This paper examines the interrelations between sectoral composition of government spending and macroeconomic (in)stability in a two-sector real business cycle model with positive productive externalities in investment and a balanced-budget fiscal policy rule, whereby endogenous public expenditures are financed by the distortionary constant tax rate. Under the benchmark parameterization, our model always exhibits indeterminacy and sunspots provided the tax rate does not exceed a critical value. When the tax rate is raised to a higher level, a sufficiently high public-consumption share can destabilize the macroeconomy by generating belief-driven cyclical fluctuations. We also find that saddle-path stability and equilibrium uniqueness will prevail when the household’s labor supply elasticity is not higher than a threshold level. In addition, analytical proofs for each of the aforementioned quantitative results are provided.

Keywords: Government Spending; Distortionary Income Taxation; Equilibrium (In)determinacy.

JEL Classification: E32; E62; O41.
1 Introduction

Recently, there has been a growing literature examining the stabilization role of various fiscal policy rules within an otherwise standard real business cycle (RBC) model that exhibits multiple, indeterminate competitive equilibria under laissez faire.\(^1\) As it turns out, many previous articles have focused on the macroeconomic (in)stability effects of changing the revenue or taxation side of the national budget,\(^2\) hence left the impact of the government’s spending policies mostly underexplored.\(^3\) In our earlier work, Chang et al. (2015) study the quantitative interrelations between sectoral composition of public expenditures and equilibrium (in)determinacy in a two-sector representative-agent macroeconomy with positive productive externalities in investment. However, when government purchases of consumption and investment goods are specified as constant fractions of total public spending, our postulated setting with lump-sum taxation will not have enough equations to pin down the model’s equilibrium allocations. It is left as a topic for future research to “investigate this formulation under [the more realistic] distortionary income tax policies” (Chang et al., 2015, p. 26). Accordingly, we will address this research question in the current paper.

Our analysis begins with embedding government purchases of goods and services into a discrete-time two-sector RBC model, as in Harrison (2001), with positive productive externalities present in the investment sector. The period utility function is characterized by logarithmic in private consumption and additively separable with labor hours. Next, distortionary income taxation is incorporated through a stylized balanced-budget rule that is commonly adopted in the existing literature: Guo and Harrison (2004, henceforth GH) consider endogenous government spending financed by a constant tax rate levied on the household’s total income. This analytical framework allows us to examine how the sectoral distribution of public expenditures affects the economy’s local stability properties under parameter values that are consistent with post Korean-war U.S. time series data.

Generally speaking, our model’s equilibrium dynamics are governed by the relative strength of two opposing effects on the household’s intertemporal private-consumption Euler equation. Start the laissez-faire economy from its non-stochastic steady state, and suppose that agents

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\(^1\)See Benhabib and Farmer (1999) for an excellent survey on this RBC-based indeterminacy literature in which the terms “animal spirits”, “sunspots” and “self-fulfilling prophecies” are used interchangeably.


\(^3\)Previous macroeconomics research that has examined the (de)stabilizing effects of public expenditures includes Schmitt-Grohé and Uribe (1997, section III), Raurich (2001), Gokan (2006), and Lloyd-Braga, Modesto and Seegmuller (2008), among others.
become optimistic about the economy’s future. Acting upon this belief, the representative household will consume less and invest more today, thus raising next period’s capital stock and output. On the one hand, an increase in today’s private investment expenditures will decrease next period’s real interest rate because of diminishing marginal product of capital (the MPK effect). This channel in turn invalidates agents’ initial optimism. On the other hand, due to the presence of positive externalities from producing investment goods, the social production possibility frontier that traces out the trade-off between private consumption and investment spending is convex to the origin. As a result, the price effect refers to a reduction in the relative price of investment as agents’ rosy expectation reallocates more capital and labor inputs into the investment sector.

