Theory Appendix to

The Bright Side of Distress Risk

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This version: May 2021

Abstract

This document contains the formal derivation of the aggregate volatility explanation of the distress risk puzzle in "The Bright Side of Distress Risk". Section 1 of this document presents the setup of the model and derives the main predictions. Section 2 collects the proofs of the propositions in Section 1. Section 3 presents simulations to back up the proofs and to evaluate the magnitude of the effects in the model.

1 The Real Options Model

Johnson (2004) and Barinov (2008) develop and successfully test two related models that predict and explain the negative relation between option-likeness of equity and expected returns. The main idea of these models is that real options can transform firm-specific uncertainty into lower systematic risk. In the Johnson setup, the natural empirical proxy for uncertainty is the dispersion of analyst forecasts, which measures analyst disagreement. In the Barinov model, which extends the Johnson model by adding the time-series dimension and highlighting the second channel linking leverage to aggregate volatility risk (higher value of levered firms, all else equal, in periods of high volatility) the empirical proxy for uncertainty is idiosyncratic volatility. In this section, I use the Johnson setup, but the model can be restated in terms of idiosyncratic volatility. I extend the Johnson model by showing that high firm-specific uncertainty means lower aggregate volatility risk (as Barinov, 2008, also does in his setup with growth options instead of the option created by leverage).

The model is cast in terms of leverage; however, the main economic mechanism in the paper is limited liability, which creates the convexity in the firm value. The decision to declare bankruptcy is not modelled explicitly either in Johnson (2004) or Barinov (2008), but it is intuitive that if a highly levered company is unlikely to go bankrupt, the convexity in its value created by leverage is not as important. Hence, the empirical tests in the paper use measures of distress (credit rating and O-score) rather than leverage, and "highly levered" and "distressed" firms are treated in the discussion below as synonyms.

Empirical proxies for firm-specific uncertainty are highly correlated; while in the model firm-specific uncertainty is uncertainty about the value of underlying assets, in empirical tests I use idiosyncratic volatility of firm returns (i.e., standard deviation of the CAPM residuals). Barinov (2008) has a formal proof that in the model idiosyncratic volatility of firm returns is monotonically related to uncertainty about the underlying asset of real options the firm owns. The empirical Robustness Appendix unlevers firm-specific uncertainty measures the same way one unlevers market beta and also orthogonalizes them to leverage, arriving at the conclusion that in the data assets of distressed firms have higher uncertainty measures than assets of healthy firms.

1.1 Leverage and Uncertainty in the Johnson Model

Consider a firm with unobservable true value of assets C_t that has issued risky debt with the face value K. Asssume that the true value of assets follows

$$dC_t = \mu_C C_t dt + \sigma_C C_t dW_C \tag{1}$$

Investors cannot observe the process C_t (the true value of assets) and observe U_t instead, which is C_t contaminated by a stationary noise process η_t. U_t is given by

$$U_t = C_t \cdot \exp(\eta_t) \tag{2}$$

The noise process η_t is an unobservable stationary diffusion process

$$d\eta_t = -\kappa \eta_t dt + \sigma_\eta dW_\eta \tag{3}$$

In addition to U_t, investors observe the stochastic discount factor Λ_t that follows

$$d\Lambda_t = -r\Lambda_t dt + \sigma_\Lambda \Lambda_t dW_\Lambda \tag{4}$$

Johnson (2004) shows that in this economy S_t, the observable price of unlevered claim on C_T (firm's assets)[1], and V_t, the observed value of equity, follow

$$dS_t = (r + \pi_S) S_t dt + \sigma_C S_t d\widetilde{W}_C \tag{5}$$

$$V_t = S_t \Phi(d_1) - \exp(r(T-t)) K \Phi(d_2) \tag{6}$$

where $\pi_S = -\rho_{C\Lambda}\sigma_C\sigma_\Lambda$ is the risk premium, $\Phi(\cdot)$ is the normal cdf,

$$d_1 = \frac{log(S/K) + (r(T-t) + \widetilde{\sigma}^2/2)}{\widetilde{\sigma}}, \; d_2 = d_1 - \widetilde{\sigma} \tag{7}$$

$$\widetilde{\sigma}^2 = \omega + \sigma_C^2 (T-t) \tag{8}$$

$d\widetilde{W}_C$ is the posterior belief of investors about the process governing C_t given the signals they have received, and ω reflects firm-specific uncertainty. Johnson (2004) also shows that

$$d\widetilde{W}_C = F(X)(\sigma_C dW_C + \sigma_\eta dW_\eta) + G(X) dW_\Lambda, \tag{9}$$

[1] Note that C_t is the true (unobservable) value of the underlying asset, and S_t is the observable value of the underlying asset, which moves according to the information about the underlying asset investors are able to filter out of the price and the economy structure.

where $F(X)$ and $G(X)$ are some functions of the model primitives κ, σ_V, σ_C, σ_Λ, $\rho_{C\Lambda}$. (Explicit expressions of $F(X)$ and $G(X)$ are given in Johnson, 2004).

