms do have the large negative elasticity of the risk premium with respect to uncertainty, which is much higher than the elasticity of most firms.

Simulations for Proposition 3 In Proposition 3 I look at the elasticity of the firm value with respect to uncertainty, which is always positive, because higher uncertainty increases the value of the option created by risky debt. I claim that the elasticity increases with uncertainty and leverage. Algebraically, it means that the derivative of the elasticity with respect to uncertainty, ω, is positive, and the cross-derivative with respect to uncertainty and the value of the assets, S, should be negative.

Economically, the sign of the first derivative means that the value of high uncertainty firms increases as uncertainty increases and the economy slides into recession. According to the cross-derivative, the increase in the value of high uncertainty firms is even stronger if these firms are also highly levered. In the papers listed in the title, I use Proposition 3 as another way to explain why high turnover firms, high disagreement firms, and high turnover variability firms are a hedge against aggregate volatility risk, and why this hedging ability increases with leverage and decreases with credit rating.

In the bottom left graph in Figure 1 I plot the derivative of the firm's value elasticity with respect to uncertainty and find that it is always positive. It the bottom right graph in Figure 1 I plot the cross-derivative of the elasticity with respect to uncertainty and the value of the assets. This cross-derivative is also negative almost everywhere, as predicted. The cross-derivative is positive for a small subset of firms with idiosyncratic volatility exceeding 60% per annum and leverage exceeding 0.65, but even for these firms the elasticity of the firm value with respect to uncertainty is much higher than for the firms with low leverage or low uncertainty. I conclude therefore that when uncertainty increases (and everything else remains the same), high uncertainty firms, especially high uncertainty, high leverage firms and high uncertainty, bad credit rating firms, gain more than other firms.

The graph of the elasticity of the firm value with respect to idiosyncratic volatility (unplotted) also suggests that the elasticity is substantial. The elasticity values of 0.5 and higher are not unusual and start at reasonable parameter values. The elasticity of 0.5 implies that the volatility increase in recessions can increase the firm value by 10%, just because the option created by risky debt is more valuable in an uncertain environment.

3.2 The Model with Growth Options

Parameter Values In the simulations, I fix two sets of parameter values. The first set is the moments of the three processes: the pricing kernel, Λ_t, the value of the assets in place, B_t, and the value of the asset behind the growth options, G_t. The values of the parameters are chosen so that the value effect roughly matches its empirical magnitude (about 6% per year). In the current setup, to keep things simple, I assume that the difference in expected returns between B_t and G_t is large enough to produce the positive value effect. It turns out that because the growth options are a highly levered claim on G_t, I have to assume quite large difference in the expected returns to G_t and B_t. While my story can informally explain the value effect, it is still outside of the current model.

I fix the volatility of the pricing kernel, σ_Λ, at 50% per year, the volatility of the asset behind the growth options, σ_G, at 10% per year, and the correlation between the asset behind the growth options and the pricing kernel, $\rho_{G\Lambda}$, at -0.8, which yields the risk premium $\pi_G = -\rho_{G\Lambda}\sigma_\Lambda\sigma_G = 4\%$. I also fix the volatility of the assets in place, σ_B, at 40% per year, and their correlation with the pricing kernel, $\rho_{B\Lambda}$, at -0.7, which yields the risk premium $\pi_B = -\rho_{B\Lambda}\sigma_\Lambda\sigma_B = 14\%$. All simulations produce similar results for other combinations of the parameters values that yield the risk premiums of 4% and 14%.

The second set of parameters describes the growth options. I assume that the current value of the asset behind them, G, is 100 and the strike price, I, is 90. My model is scale-invariant, so these values only mean that the growth options are slightly in the money. The expiration period is set at 4 years. In unreported results, I play with the assumed maturity and the assumed moneyness of the growth options and conclude that my results are robust to reasonable variations in the maturity and the moneyness.