Under the benchmark parameterization, we first show that indeterminacy and sunspots always occur in the economy with “relatively low” (constant) tax rates, regardless of how the government divides its purchases between consumption and investment goods. In this environment, positive income taxation shifts the convex social production possibility frontier downward, which will induce households to reduce their optimism-driven consumption as well as investment expenditures. It follows that the aforementioned price effect that helps make for multiple equilibria becomes weaker, whereas the corresponding after-tax MPK effect is strengthened because of a lower net real interest rate. As it turns out, the price effect continues to dominate provided the income tax rate does not exceed a certain threshold. When the tax rate is increased to a higher value, the model’s steady state can either be a saddle point or a sink depending on the fraction of government spending in the consumption sector; and that the minimum level for the public-consumption share which yields aggregate instability is monotonically increasing with respect to the income tax rate. Our numerical experiments illustrate that as long as the proportion of public expenditures on consumption goods is zero or “relatively small”, the associated after-tax MPK effect quantitatively outweighs the price effect, thus leading to saddle-path stability and equilibrium uniqueness. It follows that raising the public-consumption share to be above a corresponding critical value will produce a dominating price effect. This in turn destabilizes the macroeconomy by generating expectations-driven business cycle fluctuations. Finally, we find that independent of the government’s tax and spending policies, the model’s steady state is a locally determinate saddle point when the labor-supply elasticity parameter is “sufficiently high” to generate a relatively stronger after-tax MPK effect. For the sake of analytical completeness, we also derive the explicit theoretical expressions for key model parameters which ensure that each of the aforementioned quantitative results will hold.

Regarding the comparisons between our earlier work versus this paper, Chang et al. (2015)
consider lump-sum income taxation and postulate that government purchases of consumption and investment goods are constant fractions of (i) their respective sectoral output or (ii) the economy’s total output; whereas we now examine an identical model economy under a time-invariant tax rate together with fixed propositions of public spending from the consumption and investment sectors. As a result, there exists an important difference in terms of how the sectoral distribution of government spending is formulated: the public-consumption and public-investment shares do not add up to one in Chang et al. (2015), but their sum is equal to one in the current piece. In addition, since the fundamental nature of income taxation is not the same, the quantitative findings obtained in these two papers are not directly comparable per se. For example, a novelty of this paper is its ability to examine (de)stabilization effects of (the more empirically relevant) a distortionary fiscal policy rule, while our earlier work with non-distortionary taxation cannot. Nevertheless, we have found qualitatively similar results within both settings. In particular, the minimum level of investment externalities needed for aggregate instability is monotonically increasing in the public-investment share; and the economy is less susceptible to indeterminacy and sunspots when the household’s labor supply becomes more inelastic. In sum, our research shows that in the context of a two-sector RBC model with positive productive externalities for investment, whether and how the government’s spending policies affect macroeconomic (in)stability depend crucially on the exact formulations of the underlying income taxation as well as the sectoral composition of public expenditures.

The remainder of this paper is organized as follows. Section 2 describes the economy and analyzes its equilibrium conditions under the GH fiscal policy rule. Section 3 undertakes a numerical study of local dynamics within calibrated versions of our model, supplemented by analytical proofs for all quantitative results. Section 4 concludes.

2 The Economy

We incorporate useless government purchases of goods and services into the discrete-time two-sector real business cycle (RBC) model à la Harrison (2001) under perfect foresight. Households live forever, and derive utility from consumption and leisure. The production side of the economy consists of two distinct sectors, consumption and investment. Based on the empirical findings of Harrison (2003), competitive firms in each sector use identical private technologies to produce their respective output, and positive productive externalities are present within the investment sector. The government balances its budget each period by levying distortionary income taxes to finance its expenditures. In particular, we consider the GH balanced-budget fiscal formulation whereby the proportional income tax rate is a constant
and public spending endogenously changes over time.

2.1 Firms

In the consumption sector, output is produced by competitive firms with the following constant returns-to-scale Cobb-Douglas technology:

\[ Y_{ct} = K_{ct}^\alpha L_{ct}^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1} \]

where \( K_{ct} \) and \( L_{ct} \) are the capital and labor inputs utilized in the production of consumption goods. Under the assumption that factor markets are perfectly competitive, the first-order conditions for these firms’ profit maximization are

\[ r_t = \alpha \frac{Y_{ct}}{K_{ct}} \quad \text{and} \quad w_t = (1 - \alpha) \frac{Y_{ct}}{L_{ct}}, \tag{2} \]

where \( r_t \) is the capital rental rate and \( w_t \) is the real wage rate.