The expression (8) for the total volatility shows that in the model the uncertainty is resolved all at once at the last instant, when S_T jumps to V_T. It is the reason why the firm-specific uncertainty, which represents the jump risk for the underlying asset, is priced only for the levered claim. Johnson (2004) notes that the fact that the risk measured by ω is resolved by a jump means that this risk is truly idiosyncratic.

Proposition 1. The risk premium of the firm equals

$$\pi_V = \pi_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{10}$$

and its derivatives with respect to firm-level uncertainty and the assets value have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{11}$$

$$\frac{\partial \pi_V}{\partial S} = \pi_S \cdot \frac{\partial}{\partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{12}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial S} = \pi_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{13}$$

Proof: See Section 2.

The sign of (11) implies that firm-specific uncertainty is negatively related to expected returns, which is the main result established in Johnson (2004). The intuition for the sign is that the elasticity of the call option with respect to the underlying asset value, $\Phi(d_1)S_t/V_t$, decreases in uncertainty about the underlying asset, because more uncertainty about the underlying asset means that its current value is less informative about the value of the option at the expiration date. Therefore, the current value of the option responds less to the same percentage change in the current value of the underlying asset if there is more uncertainty about the underlying asset's true value.

Johnson (2004) also notices that the effect of firm-specific uncertainty ω on the firm's expected returns should be stronger for highly levered firms. The intuition is that uncertainty derives its pricing impact from the fact that a levered firm is a call option on the assets, and therefore volatility should matter more for highly levered firms. In algebraic terms this means that as the assets value increases (or as the face value of the debt decreases, or, equivalently, as leverage decreases), the first derivative (11) becomes less negative, i.e. (13) is positive. Evidently, as the face value of the debt and therefore leverage reach zero, (11) also reaches zero.

Higher leverage in the Johnson model makes equity more risky by levering up its beta, as the negative sign of (12) indicates, but the caveat is that (12) captures a partial derivative effect. The empirical correlation between leverage and expected returns that ignores (13) can become negative in the data if the cross effect captured by (13) is large enough and if highly levered firms are disproportionately volatile. In other words, high leverage per se makes the firm more risky, but the combination of high leverage and high uncertainty makes the firm less risky for the reasons described in the previous two paragraphs, and controlling for the cross-effect is vital for discovering the true positive relation between leverage and expected returns.

As mentioned in the introductory paragraphs, in the empirical tests I proxy for S by distress measures, based on the idea that it is limited liability and probability of bankruptcy that matter, and leaning on empirical evidence (see Panel A of Table 1 in the paper) that leverage more than triples as one goes from the best to the worst credit rating quintile and thus empirically credit rating and leverage can be close substitutes. The empirical proxy for ω is idiosyncratic volatility of firm returns, based on the result in Barinov (2008) that establishes a monotonic theoretical relation between the two, and the empirical evidence in Robustness Appendix (see Tables 9A and 10A) that distressed firms have higher uncertainty measures even after leverage is controlled for.

1.2 Extending the Johnson Model

In this subsection I extend the Johnson (2004) model to show that the expected returns effects he finds (Proposition 1 above) arise because changes in firm-specific uncertainty create changes in systematic risk, namely, in aggregate volatility risk.

Corollary 1. The equity beta equals

$$\beta_V = \beta_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{14}$$

and its derivatives have the following signs:

$$\frac{\partial \beta_V}{\partial \omega} = \beta_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{15}$$

$$\frac{\partial \beta_V}{\partial S} = \beta_S \cdot \frac{\partial}{\partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{16}$$

$$\frac{\partial^2 \beta_V}{\partial \omega \partial S} = \beta_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{17}$$

Proof: See Section 2.

Corollary 1 stresses the fact that while firm-specific uncertainty creates only idiosyncratic risk at the level of firm assets and does not change their systematic risk, this idiosyncratic risk reduces systematic risk of equity, which is a call option on assets, by making it less responsive to changes in the assets value. This is the reason why the idiosyncratic risk created by firm-specific uncertainty is diversifiable at the level of assets, but is not diversifiable at the level of firm equity.