The two other parameters that vary freely in my tests are ω, the measure of uncertainty about the asset behind the growth options, and B, the value of the assets in place. Varying these two parameters gives me a rich cross-section in terms of idiosyncratic volatility and market-to-book, V/B. As ω varies from 1% to 64% per annum, and B varies from 10 to 150, the idiosyncratic volatility varies between 20% and 80% per annum, and the market-to-book varies from 1.5 to above 6[4].

The Magnitude of the Uncertainty Effect The graph in the top left corner of Figure 2 shows the variation in the expected return as a function of the uncertainty parameter, ω, and the value of the assets in place, B. First, I notice that uncertainty (the theoretical counterpart of turnover, or analyst disagreement, or turnover variability) is always negatively related to returns. For the median firm with the market-to-book of 2, the uncertainty effect is around 3% per annum. The uncertainty effect varies from 5% per year for growth firms ($B = 10$, $V/B \in [4,\ 7]$) to 1% per year for value firms ($B = 150$, $V/B \in [1.2,\ 1.4]$).

Overall, my model produces numerically large effects of uncertainty on expected returns. These effects are smaller than their empirical counterparts (the turnover effect, the analyst disagreement effect), because the simulations do not account for the aggregate volatility risk. I also fix the baseline parameters quite conservatively. For example, for some firms in the data the risk premium spread between the assets in place and the asset behind the growth options can be larger, which would magnify the uncertainty effects.

Simulations for Proposition 4 In Proposition 4 I claim that the uncertainty effect is stronger for the firms with abundant growth options (high market-to-book). Algebraically, it means that the second cross-derivative of the expected return with respect to uncertainty, ω, and the value of the assets in place, B, is positive. The more assets in place the firm has, the weaker is the negative relation between the expected return and the uncertainty, because the uncertainty effect works through the growth options.

In the top left corner of Figure 2, I look at the cross-derivative graph and, expectedly, find that the derivative is positive everywhere. Hence, the effect of uncertainty on expected returns is more negative for growth firms. I predict that the empirical counterparts of the

[4]The lowest possible value of market-to-book in my model is 1. The market value, or the firm value V_t, differs from the book value, or the value of the assets in place B_t, by the always positive value of the growth options, P_t.

uncertainty effect - the turnover effect, the analyst disagreement effect, and the turnover variability effect - are stronger for growth firms.

Simulations for Proposition 5 Proposition 5 asserts that the elasticity of the risk premium with respect to uncertainty decreases in uncertainty. I use this fact to state that the increase in the expected risk premium in recessions, when uncertainty is high, is the smallest for high uncertainty firms. Proposition 5 implies that these firms have lower betas in recessions and the value of these firms decreases the least when the economy slides into recession. In the papers mentioned in the title of this document, I use this fact to predict that high uncertainty firms (high turnover firms, high analyst disagreement firms, high turnover variability firms) hedge against aggregate volatility risk and have procyclical market betas.

In the simulations, I need to determine the sign of the derivative of the elasticity with respect to uncertainty. The left graph in the middle of Figure 2 shows that indeed the elasticity declines (increases in the absolute magnitude) in uncertainty. Also from simulations (unplotted) I learn that the elasticity can reach -0.5. Given that the 20% increase in uncertainty is not uncommon in recessions (see Table 1 in Barinov, 2009), the expected risk premium of high uncertainty growth firms can easily be cut by 10% in bad times compared to what it could have been in the static CAPM.

The second assertion in Proposition 5 is that the elasticity of the risk premium with respect to uncertainty decreases in both uncertainty and market-to-book. That is, the second cross-derivative of the elasticity with respect to assets in place, B, and uncertainty, ω, is positive (low B means high market-to-book).

In the right graph in the middle of Figure 2 I plot the cross-derivative of the elasticity with respect to uncertainty and assets in place. I see in the graph that the derivative is positive almost everywhere, but it turns negative for the firms with very high uncertainty and abundant real options. The derivative can in fact be negative for either idiosyncratic volatility topping 80% per year and market-to-book topping 3, or idiosyncratic volatility exceeding 30% per year and market-to-book exceeding 3.5, clearly a small set of firms. Even then, the graph of the elasticity itself (unreported) shows that high uncertainty growth firms do have the large negative elasticity of the risk premium with respect to uncertainty, which is much higher than the elasticity of most firms.