Similarly, investment goods are produced by a unit measure of identical competitive firms using the private technology

\[ Y_{It} = A_t K_{It}^\alpha L_{It}^{1-\alpha}, \tag{3} \]

where \( K_{It} \) and \( L_{It} \) are physical capital and labor hours in the investment sector, and \( A_t \) denotes productive externalities that each individual firm takes as given. Moreover, \( A_t \) is postulated to take the following specification:

\[ A_t = (\bar{K}_{It}^\alpha \bar{L}_{It}^{1-\alpha})^\theta, \quad \theta \geq 0, \tag{4} \]

where \( \bar{K}_{It} \) and \( \bar{L}_{It} \) represent the within-sector average levels of capital and labor services devoted to producing investment goods, and \( \theta \) measures the degree of sector-specific externalities for investment. In a symmetric equilibrium, all firms in the investment sector make the same decisions such that \( K_{It} = \bar{K}_{It} \) and \( L_{It} = \bar{L}_{It} \), for all \( t \). As a result, (4) can be plugged into (3) to obtain the following aggregate production function for investment goods that may exhibit increasing returns-to-scale:

\[ Y_{It} = K_{It}^{\alpha(1+\theta)} L_{It}^{(1-\alpha)(1+\theta)}, \tag{5} \]

where \( \alpha(1 + \theta) < 1 \) to rule out the possibility of sustained economic growth. The first-order conditions that govern firms’ demand for capital and labor in the investment sector are

\[ r_t = \alpha \frac{p_t Y_{It}}{K_{It}} \quad \text{and} \quad w_t = (1 - \alpha) \frac{p_t Y_{It}}{L_{It}}, \tag{6} \]
where $Y_t$ refers to the social technology for investment given by (5), and $p_t$ denotes the relative price of investment to consumption goods at time $t$. Notice that firms in each sector will face the same equilibrium factor prices since capital and labor inputs are assumed to be perfectly mobile across the two production sectors.

2.2 Households

The economy is populated by a unit measure of identical infinitely-lived households. Each household maximizes its present discounted lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \left[ \log C_t - \frac{L_t^{1+\gamma}}{1+\gamma} \right], \quad 0 < \beta < 1, \text{ and } \gamma \geq 0,$$

(7)

where $C_t$ and $L_t$ are the household’s private consumption and hours worked, respectively; $\beta$ is the discount factor, and $\gamma$ denotes the inverse of the wage elasticity for labor supply. Notice that the period utility function in (7) is consistent with long-run balanced growth, a feature that is commonly adopted in the real business cycle literature. The budget constraint faced by the representative agent is given by

$$C_t + p_t I_t = (1 - \tau) \left[ r_t K_t + w_t L_t \right],$$

(8)

where $I_t$ is gross investment, $\tau \in [0, 1)$ represents the distortionary income tax rate that is a fixed constant, $Y_t$ is national income or GDP, and $K_t$ is the household’s capital stock that evolves according to the law of motion

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad K_0 > 0 \text{ given},$$

(9)

where $\delta \in (0, 1)$ is the capital depreciation rate. When $\tau = 0$, our model collapses to the original Harrison (2001) economy under laissez faire.

The first-order conditions for the household’s dynamic optimization problem are

$$C_t L_t^\gamma = (1 - \tau)w_t,$$

(10)

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}} \left[ \frac{(1 - \tau)r_{t+1} + (1 - \delta)p_{t+1}}{p_t} \right],$$

(11)

$$\lim_{t \to \infty} \beta^t \frac{K_{t+1}}{C_t} = 0,$$

(12)

where (10) equates the slope of household’s indifference curve to the after-tax real wage rate, (11) is the Euler equation for intertemporal choices of private consumption, and (12) is the transversality condition.
2.3 Government

As in Guo and Harrison (2004), the government collects its tax revenues \( \tau Y_t \) to pay for public spending on goods and services \( G_t \) that is produced by the consumption and investment sectors; and balances its budget each period. Hence, its period budget constraint is given by

\[ \tau Y_t = G_t = G_{ct} + p_t G_{It}, \quad (13) \]

where \( G_{ct} \) and \( G_{It} \) are quantities of consumption and investment goods, respectively, purchased by the government. In addition, public expenditures on the consumption and investment goods are postulated to be constant fractions of the government’s total spending:

\[ \frac{G_{ct}}{G_t} = \phi \quad \text{and} \quad \frac{p_t G_{It}}{G_t} = 1 - \phi, \quad (14) \]

where \( 0 \leq \phi \leq 1 \). Finally, combining (8) and (13) yields the following aggregate resource constraint for the economy

\[ C_t + p_t I_t + G_t = Y_t. \quad (15) \]

2.4 Competitive Equilibrium and Local Dynamics

Since firms use identical private technologies and face the same factor prices across the two sectors, the fractions of capital and labor inputs utilized in the consumption sector are equal,

\[ \frac{K_{ct}}{K_t} = \frac{L_{ct}}{L_t} \equiv \mu_t. \quad (16) \]