Hypothesis 1a *Future returns are negatively related to idiosyncratic volatility, and this effect is stronger for distressed firms.*

Hypothesis 1b *Future returns are positively related to distress if idiosyncratic volatility is low/kept constant. If distressed firms are significantly more volatile than financially healthy firms, then future returns are negatively related to distress. This negative relation is stronger in the high idiosyncratic volatility subsample.*

The negative correlation between firm-specific uncertainty and the exposure of the firm's equity to systematic risk is useful during periods of high aggregate volatility. These periods usually coincide with the periods of high idiosyncratic volatility and high dispersion of analyst forecasts (see Barinov, 2013, and references therein). The next proposition shows that, all else equal, the increased uncertainty makes the risk premium of high uncertainty, distressed firms increase less and makes their value drop less during periods of aggregate volatility.

Proposition 2 The elasticity of the equity risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{18}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is positive:

$$\frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) > 0 \tag{19}$$

Similar statements are true with respect to market beta:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \beta_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{20}$$

$$\frac{\partial^2}{\partial\omega\partial S}\Big(\frac{\partial\beta_V}{\partial\omega}\cdot\frac{\omega}{\pi_V}\Big) > 0 \tag{21}$$

Proof: See Section 2.

Hypothesis 2a *Market beta of firms with higher idiosyncratic volatility are more procyclical, which makes them less risky in the Conditional CAPM sense, and this effect is stronger for distressed firms.*

Hypothesis 2b *If distressed firms are significantly more volatile than financially healthy firms, then market betas of more distressed firms will be more procyclical. This relation is stronger in the high idiosyncratic volatility subsample.*

As Campbell (1993) and Chen (2002) show, investors require lower risk premium from the stocks that react less negatively to aggregate volatility increases. Proposition 2 implies that the firm's exposure to aggregate volatility risk decreases with firm-specific uncertainty, since lower beta in bad periods of time (which are usually periods of high aggregate volatility, as Barinov (2013), Bartram et al. (2016) and Herskovic et al. (2016) show) implies a smaller increase in the cost of capital and consequently a smaller drop in the present value of future cash flows, i.e., the value of equity. (Firms with procyclical (low in recessions) betas can still have higher cost of capital in recessions than in expansions, since the cost of capital is the market beta times the market risk premium, and the market risk premium is higher in recessions).

Because the uncertainty is transformed into lower aggregate volatility risk through the real option created by leverage, distressed firms with highest firm-specific uncertainty have the most procyclical market betas and the best hedging ability against aggregate volatility risk.

Similar to the previous section, the interaction between leverage and firm-specific uncertainty in the previous paragraph suggests that if distressed firms are significantly more volatile than financially healthy firms, then distressed firms can have more procyclical market beta and lower exposure to aggregate volatility risk, and this effect will be stronger in the subsample with high firm-specific uncertainty.

Hypothesis 3a *In cross-section, exposure to aggregate volatility risk decreases (beta with respect to a portfolio tracking changes in aggregate volatility becomes more positive) as idiosyncratic volatility increases. This decrease in risk exposure is stronger for distressed firms.*

Hypothesis 3b *If distressed firms are significantly more volatile than financially healthy firms, then exposure to aggregate volatility risk of more distressed firms will be smaller (and their beta with respect to a portfolio tracking changes in aggregate volatility will be more positive). These relations are stronger in the high idiosyncratic volatility subsample.*

1.3 The Second Channel Linking Option-Likeness and Aggregate Volatility Risk

Another effect that ties real options and aggregate volatility risk comes from the fact that, all else equal, higher uncertainty means higher value of the real option[2]. When both aggregate volatility and firm-specific uncertainty increase, this effect makes the value of the real option created by leverage increase in value (holding other effects fixed). In Proposition 3, I show that, holding constant all other (usually negative) cash flow effects of the aggregate volatility increase, the positive effect of uncertainty on the real option value is larger for high uncertainty firms, especially if they are also distressed.

Proposition 3 The elasticity of the equity value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial\omega}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) > 0 \tag{22}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is negative:

$$\frac{\partial^2}{\partial\omega\partial S}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) < 0 \tag{23}$$

Proof: See Section 2.

The second cross-derivative result is different from the well-known result from the options literature that vega ($\frac{\partial V}{\partial\omega}$) reaches its maximum when the option is at the money. Proposition 3 looks at the elasticity of the option's/firm's value with respect to the volatility parameter ω, and the elasticity is the vega times ω divided by the option's/firm's value. Proposition 3 then shows that this elasticity behaves differently from vega.