Simulations for Proposition 6 In Proposition 6 I look at the elasticity of the firm value with respect to uncertainty, which is always positive, because higher uncertainty increases the value of growth options. I claim that the elasticity increases with uncertainty and market-to-book. Algebraically, it means that the derivative of the elasticity with respect to uncertainty, ω, is positive, and the cross-derivative with respect to uncertainty and the value of the assets in place, B, should be negative.

Economically, the sign of the first derivative means that the value of high uncertainty firms increases as uncertainty increases and the economy slides into recession. According to the cross-derivative, the increase in the value of high uncertainty firms is even stronger if these firms also possess valuable growth options. In the papers listed in the title, I use Proposition 6 as another way to explain why high turnover firms, high disagreement firms, and high turnover variability firms are a hedge against aggregate volatility risk, and why this hedging ability increases in market-to-book.

In the bottom left graph in Figure 2 I plot the derivative of the firm's value elasticity with respect to uncertainty and find that it is always positive. It the bottom right graph in Figure 2 I plot the cross-derivative of the elasticity with respect to uncertainty and the value of the assets in place. This cross-derivative is also negative everywhere, as predicted. I conclude therefore that when uncertainty increases (and everything else remains the same), high uncertainty firms, especially high uncertainty growth firms, gain more than other firms.

The graph of the elasticity of the firm value with respect to idiosyncratic volatility (unplotted) also suggests that the elasticity is substantial. The elasticity values of 0.5 and higher are not unusual and start at reasonable parameter values. The elasticity of 0.5 implies that the volatility increase in recessions can increase the firm value by 10%, just because growth options are more valuable in a uncertain environment.

References

[1] Barinov, Alexander, 2010, Idiosyncratic Volatility, Growth Options, and the Cross-Section of Returns, *Working paper*, Univesity of Georgia

[2] Barinov, Alexander, 2009, Analyst Disagreement and Aggregate Volatility Risk, *Working paper*, Univesity of Georgia

[3] Black, Fischer, and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, v. 81, pp. 637-654

[4] Campbell, John Y., 1993, Intertemporal Asset Pricing without Consumption Data, *American Economic Review*, v. 83, pp. 487-512

[5] Campbell, John Y., Martin Lettau, Burton G. Malkiel, and Yexiao Xu, 2001, Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk, *Journal of Finance*, v. 56, pp. 1-43

[6] Campbell, John Y., and Tuomo Vuolteenaho, 2004, Good Beta, Bad Beta, *American Economic Review*, v. 94, pp. 1249-1275

[7] Chen, Joseph, 2002, Intertemporal CAPM and the Cross-Section of Stock Returns, *Working Paper*, University of Southern California

[8] Chordia, Tarun, Avanidhar Subrahmanyam, and V. Ravi Anshuman, 2001, Trading Activity and Expected Stock Returns, *Journal of Financial Economics*, v. 59, pp. 3-32

[9] Datar, Vinay T., Narayan Y. Naik, and Robert Radcliffe, 1998, Liquidity and Stock Returns: An Alternative Test, *Journal of Financial Markets*, v. 1, pp. 203-219

[10] Diether, Karl, Christopher Malloy, and Anna Scherbina, 2002, Differences of Opinion and the Cross-Section of Returns, *Journal of Finance*, v. 57, pp. 2113-2141

[11] Galai, Dan, and Ronald W. Masulis, 1976, The Option Pricing Model and the Risk Factor of Stock, *Journal of Financial Economics*, v. 3, pp. 53-81

[12] Grullon, Gustavo, Evgeny Lyandres, and Alexei Zhdanov, 2007, Real Options, Uncertainty, and Stock Returns, *Working Paper*, Rice University and University of Lausanne

[13] Johnson, Timothy C., 2004, Forecast Dispersion and the Cross-Section of Expected Returns, *Journal of Finance*, v. 59, pp. 1957-1978