Using (1)-(2), (5)-(6) and (16), the equilibrium relative price of investment goods can be expressed as

\[ p_t = \frac{1}{[(1 - \mu_t) K_t^\alpha L_t^{1-\alpha}]^\phi}. \quad (17) \]

We focus on competitive symmetric equilibria that consist of a set of prices \( \{p_t, r_t, w_t\}_{t=0}^\infty \) and allocations \( \{C_t, L_t, K_{t+1}\}_{t=0}^\infty \) which satisfies the household’s and firms’ first-order conditions. The equalities of demand by households and supply by firms in the consumption and investment sectors are given by \( C_t + G_{ct} = Y_{ct} \) and \( I_t + G_{It} = Y_{It} \). Moreover, both the capital and labor markets will clear, hence \( K_{ct} + K_{It} = K_t \) and \( L_{ct} + L_{It} = L_t \).

Next, it is straightforward to show that our model’s unique interior steady state at which the fraction of factor inputs allocated to producing consumption goods is
and that the corresponding expressions of all remaining endogenous variables can be easily derived. We then take log-linear approximations to the economy’s equilibrium conditions in a neighborhood of this steady state to obtain the following dynamical system:

\[
\begin{bmatrix}
\hat{K}_{t+1} \\
\hat{C}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{K}_t \\
\hat{C}_t
\end{bmatrix}, \quad \hat{K}_0 \text{ given},
\]

(19)

where hat variables represent percentage deviations from their respective steady-state values, and \( J \) is the Jacobian matrix of partial derivatives of the transformed dynamical system. After tedious (but manageable) algebra, the elements that make up our model’s Jacobian matrix are given by

\[
J_{11} = 1 - \delta[1 - \alpha(1 + \theta)] + \frac{\alpha \delta \left(1 - \alpha\right) \left(1 + \Theta \right) + \frac{(1 + \gamma) \Omega_1}{\alpha \beta \delta} + \frac{\theta \left(1 - \beta \left(1 - \delta\right)\right)}{\Upsilon_2}}{\alpha + \gamma},
\]

(20)

\[
J_{12} = -\frac{\delta \left(1 - \alpha\right) \left(1 + \Theta \right) + \frac{(1 + \gamma) \Omega_1}{\alpha \beta \delta} + \frac{\theta \left(1 - \beta \left(1 - \delta\right)\right)}{\Upsilon_2}}{\alpha + \gamma},
\]

(21)

\[
J_{21} = \alpha J_{11} + \frac{\alpha \delta \left(1 - \alpha\right) \left(1 + \Theta \right) + \frac{(1 + \gamma) \Omega_1}{\alpha \beta \delta} + \frac{\theta \left(1 - \beta \left(1 - \delta\right)\right)}{\Upsilon_2}}{\alpha + \gamma + (1 - \alpha) \left[1 - \beta \left(1 - \delta\right)\right] - \beta \theta \left(1 - \delta\right)} \left(1 - \alpha + \frac{(1 - \tau)(1 + \gamma) \Omega_1}{\Upsilon_2}\right),
\]

(22)

\[
J_{22} = \alpha J_{12} + \frac{\alpha + \gamma - \theta \left(1 - \alpha + \frac{(1 - \tau)(1 + \gamma) \Omega_1}{\Upsilon_2}\right)}{\alpha + \gamma + (1 - \alpha) \left[1 - \beta \left(1 - \delta\right)\right] - \beta \theta \left(1 - \delta\right)} \left(1 - \alpha + \frac{(1 - \tau)(1 + \gamma) \Omega_1}{\Upsilon_2}\right),
\]

(23)

where \( \Omega_1 \equiv 1 - \beta + \beta \delta (1 - \alpha) > 0, \; \Omega_2 \equiv \tau (1 - \phi) [1 - \beta(1 - \delta)] + \alpha \beta \delta (1 - \tau) > 0 \) and \( \Omega_3 \equiv 1 - \beta (1 - \delta) [1 - \alpha(1 + \theta)] > 0 \). It follows that the expressions for the determinant and trace of \( J \) are

\[
Det(J) = \frac{(\alpha + \gamma) \left\{1 - \delta \left[1 - \alpha(1 + \theta)\right] + \alpha \delta (1 + \Theta) \left(1 - \alpha + \frac{\Omega_1}{\alpha \beta \delta}\right) - \theta \left(1 - \delta\right) \left(1 - \alpha + \frac{(1 - \tau)(1 + \gamma) \Omega_1}{\Upsilon_2}\right)\right\}}{\alpha + \gamma + (1 - \alpha) \left[1 - \beta \left(1 - \delta\right)\right] - \beta \theta \left(1 - \delta\right) \left(1 - \alpha + \frac{(1 - \tau)(1 + \gamma) \Omega_1}{\Upsilon_2}\right)},
\]