In the paper, I define aggregate volatility factor, FVIX, as the portfolio that tracks changes in expected aggregate volatility (changes in the VIX index). The positive exposure

[2] A recent paper by Grullon, Lyandres, and Zhdanov (2012) presents supporting empirical evidence.

to aggregate volatility factor is then desirable, because it means (relative) gains in response to aggregate volatility increases. I can formulate the following empirical hypotheses, similar to Hypothesis 3a and 3b above, but stemming from a different channel:

Hypothesis 4a *In cross-section, exposure to aggregate volatility risk decreases and FVIX beta becomes more positive as idiosyncratic volatility increases. This decrease in risk exposure and increase in FVIX beta is stronger for distressed firms.*

Hypothesis 4b *If distressed firms are significantly more volatile than financially healthy firms, then exposure of more distressed firms to aggregate volatility risk will be smaller and their FVIX beta will be more positive. Both effects will be stronger in the high idiosyncratic volatility subsample.*

I would like to stress that Propositions 2 and 3 are formulated in terms of partial derivatives. It is beyond doubt that, as almost all risky assets, distressed firms lose value when the market goes down and aggregate volatility increases. During these periods, the risk premium of distressed firms also increases. Moreover, since equity of distressed firms is a levered claim on their assets, the negative reaction of distressed firms to an increase in aggregate volatility can be stronger than average. What Propositions 2 and 3 state is that all else equal, distressed firms and high uncertainty react to aggregate volatility increases less negatively than other firms with similar market betas.

Using credit rating and O-score as empirical proxies for importance of optionality created by leverage and idiosyncratic volatility of firm returns as an empirical proxy for firm-specific uncertainty, I can formulate the following empirical hypothesis:

Hypothesis 4c *When aggregate volatility increases, distressed firms lose value, but beat the CAPM, which is indicated by their positive FVIX beta. This effect is stronger in the high idiosyncratic volatility subsample.*

1.4 Leverage and Growth Options in One Model

The vast majority of firms have both debt and growth options. Therefore, for a typical firm higher firm-specific uncertainty should create a hedge against aggregate volatility risk both through the above-described leverage channel and the growth options channel studied in Barinov (2008) along similar lines and with similar results (e.g., Barinov (2008) shows in his model that growth firms are less exposed to aggregate volatility risk, this exposure is even smaller for volatile growth firms, and even zero-leverage firms with high idiosyncratic volatility can have low exposure to aggregate volatility risk if the said firms have growth options).

Leverage and market-to-book are strongly negatively correlated in the data, both mechanically via market cap and economically via the free cash flow story and the underinvestment story. Thus, the growth options channel will work, empirically, against the mechanism described above: as Barinov (2008) suggests, growth firms are less exposed to aggregate volatility risk, but if distressed firms are more likely to be value firms, they should be more exposed to aggregate volatility risk, while hypotheses 3b and 4b above say otherwise. Still, one may wonder if both channels can coexist in my model. In this subsection, for the sake of brevity, I will provide the intuition of why the conclusions will be the same if we study both real options simultaneously.

There are two ways to introduce both growth options and leverage into one model. First, one can assume that only assets in place are pledgable and leverage creates a call option on their value only, while growth options are written on some other "future projects". If this is the case, the firm will consist of two separate real options. When aggregate volatility and the firm-specific uncertainty simultaneously increase (as they tend to do), the risk of both options will decrease, all else equal, and the value of both options will increase. Hence, in the model with both leverage and growth options more firm-specific uncertainty will still mean less aggregate volatility risk. More leverage will mean higher relative weight of the option created by leverage and will make that option more sensitive to changes in uncertainty. Thus, distressed firms with large firm-specific uncertainty will have lower aggregate volatility risk.

The second way to model leverage and growth options jointly is to assume that the whole firm value is pledgable and leverage creates a call option on the sum of growth options and assets in place. Assume for a minute that the firm has no assets in place and the call option created by leverage is written on growth options only. All else equal, if uncertainty increases, the risk of growth options decreases and their value increases. The same happens to the value of the firm, because growth options are in turn an underlying asset for the call option created by leverage, and an increase in their value and decrease in their risk means the same for the call option on them. On top of that, all else equal, higher uncertainty would lower the risk of the call option created by leverage and increase its value even if nothing happened to the growth options. Hence, the model is effectively reduced to the model described above, with the only difference that S stands for the value of growth options, but the results in the model above are preserved.

If I add assets in place back to this version of the model, the results are still similar. If, all else equal, an increase in uncertainty lowers the risk of the mix of assets in place and growth options and increases the total value of the mix (as Barinov, 2008, shows), the equity value, i.e. the value of the call option on the mix, will go up, and the equity risk will go down. On top of that, all else equal, the equity value would increase with uncertainty and the equity risk would decrease with uncertainty even if the risk and the value of the mix stayed constant.

It is easy to see that one can introduce the recovery rate into the model without changing the results. For example, it is possible to assume that only half of the growth options' value is recovered after the default, and the other half is either destroyed, or accrues to shareholders. The asset-pricing implications of my model will still be the same under both scenarios.