[14] Zhang, Lu, 2005, The Value Premium, *Journal of Finance*, v. 60, pp. 67-103

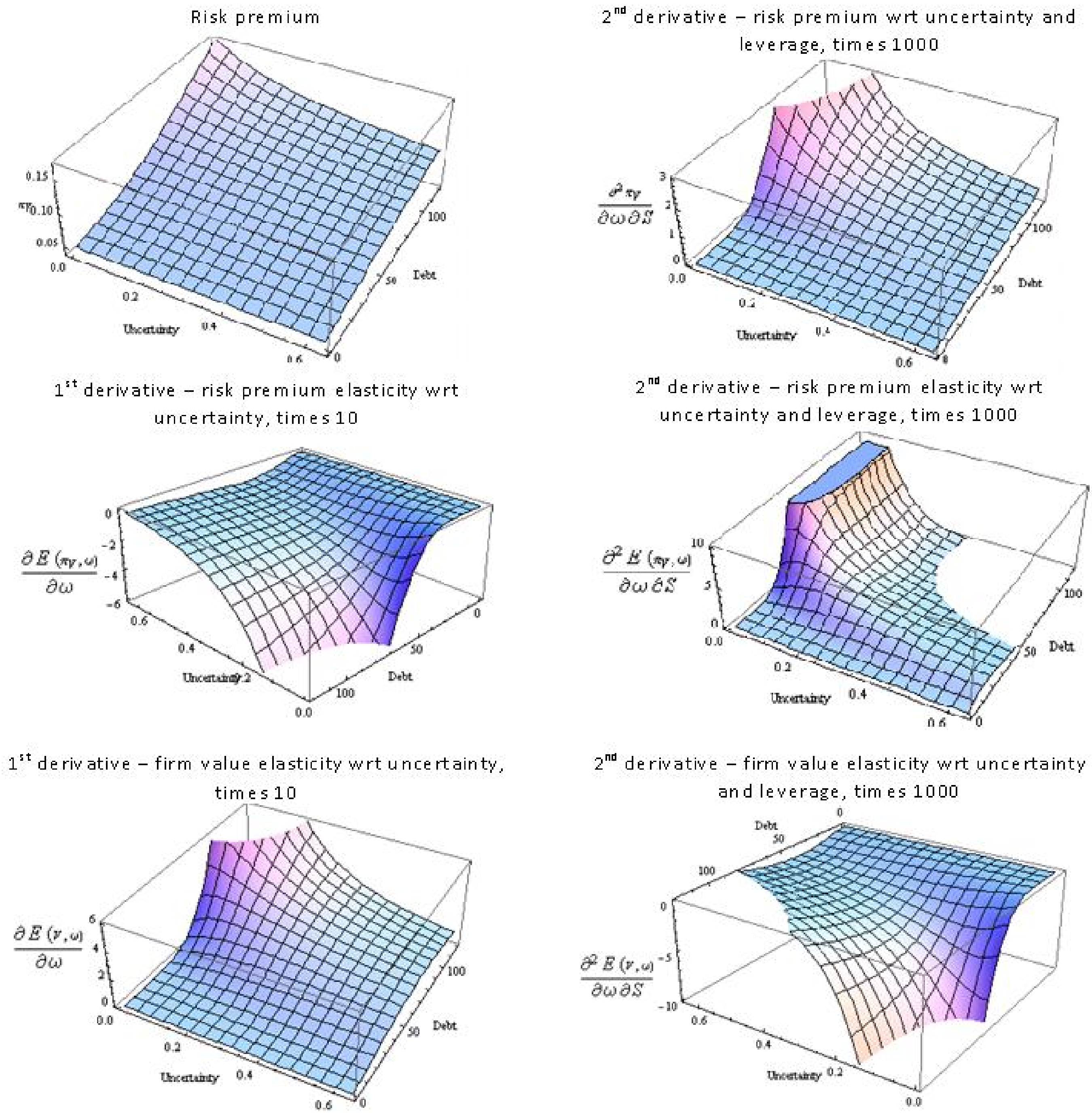