(24)
and

\[ Tr(J) = 1 - \delta [1 - \alpha(1 + \theta)] + \frac{\alpha + \gamma + \delta \Omega_3 [1 - \alpha + \frac{(1+\gamma)\Omega_1}{\alpha \beta \delta}] - (1 - \delta \Omega_3) [1 - \alpha + \frac{(1-\tau)(1+\gamma)\Omega_1}{\Omega_3}]}{\alpha + \gamma + (1 - \alpha)[1 - \beta(1 - \delta)] - \beta \theta(1 - \delta)} \left[1 - \alpha + \frac{(1-\tau)(1+\gamma)\Omega_1}{\Omega_2}\right]. \]

(25)

### 3 Macroeconomic (In)stability

Since there is one initial condition represented by \( K_0 \), the economy will exhibit saddle-path stability and equilibrium uniqueness when one eigenvalue of \( J \) lies inside and the other outside the unit circle. When both eigenvalues are inside the unit circle, the steady state becomes an indeterminate sink around which there are a continuum of stationary equilibrium trajectories that display cyclical fluctuations driven by agents’ animal spirits or sunspots. This will take place if and only if conditions (i) \(-1 < \text{Det}(J) < 1\) and (ii) \([-1 + \text{Det}(J)] < Tr(J) < 1 + \text{Det}(J)\) hold simultaneously.

To facilitate comparisons with our earlier work Chang et al. (2015) and other previous research, this section examines the quantitative interrelations between the government’s tax/spending policy and equilibrium (in)determinacy within a calibrated version of our two-sector RBC model under the GH balanced-budget formulation. Each period in the model is taken to be one quarter. As in Benhabib and Farmer (1996), Harrison (2001), Chang et al. (2015) and many earlier studies in the real business cycle literature, the labor share of national income, \( 1 - \alpha \), is chosen to be 0.7; the discount factor, \( \beta \), is set to be \( \frac{1}{1.01} \); the labor supply elasticity, \( \gamma \), is equal to 0 (i.e. indivisible labor, à la Hansen [1985] and Rogerson [1988], that is infinitely elastic); and the capital depreciation rate, \( \delta \), is fixed at 0.025. Based on Harrison’s (2003) empirical findings on the U.S. economy, we set the degree of productive externalities for investment \( \theta \) to be 0.108.

We find that under all feasible combinations of \( \tau \in [0, 1] \) and \( \phi \in [0, 1] \), together with the aforementioned benchmark parameterization, the most binding restriction among the necessary and sufficient conditions as in (i) and (ii) for local indeterminacy turns out to be

\[ \text{Det}(J) + Tr(J) > -1. \]

(26)

Using equations (24) and (25), this inequality condition will be used below for the theoretical analyses of macroeconomic (in)stability within our model. Given the baseline calibrations on \( \alpha, \beta, \gamma, \delta \) and \( \theta \) described above, Figure 1 depicts the economy’s local stability properties in
terms of the fraction of government spending from the consumption sector $\phi$ versus the income tax rate $\tau$. In particular, the $\phi - \tau$ space can be divided into regions of “Saddle” and “Sink” by the upward-sloping curve characterized by $Det(J) + T\tau(J) = -1$. Below are our findings:

**Result 1.** When $0 \leq \tau \leq 0.098$, our model always exhibits an indeterminate steady state, regardless of how public expenditures are separated between consumption and investment goods.

To understand the intuition behind this result, we note that the private-consumption Euler equation (11) can be rewritten as

$$\frac{C_{t+1}}{C_t} = \beta \left[ \frac{(1 - \tau)r_{t+1} + (1 - \delta)p_{t+1}}{p_t} \right].$$

(27)

Start the laissez-faire ($\tau = 0$) model from its steady state at period $t$, and suppose that agents become optimistic about the economy’s future. Acting upon this non-fundamental belief change, the representative household will consume less and invest more today, which in turn raises next period’s capital stock, hours worked, output, and consumption. As a result, the left-hand side of (27) will rise. For this alternative dynamic path to be justified as a self-fulfilling equilibrium, the (price-weighted) rate of return on $K_{t+1}$ net of depreciation, i.e. the right-hand side of (27), needs to increase as well.