To sum up, I conclude that it is possible to unite the model in the sections above and the model in Barinov (2008) into one model without changing any of the conclusions. The second setup, with growth options being at least partially pledgable, is also likely to lead to conclusion that leverage/distress creates greater hedge against aggregate volatility risk for firms with abundant growth options, because for such firms the option created by leverage is an option on an option.

Hypothesis 5 *In the subsample of firms with abundant growth options (e.g., firms with high market-to-book) exposure of more distressed firms to aggregate volatility risk will be even smaller than usual and their FVIX beta will be more positive than in the subsample of firms without growth options (e.g., value firms).*

2 Proofs

Proposition 1. The risk premium of the firm equals

$$\pi_V = \pi_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V} \tag{24}$$

and its derivatives with respect to firm-level uncertainty and the assets value have the following signs:

$$\frac{\partial \pi_V}{\partial \omega} = \pi_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{25}$$

$$\frac{\partial \pi_V}{\partial S} = \pi_S \cdot \frac{\partial}{\partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{26}$$

$$\frac{\partial^2 \pi_V}{\partial \omega \partial S} = \pi_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{27}$$

Proof: The risk-premium equation (24) follows from a straightforward application of Ito's lemma and no-arbitrage condition to the Black-Scholes (1973) formula (6), which gives the value of equity for a levered firm. The Ito's lemma and no-arbitrage condition applied to (6) yield

$$dV_t/V_t = (r + \pi_S \cdot \Phi(d_1)\frac{S_t}{V_t})dt + \sigma_C \Phi(d_1)\frac{S_t}{V_t} d\widetilde{W}_C \tag{28}$$

and the drift less the risk-free rate is, by definition, the risk premium.

The fact that the first derivatives of the risk premium (25) and (26) are negative is well-known. The exact expression and the reason why they are negative were first given in appendix to Galai and Masulis (1976).

The expression for the second derivative (27) is complicated and can be signed only by simulations. The simulations in Section 3 show that for all plausible parameter values (27) is positive.

Corollary 1. The equity beta equals

$$\beta_V = \beta_S \cdot \frac{\partial V}{\partial S} \cdot \frac{S}{V}, \tag{29}$$

and its derivatives have the following signs:

$$\frac{\partial \beta_V}{\partial \omega} = \beta_S \cdot \frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{30}$$

$$\frac{\partial \beta_V}{\partial S} = \beta_S \cdot \frac{\partial}{\partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) < 0 \tag{31}$$

$$\frac{\partial^2 \beta_V}{\partial \omega \partial S} = \beta_S \cdot \frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial V}{\partial S} \cdot \frac{S}{V}\Big) > 0 \tag{32}$$

Proof: The Ito's lemma implies that in discrete time

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial S^2} \cdot \sigma^2 S^2 \Delta t + \frac{\partial V}{\partial t}\Delta t \tag{33}$$

Dividing both sides of (39) by V and using the definition of returns, I obtain that when $\Delta t \to 0$

$$R_V = \frac{\Delta V}{V} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \frac{\Delta S}{S} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot R_S \tag{34}$$

By definition of beta

$$\beta_V = \frac{cov(R_V, R_M)}{Var(R_M)} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \frac{cov(R_S, R_M)}{Var(R_M)} = \frac{\partial V}{\partial S} \cdot \frac{S}{V} \cdot \beta_S \tag{35}$$

The rest of the proof copies the proof of Proposition 1.

Proposition 2 The elasticity of the equity risk premium decreases (increases in the absolute magnitude) as firm-specific uncertainty increases:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{36}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is positive:

$$\frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial \pi_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) > 0 \tag{37}$$

Similar statements are true with respect to market beta:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial \beta_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) < 0 \tag{38}$$

$$\frac{\partial^2}{\partial \omega \partial S}\Big(\frac{\partial \beta_V}{\partial \omega} \cdot \frac{\omega}{\pi_V}\Big) > 0 \tag{39}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for all reasonable parameter values the signs of (36) and (37) are negative and positive, respectively.

Proposition 3 The elasticity of the equity value with respect to firm-specific uncertainty increases with the uncertainty:

$$\frac{\partial}{\partial \omega}\Big(\frac{\partial V}{\partial \omega} \cdot \frac{\omega}{V}\Big) > 0 \tag{40}$$

The second cross-derivative of the elasticity with respect to uncertainty and the assets value is negative:

$$\frac{\partial^2}{\partial\omega\partial S}\Big(\frac{\partial V}{\partial\omega}\cdot\frac{\omega}{V}\Big) < 0 \tag{41}$$

Proof: The analytical expressions for the derivatives are very complicated and cannot be signed without simulations. The simulations in Section 3 show that for all reasonable parameter values the signs of (40) and (41) are negative and positive, respectively.