Figure 1. Uncertainty, Leverage, and Expected Returns

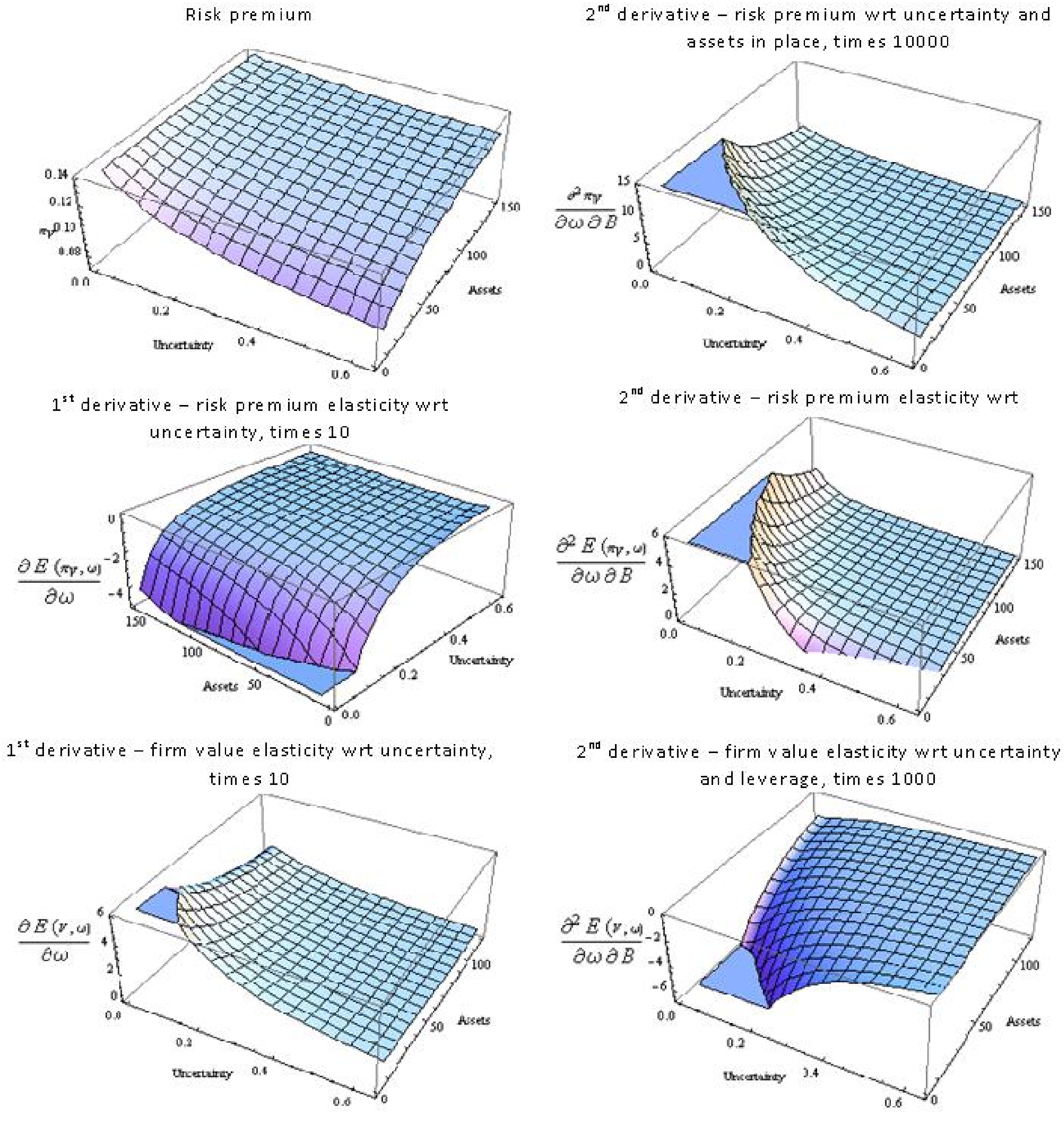

Figure 2. Uncertainty, Market-to-Book, and Expected Returns