Generally speaking, our model’s local dynamics are governed by the relative strength of two opposite effects. On the one hand, an increase in today’s private investment that raises $K_{t+1}$ will lead to a lower real interest rate $r_{t+1}$ because of diminishing marginal product of capital as $\alpha(1 + \theta) < 1$. Therefore, this MPK effect causes the right-hand side of (27) to fall. On the other hand, due to the presence of positive productive externalities in the investment sector, the economy’s social production possibility frontier which traces out the trade-off between private consumption and investment spending is convex to the origin. As a result, the relative price of investment $p_t$ will decrease upon agents’ optimism that shifts more capital and labor inputs into producing investment goods ($\frac{\partial p_t}{\partial (1 - \mu_t)} < 0$; see equation 17). Consequently, this price effect causes right-hand side of (27) to rise.

As in the no-government economy of Harrison (2001, p. 756) with identical calibrations on $\alpha$, $\beta$, $\gamma$ and $\delta$, the origin of Figure 1 with $\tau = G_t = 0$ illustrates that since price effect is quantitatively stronger, the model’s steady state is an indeterminate sink. Intuitively, the equilibrium rate of return on capital will rise to fulfill the household’s initial rosy expectations in that the calibrated degree of investment sector-specific externalities ($\theta_{US} = 0.108$) is higher than the critical level of 0.0773. In this environment, positive income taxation ($\tau > 0$) generates a downward shift of the convex social production possibility frontier, which in turn induces agents to reduce their optimism-driven consumption as well as investment expendi-
tures. It follows that the above-mentioned price effect that helps make for multiple equilibria becomes weaker, whereas the corresponding after-tax MPK effect is strengthened because of a lower net real interest rate \((1 - \tau)r_{t+1}\). Figure 1 shows that for all values of \(\phi \in [0, 1]\) under the baseline parameterization, the price effect continues to dominate provided \(0 < \tau \leq 0.098\). That is, regardless of how the government separates its purchases between consumption and investment goods, indeterminacy and sunspots always occur in our benchmark model with “relatively low” (constant) income tax rates.

Remarks. Result 1 is obtained under the calibrated value of \(\theta = 0.108\) that is taken from Harrison’s (2003) estimation on the U.S. manufacturing data. Nevertheless, the preceding paragraph indicates that our model’s local stability properties are affected by the degree of productive externalities in investment. To derive the precise lower bound on \(\theta\) which ensures that Result 1 holds, we use condition (26) to find that the requisite \(\theta_{\min}\) as a function of \(\tau\) and \(\phi\) is

\[
\theta_{\min}(\tau, \phi) = \frac{2\alpha\beta \{\alpha(1 - \beta) + [2 - (1 - \alpha)\delta]\gamma + \Omega_1\} + (1 + \gamma)\Omega_1\Pi_1}{2\alpha\beta\Pi_2 - (1 + \gamma)[1 - \beta(1 - \delta)]\Omega_1 + \frac{2\alpha\beta(1 + \beta)(1 + \gamma)[1 - \delta][1 - \gamma]\Omega_1}{\Omega_2}},
\]

where \(\Pi_1 \equiv \alpha(1 + \beta) + \Omega_1\), and \(\Pi_2 \equiv (1 - \alpha)(1 + \beta)(1 - \delta) - \alpha\delta(1 + \gamma)\). In addition, it is straightforward to derive that \(\frac{\partial \theta_{\min}}{\partial (1 - \phi)} > 0\), which turns out to be qualitatively similar to Result 5 of Chang et al. (2015, p. 29) for an identical two-sector RBC model under lump-sum income taxation. Specifically, an increase in the public-investment share of government spending \((1 - \phi)\) ceteris paribus generates a stronger MPK effect that helps stabilize the economy against sunspot-driven business cycle fluctuations. It follows that the threshold level of investment externalities needed for aggregate instability \(\theta_{\min}\) will rise.