3 Simulations

3.1 Parameter Values

I fix two sets of parameter values to calibrate my model. First, I look at the risk premium and the risk-free rate. I set the risk free rate to 5% per year, close to its long-term average in the data. I set the risk premium of the assets of the representative firm, π_S, at 5% per year also. This is at the lower end of the long-term average for the equity premium, but in the model the correct counterpart of the equity premium will be $\Phi(d_1)\cdot S/V\cdot\pi_S$, which is larger than 5% because equity is a levered claim on the assets.

I achieve $\pi_S = 5\%$ by fixing the volatility of the pricing kernel, σ_Λ, at 40% per year, the volatility of the assets, σ_S, at 20% per year, and the correlation between the assets and the pricing kernel, $\rho_{S\Lambda}$, at -0.625, which yields the risk premium $\pi_S = -\rho_{S\Lambda}\sigma_\Lambda\sigma_S = 5\%$. All simulations produce similar results for other combinations of the parameters values that yield the risk premiums of 5%.

Second, I set the maturity (time to expiration) of the call option that represents equity to four years. The four years can be thought of as the assumed average duration of debt. Changing the time to expiration does not alter my results. Since my model is scale-invariant, I fix the value of assets in place, S, at 100 and let the value of debt, K, which is the strike price of the option, to vary from 0 to 120. When $K = 0$, leverage is zero. When $K = 120$, leverage (debt over debt plus equity) is about 0.8. The leverage is roughly proportional to K with some concavity if K is between 0 and 120.

The second parameter that I let vary is ω, the measure of uncertainty about the firm's assets. As ω varies from 1% to 64% per annum, the idiosyncratic volatility varies between 15% and 100% per annum. In the sample period in the paper (1986-2017), average idiosyncratic volatility (defined as average standard deviation of firm-level CAPM residuals) is at roughly 2.8% per day, 44.5% per year and median idiosyncratic volatility is at roughly 2% per day, 31% per year (10th percentile and 90th percentile are at about 10% and 90% per annum, respectively).

3.2 The Magnitude of the Uncertainty and Distress Risk Effects

The graph in the top left corner of Figure 1 shows the variation in the expected return as a function of the uncertainty parameter, ω, and the value of the assets, K. First, I notice that uncertainty (the theoretical counterpart of idiosyncratic volatility or analyst disagreement) is always negatively related to returns. The uncertainty effect varies from 6% per year for highly levered firms ($K = 100$, leverage is 0.7) to only a few basis points per year for low leverage firms ($K = 15$, leverage is 0.1).

The magnitude of the effect of uncertainty on expected returns (the uncertainty effect) is comparable to empirical results on the analyst disagreement effect (Diether et al., 2002) and the idiosyncratic volatility effect (Ang et al., 2006). Consistent with the empirical results in Johnson (2004) and Barinov (2013), the uncertainty effect is small unless the firm is very highly levered.

Looking at the simulated risk premium along the other dimension, one can see that holding uncertainty ω fixed, higher leverage leads to higher risk premium, but the effect is muted if uncertainty is fixed at a higher level. It is easy to find two points on the plane so that the one with higher leverage (and higher uncertainty) would have lower risk premium: since the risk premium in the model spikes for highly-levered, low-uncertainty firms, $K = 100$, $\omega = 0$ vs. $K = 120$, $\omega = 0.6$ is the most obvious example of the case when more highly levered firm has lower risk premium, but similar examples with less extreme values of K also abound, though they are not as obvious from the graph.

3.3 Simulations for Proposition 1

In Proposition 1, I claim that the distress risk puzzle is stronger for high uncertainty firms. Algebraically, that means that the second cross-derivative of the expected return with respect to uncertainty, ω, and the value of the assets, S, is positive. The more valuable are the assets (relative to the value of debt), the less levered the firm is and the less negative is the uncertainty effect, which works through the option created by leverage.

In the top left corner of Figure 1, I look at the cross-derivative graph and, expectedly, find that the derivative is positive everywhere.

3.4 Simulations for Proposition 2

Proposition 2 asserts that elasticity of the risk premium with respect to uncertainty decreases in uncertainty. I use this fact to state that the increase in the expected risk premium in recessions, when uncertainty is high, is the smallest for distressed firms with high idiosyncratic volatility. Proposition 2 implies that these firms have lower betas in recessions and the value of these firms decreases the least when the economy slides into recession.

In the simulations, I need to determine the sign of the derivative of the elasticity with respect to uncertainty. The left graph in the middle of Figure 1 shows that the elasticity indeed declines (increases in the absolute magnitude) in uncertainty for all values of ω and K. Also from simulations (unplotted) I learn that the elasticity can reach -0.2. Given that the 25 percentage points per annum increase in uncertainty is not uncommon in recessions (see Table 1 in Barinov, 2013), the expected risk premium of high uncertainty growth firms can easily be cut by 5 percentage points per annum in bad times compared to what it could have been in the static CAPM.