Next, we follow the same procedure to analytically prove that the upper bound on the tax rate \(\tau_{\max}\), as a function of \(\theta\) and \(\phi\), which leads to Result 1 is given by

\[
\tau_{\max}(\theta, \phi) = 1 - \frac{(1 - \phi)[1 - \beta(1 - \delta)]}{(1 - \phi)[1 - \beta(1 - \delta)] + \alpha\beta \left\{\frac{2\theta(1 + \beta)(1 - \delta)(1 + \gamma)\Omega_1}{2\alpha\beta(1 - \alpha)\beta + \Omega_1 - \theta\Pi_2} + (1 + \gamma)\Omega_1(\theta[1 - \beta(1 - \delta)] + \Pi_1)\right\}} - \delta.
\]

Substituting \(\phi = 0\) together with the benchmark values of \(\alpha, \beta, \gamma, \delta\) and \(\theta\) into (29) verifies that \(\tau_{\max} = 0.098\), as shown in the vertical axis of Figure 1.

Result 2. When \(\tau > 0.098\), the model’s steady state can either be a saddle point or a sink depending on the value of the public-consumption share \(\phi\). Specifically, there exists a

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4Notice that Chang et al. (2015) obtain this result when government purchases of consumption and investment goods are postulated to be constant fractions of their respective sectoral output.
threshold level, denoted as $\phi_{min}$, which is monotonically increasing with respect to the income tax rate ($\frac{\partial \phi_{min}}{\partial \tau} > 0$), such that local indeterminacy occurs when $\phi > \phi_{min}$.

We first observe that for a given level of $\tau > 0.098$, the resulting after-tax $MPK$ effect outweighs the price effect as long as the fraction of government spending on consumption goods $\phi$ is zero, as demonstrated by the vertical axis of Figure 1, or “relatively small”. In this case, our model exhibits saddle-path stability and equilibrium uniqueness since the fall in the right-hand side of (27) is sufficiently strong to invalidate agent’s initial anticipation of a higher rate of return on capital. It follows that raising the public-consumption share to the corresponding critical level $\phi_{min}$ ($= 0.5665$ when $\tau = 0.2$, which is also the steady-state ratio of government purchases to the economy’s total output) or above leads to a dominating price effect. This in turn destabilizes the macroeconomy by generating aggregate fluctuations driven by animal spirits.

Per the same reasoning for Result 1 under positive income taxation, an increase in the tax rate will enhance the after-tax $MPK$ effect with a larger decrease in $(1 - \tau) r_{t+1}$ and weaken the price effect with a smaller reduction in $p_t$. In this case, $\phi_{min}$ will rise ($\frac{\partial \phi_{min}}{\partial \tau} > 0$) to overturn the relative magnitude of these two opposing forces, which in turn change the model’s steady state from being a saddle point to a sink. In sum, Results 1 and 2 together indicate that endogenous business cycles will arise within the benchmark parameterization of our two-sector RBC model under the GH balanced-budget formulation when (i) the income tax rate $\tau \in (0, 0.098]$ and the public-consumption share $\phi \in [0, 1]$; or (ii) $0.098 < \tau < 1$ and $\phi > \phi_{min}$.

**Remarks.** Using equation (26), it can be shown that the analytical expression for $\phi_{min}$, as a function of $\theta$ and $\tau$, which satisfies Result 2 is

$$
\phi_{min}(\theta, \tau) = 1 - \frac{\alpha \beta (1 - \tau)}{\tau (1 - \beta (1 - \delta))} \left\{ \frac{2 \theta (1 + \beta)(1 + \gamma)(1 - \delta)(1 - \tau) \Omega_1}{2 \alpha \beta \{\gamma [2 - (1 - \alpha) \delta] + \Pi_1 - \theta \Pi_2 + (1 + \gamma) \Omega_1 \{\theta [1 - \beta (1 - \delta)] + \Pi_1}\}} - \delta \right\},
$$

and that $\frac{\partial \phi_{min}}{\partial \tau} = \frac{1 - \phi_{min}}{\tau (1 - \tau)} > 0$ for all levels of $\tau \in (0.098, 1)$.

Although infinitely elastic labor hours ($\gamma = 0$) are commonly postulated in many early RBC-based indeterminacy studies such as Benhabib and Farmer (1994) and Farmer and Guo (1994), recent empirical research by Chetty et al. (2011, 2012) reports that modern macroeconomic calibrations have taken on a larger labor supply elasticity than that observed in the micro-level evidence. Accordingly, the next result explores our model economy with lower labor supply elasticities or strictly positive values of $\gamma$.

**Result 3.** Given the same baseline calibrations on $\alpha$, $\beta$, $\delta$ and $\theta$, the economy always
exhibits saddle-path stability and equilibrium uniqueness when $\gamma \geq 0.208$, regardless of the income tax rate $\tau$ and the sectoral composition of government spending on goods and services $\phi$.