The second assertion in Proposition 2 is that the elasticity of the risk premium with respect to uncertainty decreases in both uncertainty and leverage. That is, the second cross-derivative of the elasticity with respect to the value of the assets, S, and uncertainty, ω, is positive (low S means distress/high leverage).

In the right graph in the middle of Figure 1, I plot the cross-derivative of the elasticity with respect to uncertainty and value of assets. I see in the graph that the derivative is positive almost everywhere, but it turns negative for firms with very high uncertainty (ω higher than 50% per annum) and very high leverage (higher than 0.5), clearly a small set of firms (Panel A of Table 1 in the paper shows that median leverage in the worst credit rating quintile is 0.44, i.e., only 10% of rated firms have leverage higher than 0.44). Even then, the graph of the elasticity itself (unreported) shows that high uncertainty, high leverage firms do have the large negative elasticity of the risk premium with respect to uncertainty, which is much higher than the elasticity of most firms.

3.5 Simulations for Proposition 3

In Proposition 3, I look at elasticity of firm value with respect to uncertainty, which is always positive, because higher uncertainty increases the value of the option created by risky debt. I claim that the elasticity increases with uncertainty and leverage. Algebraically, that means that the derivative of the elasticity with respect to uncertainty, ω, is positive, and the cross-derivative with respect to uncertainty and value of the assets, S, should be negative. As mentioned above, this result is different from the well-known result that option's vega reaches its maximum, when the option is at the money, because Proposition 3 looks at elasticity of firm value (which is closer to return rather than change in value), and vega is only a part of elasticity.

Economically, the sign of the first derivative means that, all else fixed, the value of high uncertainty firms increases as uncertainty increases and the economy slides into recession. According to the cross-derivative, the increase in the value of high uncertainty firms is even stronger if these firms are also distressed. In my paper, I use Proposition 3 as another way to explain why, controlling for market beta, distressed firms can be hedges against aggregate volatility risk, especially if they have high idiosyncratic volatility.

In the bottom left graph in Figure 1, I plot the derivative of firm value elasticity with respect to uncertainty and find that it is always positive. It the bottom right graph in Figure 1, I plot the cross-derivative of the elasticity with respect to uncertainty and the value of the assets. This cross-derivative is also negative almost everywhere, as predicted. The cross-derivative is positive for a small subset of firms with idiosyncratic volatility exceeding 60% per annum and leverage exceeding 0.65, but even for these firms the elasticity of the firm value with respect to uncertainty is much higher than for the firms with low leverage or low uncertainty. I conclude therefore that when uncertainty increases (and everything else remains the same), high uncertainty firms, distressed firms and especially distressed firms with high uncertainty perform better than other firms.

The graph of values of elasticity of the firm value with respect to idiosyncratic volatility (unplotted) also suggests that the elasticity is substantial. The elasticity values of 0.15 and higher are not unusual and start at reasonable parameter values. The elasticity of 0.15 implies that the 25 percentage points per annum volatility increase in recessions can increase the firm value by roughly 4%, just because the option created by risky debt is more valuable in an uncertain environment.

References

[1] Ang, Andrew, Robert J.Hodrick, Yuhan Xing, and Xiaoyan Zhang, 2006, The Cross-Section of Volatility and Expected Returns, *Journal of Finance* 61, 259–299.

[2] Barinov, Alexander, 2008, Idiosyncratic Volatility, Aggregate Volatility Risk, and the Cross-Section of Returns, *Doctoral thesis*, University of Rochester.

[3] Barinov, Alexander, Analyst Disagreement and Aggregate Volatility Risk, *Journal of Financial and Quantitative Analysis* 48, 1877–1900.

[4] Bartram, Söhnke M., Gregory Brown, and René M. Stulz, 2016, Why does Idiosyncratic Risk Increase with Market Risk? *NBER Working Paper* 22492.

[5] Black, Fischer, and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, v. 81, pp. 637-654

[6] Campbell, John Y., 1993, Intertemporal Asset Pricing without Consumption Data, *American Economic Review*, v. 83, pp. 487-512

[7] Chen, Joseph, 2002, Intertemporal CAPM and the Cross-Section of Stock Returns, *Working Paper*, University of Southern California

[8] Diether, Karl, Christopher Malloy, and Anna Scherbina, 2002, Differences of Opinion and the Cross-Section of Returns, *Journal of Finance*, v. 57, pp. 2113-2141

[9] Galai, Dan, and Ronald W. Masulis, 1976, The Option Pricing Model and the Risk Factor of Stock, *Journal of Financial Economics*, v. 3, pp. 53-81

[10] Grullon, Gustavo, Evgeny Lyandres, and Alexei Zhdanov, 2007, Real Options, Uncertainty, and Stock Returns, *Journal of Finance* 67, 1499–1537.