With less elastic labor supply (or when $\gamma$ is higher than zero), agents are less willing to move out of leisure into hours worked at period $t+1$ upon an expected increase in the rate of return on today’s investment. This will lead to smaller increases in $L_{t+1}$ and $r_{t+1}$ via firms’ capital demand, which in turn dampens the representative household’s initial optimism. Our numerical experiments find that independent of the government’s tax and spending policies (i.e. for all combinations of $\tau \in [0, 1]$ and $\phi \in [0, 1]$), the model’s steady state is a locally determinate saddle point provided the labor supply elasticity parameter is “sufficiently high” to generate a relatively stronger after-tax MPK effect. Since there are no shocks to economic fundamentals within our model economy, the resulting unique equilibrium with $\gamma \geq 0.208$ will not display any cyclical fluctuations.\(^5\)

**Remarks.** Result 3 illustrates that there exists a lower bound on the preference parameter $\gamma$ (or an upper bound on the household’s labor supply elasticity $\frac{1}{\alpha}$), as a function of $\tau$ and $\phi$, above (below) which the model’s steady state is always a saddle point. We use condition (26) to derive that the explicit expression for $\gamma_{\text{min}}$ is

$$
\gamma_{\text{min}}(\tau, \phi) = \frac{2\alpha\beta \left\{ \theta \left( (1 - \alpha)(1 + \beta)(1 - \delta) - \alpha\delta \right) - \Pi_1 + \frac{\theta(1+\beta)(1-\delta)(1-\tau)\Omega_1}{\Omega_2} \right\} - \Omega_1\Pi_3}{2\alpha\beta \left\{ 2 - \delta[1 - \alpha(1 + \theta)] - \frac{\theta(1+\beta)(1-\delta)(1-\tau)\Omega_1}{\Omega_2} \right\} + \Omega_1\Pi_3}, \quad (31)
$$

where $\Pi_3 \equiv 2 - [1 - \alpha(1 + \theta)][1 + \beta(1 - \delta)]$. It follows that the economy is less susceptible to indeterminacy and sunspots when the household’s labor supply becomes more inelastic. While Chang et al. (2015) do not conduct this particular sensitive analysis, it is straightforward to show that Result 3 will qualitatively continue to hold in our two-sector RBC model with non-distortionary income taxation.

### 4 Conclusion

In this paper, we have examined the interrelations between government purchases, distortionary income taxation and equilibrium (in)determinacy in Harrison’s (2001) representative-agent macroeconomy with two production sectors. Given Guo and Harrison’s (2004) balanced-budget formulation, we find that regardless of how public expenditures are divided between

\(^{5}\)By contrast, Huang and Meng (2012) find that in a one-sector representative-agent macroeconomy with monopolistic competition and unsynchronized wage adjustment, equilibrium indeterminacy can arise largely independent of labor supply elasticities. It would be worthwhile for future research to examine the robustness of our Result 3 under sticky wages.
consumption and investment goods, our model economy under the benchmark parameterization will exhibit indeterminacy and sunspots provided the (constant) income tax rate does not exceed a critical value. When the tax rate is raised to a higher level, the model’s steady state can either be a saddle point or a sink depending on the fraction of government spending in the consumption sector; and the threshold for the public-consumption share above which aggregate instability occurs is shown to be monotonically increasing with respect to the income tax rate. Finally, we find that saddle-path stability and equilibrium uniqueness always prevail when the labor-supply elasticity parameter is not lower than a certain minimum. For the sake of analytical completeness, theoretical proofs for each of the aforementioned quantitative results are also provided.

This paper can be extended in several directions. For example, it would be worthwhile to explore our model economy under a non-balanced budget with national debt à la Schmitt-Grohé and Uribe’s (1997, pp. 990-991); a progressive tax policy à la Guo and Lansing (1998) or Dromel and Pintus (2007); the presence of maintenance expenditures on existing capital stock à la Guo and Lansing (2007); a non-separable preference formulation à la Linnemann (2008); or productive/utility-generating government spending à la Guo and Harrison (2008). These possible extensions will allow us to examine the robustness of this paper’s quantitative/theoretical results and policy implications, as well as further enhance our understanding of the (de)stabilization effects of public expenditures within a multi-sector representative-agent macroeconomy. We plan to pursue these research projects in the near future.
References


Figure 1. GH Balanced-Budget Rule with Constant Tax Rates