[11] Herskovic, Bernard, Bryan Kelly, Hanno Lustig, and Stijn van Nieuwerburgh, 2016, The Common Factor in Idiosyncratic Volatility: Quantitative Asset Pricing Implications, *Journal of Financial Economics* 119, 249–283.

[12] Johnson, Timothy C., 2004, Forecast Dispersion and the Cross-Section of Expected Returns, *Journal of Finance*, v. 59, pp. 1957-1978

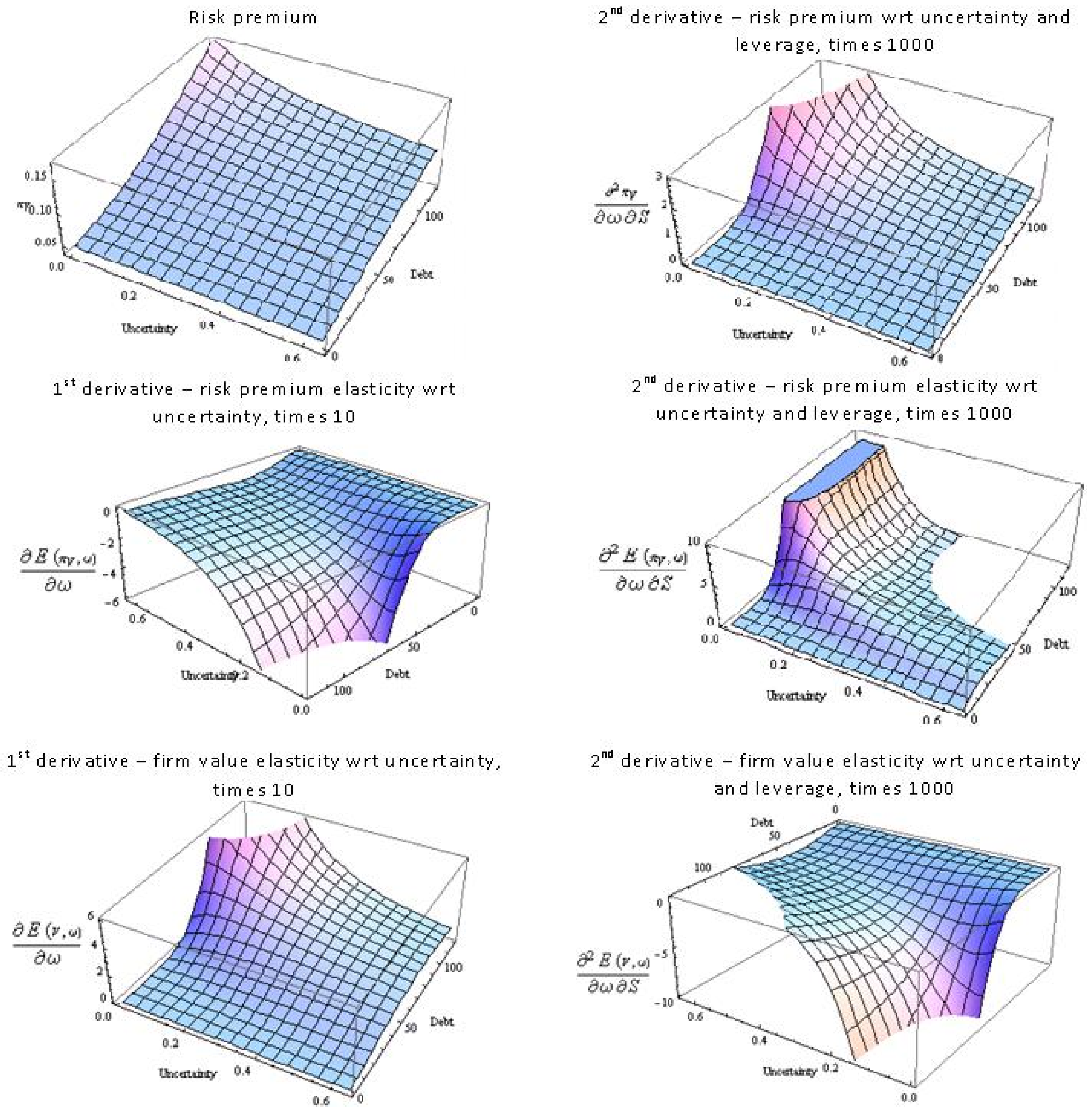

Figure 1. Uncertainty, Leverage, and Expected Returns